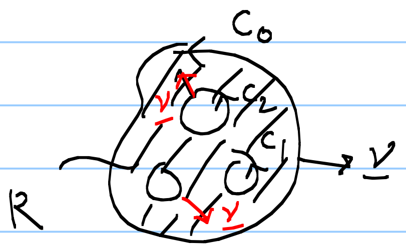
Lecture 3.3:

Torsion Analysis for Multiply-Connected Section

노트 제목



multiply-connected crossection

C_0: outer boundary ↺

C_i $(i=1,\cdots,n)$: inner boundary ↻

- Warping ftn formulation
  - ↳ no need to consider compability
- Prandtle ftn approach
  - ↳ must consider compatibility for single-valuedness

<warping ftn formulation>

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \quad \text{in } R$$

$$\frac{d\phi}{d\nu}=y\nu_x-x\nu_y \quad \text{on } C_i \quad (i=0,1,\cdots,n)$$

<Prandtle Approach>

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-2$$

$$\underbrace{\psi = \overset{\text{constant } k_i}{k_i} \text{ on } C_i \quad \text{with } k_0 = 0}_{\text{physical meaning} \quad t_z \text{ on } C_i = 0}$$

What are the values of k_i = ?

$\Rightarrow$ "n" additional conditions are needed

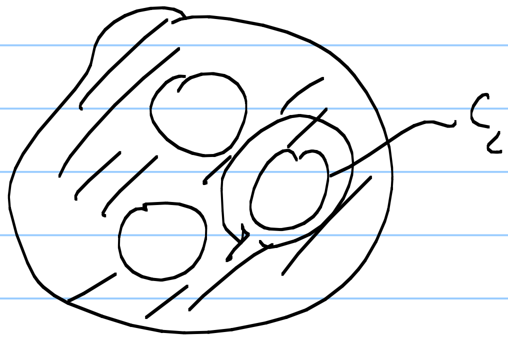
"must satisfy single-valuedness"

We will show

$$\oint_{C_i} \frac{d\psi}{d\nu}\, dl = 2A_i \quad (i = 1, \ldots, n) \quad (*)$$

A_i: Area of a hole bounded by C_i

<Derivation of (*)>



$$0 = \oint_{C_i} d\,u_z$$

$$= \oint_{C_i} \alpha \, d\varphi$$

$$= \alpha \left[\oint_{C_i} \frac{\partial \varphi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \right]$$

$$0 = \alpha \left[\oint_{C_i} \left(\frac{\partial \psi}{\partial y} + y \right) dx - \left(x + \frac{\partial \psi}{\partial x} \right) dy \right]$$

or $$\oint_{C_i} \left(\underbrace{\frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy}_{} \right) = \oint_{C_i} (\underbrace{-y\,dx + x\,dy}_{})$$

$$LHS = \oint -\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right)\cdot(dy, -dx)$$

$$= -\oint \underline{\nabla}\psi \cdot \left(\underbrace{\frac{dy}{ds}}_{\nu_x}, \underbrace{-\frac{dx}{ds}}_{\nu_y}\right) ds$$

$$= -\oint \underline{\nabla}\psi \cdot \underline{\nu}\, ds = -\oint \frac{d\psi}{d\nu} ds$$

$$RHS = \oint -y\,dx + x\,dy$$

$$= \oint (x, y)\cdot(dy, -dx)$$

$$= \oint (x, y)\cdot\left(\frac{dy}{ds}, -\frac{dx}{ds}\right) ds$$

$$= \oint_{c_i} (x, y)\cdot \underline{\nu}\, ds$$

$$\overset{\text{Div}}{=} \oint_{c_i} \underline{\nabla}\cdot(x, y)\, dA = 2\oint_{c_i} dA$$

$$= 2A_i$$

Thus

$$\boxed{\oint_{C_i} \frac{d\psi}{d\nu}\, ds = -2A_i \quad (i=1,\cdots,n)} \quad (*)$$

physical meaning of $(*)$:

use

$$\tau = -\mu\alpha \frac{d\psi}{d\nu}$$

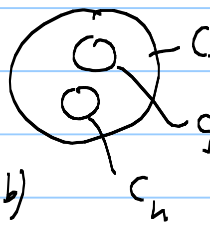
Then

$$\underbrace{\oint_{C_i} \tau\, ds = 2\mu\alpha A_i}$$

Circulation of τ // Area enclosed

Summary: (Prandtl Stress Function Approach)

$$\begin{cases} \nabla^2 \psi = -2 \quad \text{in } R & \text{(a)} \\ \left.\begin{cases} \psi = k_i \quad \text{on } C_i \quad (i=1,\cdots n) \\ \quad \hookrightarrow \text{unknown const} \\ \psi = 0 \quad \text{on } C_0 \end{cases}\right\} & \text{(b)} \end{cases}$$



with

$$\oint_{C_i} \frac{d\psi}{d\nu}\, ds = -2 A_i \qquad (i=1,\cdots,n) \qquad \text{(c)}$$

<u>Solution procedure</u> (analytic)

① $\psi = \psi_p + \sum_{i=1}^{n} \alpha_j \psi_c^{(j)}$

Select ψ_p and ψ_c as

② $\nabla^2 \psi_p = -2$

$\psi_p = 0$ for all C_i $(i=0,\cdots n)$

③ $$\begin{cases} \nabla^2 \psi_c^{(j)} = 0 \\ \psi_c^{(j)} = 0 \quad \text{on } C_k \quad (k \neq j) \\ \psi_c^{(j)} = 1 \quad \text{on } C_j \end{cases}$$

■ ①, ②, ③ satisfy (a) and (b).

Check:

PDE: $\nabla^2 \psi = \nabla^2 \psi_p + \sum_{i=1}^{n} d_i \psi_c^{(i)}$

$= -2 + 0 \Rightarrow (a)$

BC:

$$\begin{cases} \psi\big|_{C_0} = \psi_p\big|_{C_0} + \sum d_i \psi_c^{(i)}\big|_{C_0} \\ \quad = 0 + \sum d_i \, 0 = 0 \\ \psi\big|_{C_k} = \psi_p\big|_{C_k} + \sum_{i=1}^{n} d_i \psi_c^{(i)}\big|_{C_k} \end{cases}$$

$(k \neq 0)$

$= 0 + d_k (1) = d_k$

constant

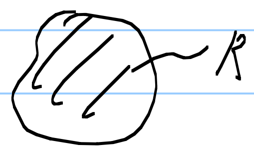
To determine d_k;

$$\oint_{C_i} \frac{d\psi}{d\nu}\, ds = -2A_i \qquad (i=1,\dots,n)$$

$$\rightarrow \boxed{\sum_{j=1}^{n} d_j \oint_{C_i} \frac{d\psi_c^{(j)}}{d\nu}\, ds + \oint_{C_i} \frac{d\psi_p}{d\nu}\, ds = -2A_i \qquad (i=1,\dots n)}$$

n additional equations

Torsional Rigidity C_t;



Recall $C_t = M_t/\alpha$

$$= \int_R (x\sigma_{yz} - y\sigma_{xz})\, dA/\alpha$$

$$= -\mu \int_R \left(x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y}\right) dA$$

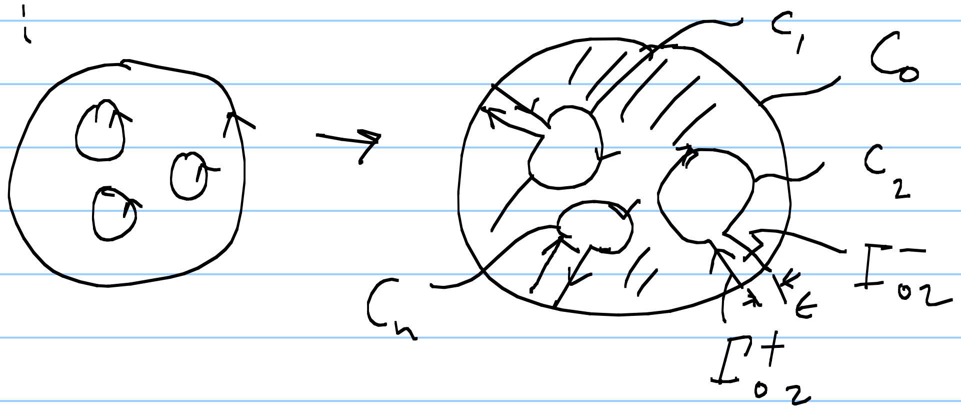
$$= -\mu \int_R \left[\frac{\partial}{\partial x}(x\psi) + \frac{\partial}{\partial y}(y\psi)\right] dA + 2\mu \int_R \psi \, dA$$

$$\underset{\text{Divergence Th}}{=} -\mu \oint_C \psi \,(x\nu_x + y\nu_y)\, dl + 2\mu \int_R \psi \, dA$$

applies only in simply connected Region

To deal with multiply-connected region:

trick:



Introduce C

$$C = C_0 - \sum_{i=1}^{n} C_i + \lim_{\epsilon \to 0} \sum_{i=1}^{n} \left(\Gamma_{0i}^{+} + \Gamma_{0i}^{-} \right)$$

$\Rightarrow$ With C, R is now treated as a simply-connected region!!

Then

$$C_t = -\mu \int_R \left(\frac{\partial}{\partial x}(x\psi) + \frac{\partial}{\partial y}(y\psi) \right) dA + 2\mu \int_R \psi \, dA$$

$$= -\mu \oint_C (x\psi\nu_x + y\psi\nu_y)\, dl + 2\mu \int_R \psi \, dA$$

$$= -\mu \left(\oint_{C_0} - \oint_{C_1} - \oint_{C_2} \cdots - \oint_{C_n} + \lim_{\epsilon \to 0} \sum_{i=1}^{n} \int_{\Gamma_{0i}^{+}} + \int_{\Gamma_{0i}^{-}} \right) \psi (x\nu_x + y\nu_y)\, dl$$

$$+ 2\mu \int_R \psi \, dA$$

$$= -\mu \left[\cancel{k_0} \oint (x\nu_x + y\nu_y)\, dl - \sum_{i=1}^{n} k_i \oint (x\nu_x + y\nu_y)\, dl \right] + 2\mu \int_R dA$$

$$= \mu \sum_{i=1}^{n} \oint_{C_i} k_i (x\nu_x + y\nu_y)\, dl + 2\mu \int_R dA$$

$$= 2\mu \left(\sum_{i=1}^{n} k_i A_i + \int_R \psi\, dA \right)$$

A_i = Area enclosed by C_i $(i = 1, \dots, n)$

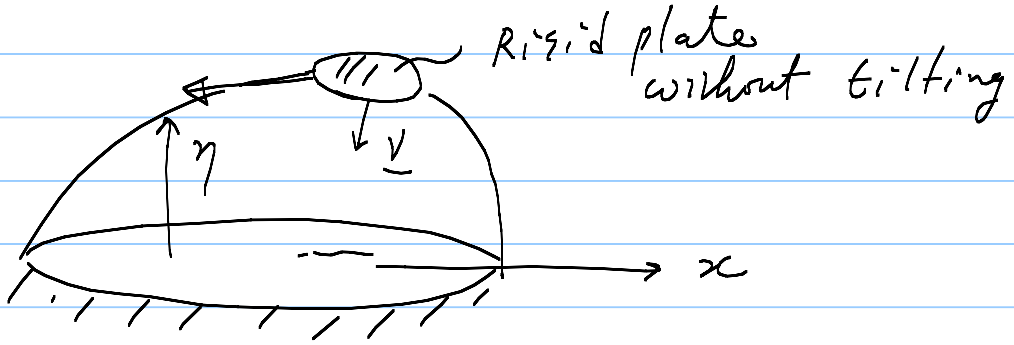
<Membrane Analogy>

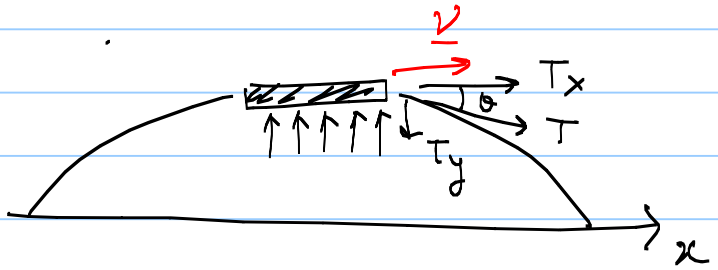
$$\oint_{C_i} \frac{d\psi}{d\nu}\, ds = -2A_i$$

$\oplus$ $$\psi = \frac{2T}{p}\eta$$

$$\oint_{C_i} \left(-T\frac{d\eta}{d\nu}\right) ds = p A_i$$

(η = CONST on C_i)





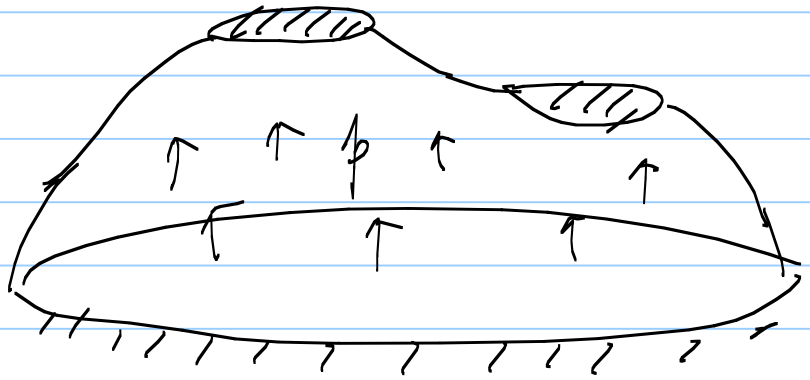
$$T_y = T\sin\theta = T\left(-\frac{\Delta\eta}{\Delta x}\right) = -T\frac{d\eta}{d\nu}$$

$$\oint T_y\, ds = p A_i$$

or $$\int \left(-T\frac{d\eta}{d\nu}\right) ds = p A_i$$

Eqm condition

$\Rightarrow$ Hole surrounded by C_i $\equiv$ tiltingless rigid plate



Torsional Rigidity

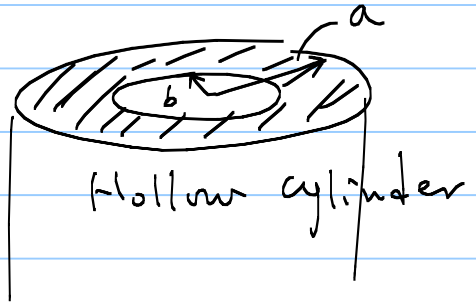
$$C_t = 2\mu\left(\int \sum_{i=1}^{n} k_i A_i + \int_R \psi \, dA\right)$$

$$= \frac{4\mu T}{\varphi}\left(\sum_{i=1}^{n} K_i A_i + \int_R \eta \, dA\right)$$

value of η along C_i

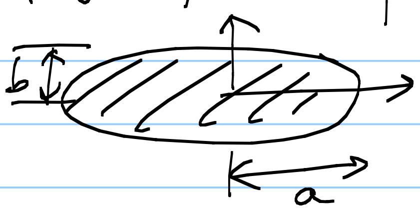
"Total volume enclosed by R with rigid plates"

Special Case



$$\psi = \frac{1}{2}(a^2 - r^2) \qquad (a)$$

solution from elliptic section



$$\psi = -\frac{a^2 b^2}{a^2 + b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

(a) ψ satisfies $\nabla^2\psi = -2$ yes

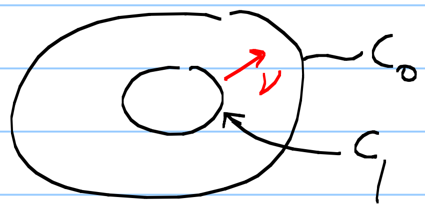
① $\psi = 0$ on C_0 $(r=a)$ yes

② $\psi = k_1$ on C_1 $(r=b)$ yes

$= \frac{1}{2}(a^2-b^2)$

③ $\boxed{\oint_{C_1} \frac{d\psi}{d\nu}\, ds \overset{?}{=} -2A_1}$

$$\frac{d\psi}{d\nu} = \frac{d\psi}{dr} = -r$$



$$\therefore \oint_{C_1} \frac{d\psi}{d\nu}\, ds = \int_0^{2\pi} (-b)\,(b)\,d\theta$$

$$= -2\pi b^2 = -2A_1$$

(yes)

In this case

$$C_t = \frac{\mu\pi}{2}\left(a^4 - b^4\right)$$