29 Laurent Series

29.1 Laurent Series

- Expansion of $f(z)$ around a point where $f(z)$ has a singularity.

Theorem 1 Laurent's theorem.

If $f(z)$ is analytic on two concentric circles C_1 and C_2 with center z_0 and in the annulus between them, then $f(z)$ can be represented by the Laurent series.

(1)

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \\ &= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots \end{aligned}$$

consisting of nonnegative powers and the principal part (the negative powers). The coefficients of this Laurent series are given by the integrals.

(2)

$$a_n = \frac{1}{2\pi i}\oint_c \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*, \qquad b_n = \frac{1}{2\pi i}\oint (z^*-z_0)^{n-1} f(z^*) dz^*.$$

taken counterclockwise around any simple closed path c that lies in the annulus and encircles the inner circle.

This series converges and represents $f(z)$ in the open annulus by continuously increasing the outer circle C_1 and decreasing C_2 until each of the two circles reaches a point where $f(Z)$ is singular.

In the important special case that z_0 is the only singular point of $f(z)$ inside C_2, this circle can be shrunk to the point z_0, giving convergence in a disk except at the center.

(1')

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

(2')

$$a_n = \frac{1}{2\pi i}\oint_c \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^* \quad (n = 0, \pm 1, \pm 2, \cdots)$$

Proof. (a) positive powers.

(3)

$$f(z) = g(z) + h(z) = \frac{1}{2\pi i}\oint_{c_1}\frac{f(z^*)}{z^*-z}dz^*dz^* - \frac{1}{2\pi i}\oint_{c_2}\frac{f(z^*)}{z^*-z}dz^*$$

1^{st} integral : Taylor series of $g(z)$.

(4)

$$g(z) = \frac{1}{2\pi i}\oint_{c_1}\frac{f(z^*)}{z^*-z}dz^* = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

(5)

$$a_n = \frac{1}{2\pi i}\oint_{c_1}\frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$$

C_1 can be replaced by C, by the principle of deformation of path.

(b) The negative powers. (the principal parts)
(Since z lies in the annulus)

(6)

$$\text{(a)}\ \left|\frac{\mathrm{z}-\mathrm{z}_0}{\mathrm{z}^*-\mathrm{z}_0}\right| < 1, \qquad \text{(b)}\ \left|\frac{\mathrm{z}^*-\mathrm{z}_0}{\mathrm{z}-\mathrm{z}_0}\right| < 1$$

$(\because z^*$ lied on the $C_1)$ $\qquad$ $(\because z^*$ lies on the $C_2)$

$$\frac{1}{z^*-z} = \frac{1}{z^*-z_0-(z-z_0)} = \frac{-1}{(z-z_0)\left(1-\dfrac{z^*-z}{z-z_0}\right)}$$

$$\frac{1}{z^*-z} = -\frac{1}{z-z_0}\left\{1+\frac{z^*-z}{z-z_0}+\left(\frac{z^*-z}{z-z_0}\right)^2+\cdots+\left(\frac{z^*-z_0}{z-z_0}\right)^n\right\} - \frac{1}{z-z^*}\left(\frac{z^*-z_0}{z-z_0}\right)^{n+1}$$

Multiplication by $-f(z^*)/2\pi i$ and integration over C_2 on both sides now yield.

$$\begin{aligned}
h(z) &= -\frac{1}{2\pi i}\oint_{c_2}\frac{f(z^*)}{z^*-z}dz^* \\
&= \frac{1}{2\pi i}\left\{\frac{1}{z-z_0}\oint_{c_2}f(z^*)dz^* + \frac{1}{(z-z_0)^2}\oint_{c_2}(z^*-z_0)f(z^*)dz^*\right. \\
&\quad \left. +\cdots+\frac{1}{(z-z_0)^{n+1}}\oint_{c_2}(z^*-z_0)^n f(z^*)dz^*\right\} + R_n^*(z).
\end{aligned}$$

(7)

$$R_n^*(z) = \frac{1}{2\pi i(z-z_0)^{n+1}}\oint_{c_2}\frac{(z^*-z_0)^{n+1}}{z-z^*}f(z^*)dz^*$$

$$\text{from (2)} \quad b_n = \frac{1}{2\pi i} \oint_{c_2} (z^* - z_0)^{n-1} f(z^*) dz^*$$

This establishes Laurent's theorem, provided
(8)

$$\lim_{n\to\infty} R_n^*(z) = 0$$

(c) $f(z^*)$ is analytic in the annulus and on C_2, and z^* lies on C_2, and z outside, so that $z - z^* \neq 0$

$$\left|\frac{f(z^*)}{z - z^*}\right| < \tilde{M} \quad \text{for all } z^* \text{ on } C_2$$

$$|R_n^*(z)| \leq \frac{1}{2\pi|z - z_0|^{n+1}} |z^* - z_0|^{n+1} \tilde{M} L = \frac{\tilde{M}L}{2\pi} \left|\frac{z^* - z_0}{z - z_0}\right|^{n+1}$$

$$\text{Since} \quad \left|\frac{z^* - z_0}{z - z_0}\right| < 1, \quad \overset{n\to\infty}{\longrightarrow} \quad R_n^*(z) \to 0$$

(d) Convergence of (1) in the larger annulus.
The first series in (1) is a Taylor series ; hence it converges in the disk D with center z_0 whose radius equals the distance of that singularity of $g(z)$ which is close to z_0.
The second series in (1), representing $h(z)$, is a power series in $z = 1/(z - z_0)$

$$r_2 < |z - z_0| < r_1, \quad r_1 \text{ of } c_1 \ \& \ r_2 \text{ of } c_2.$$

$$1/r_2 > z > 1/r_1$$

This power series in z must converge at least in the disk $|z| < 1/r_2$. This corresponds to the exterior $|z - z_0| > r_2$ of c_2, so that $h(z)$ is analytic for all z outside c_2.

$\therefore$ The domain is the open annulus characterized near the end of Laurent's theorem.

Uniqueness : The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique. However, $f(z)$ may have different Laurent series in two annuli with the same center.

Example 1. Use of Maclaurin series.

Find the Laurent series of $z^{-5} \cdot \sin z$ with center 0.
Solution.

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$z^{-5} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} = \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{z^2}{5040} + \cdots$$

$$(|z| > 0)$$

Example 2. Substitution.

Find the Laurent series of $z^2 \cdot e^{1/z}$ with center O.
Solution.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$

$$z^2 e^{1/z} = z^2 \left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots$$

$$(|z|) > 0)$$

Example 3. $1/(1-z)$

(a) in nonnegative power of z.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{(varid if } |z| < 1)$$

(b) in negative power of z

$$\frac{1}{1-z} = \frac{-1}{z(1-1/z)} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots$$

$$\text{(varid if } |z| > 1)$$

Example 4. Laurent expansions in different concentric annuli.

Find the Laurent series of $1/(z^3 - z^4)$ with center O
Solution.
(I)

$$\frac{1}{z^3 - z^4} = \frac{1 \cdot 1}{z^3(1-z)} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots$$

$$(0 < |z| < 1)$$

(II)

$$\frac{1}{z^3 - z^4} = -\frac{1}{z^4(1-1/z)} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots$$

$$(|z| > 1)$$

Example 5. Use of partial fractions.

Find all Taylor and Laurent series of $f(z) = \dfrac{-2z+3}{z^2 - 3z + 2}$ with center O

Solution.

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{1-z} + \frac{1}{2-z}$$

From example 3.
a)

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

b)

$$\frac{1}{1-z} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots$$

c)

$$+\frac{1}{2-z} = \frac{1}{2\left(1-\frac{z}{2}\right)} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \qquad (|z| < 2)$$

d)

$$-\frac{1}{z-2} = -\frac{1}{z\left(1-\frac{2}{z}\right)} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \qquad (|z| > 2)$$

(I) From (a) and (c), valid for $|z| < 1$,

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n = \frac{3}{2} + \frac{5z}{4} + \frac{9z^2}{8} + \cdots$$

(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots$$