30 Singularity, Residue

30.1 Singularities and Zeroes. Infinity.

- singular point : a z at which $f(z)$ ceases to be analytic.
- zero : a z at which $f(z) = 0$.

isolated singularity : if $z = z_0$ has a neighborhood without further singularities of $f(z)$

$$\text{Example)} \ \tan z, \quad z = \pm\pi/2, \pm 3\pi/2, \ \text{etc}$$

nonisolated singularity : Ex.) $\tan(1/z)$ at $z = 0$

$$f' = -\frac{1}{z^2}\sec^2(1/z)$$

(1)

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \text{valid in the immediate}$$

$$\text{neighborhood of} \quad 0 < |z - z_0| < R$$

The 1^{st} series is analytic at $z = z_0$. The 2^{nd} series is called the principal part of (1)

(2)

$$\frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad\qquad (b_m \neq 0)$$

$$\left(\begin{array}{l} \text{pole} \ : z = z_0 \\ \text{order} \ : m \end{array} \right. \qquad \begin{array}{l} \text{isolated essential singularity} \\ \text{when principal part of (1) has infinitely many terms.} \end{array}$$

simple poles $\quad (m = 1)$

Example 1. Poles. Essential singularities.

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2} \quad \begin{array}{l} \text{a simple pole at } z = 0 \\ \text{a pole of fifth order at } z = 2 \end{array}$$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots \quad \text{isolated essential singularity at } z = 0$$

$$\text{also,} \quad \sin\frac{1}{z} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!z^{2n+1}} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} + \cdots$$

Example 2. Behavior near a pole

$f(z) = 1/z^2$ has a pole at $z = 0$, and $|f(z)| \to \infty$ as $z \to 0$ in any manner.

Theorem 1. (Poles)
If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.

Example 3. Behavior near an essential singularity

$f(z) = e^{1/z}$ has an essential singularity at $z = 0$
No limit for approach along the imaginary axis

$$(\because e^{1/iy} = e^{-i/y} = \cos(1/y) - i\sin(1/y))$$

$$e^{1/z} \to \infty \text{ as } z \to +0 \;\;\&\;\; e^{1/z} \to 0 \text{ as } z \to -0$$

$$e^{1/z} = e^{(\cos\theta - i\sin\theta)/r} = c_0 e^{i\alpha} \neq 0$$

$$e^{\cos\theta/r} = c_0 \Rightarrow \cos\theta = r\ln c_0 \;\;\& \;\; -\sin\theta = \alpha r.$$

$$\cos^2\theta + \sin^2\theta = r^2(\ln c_0)^2 + \alpha^2 r^2 = 1$$

$$r^2 = \frac{1}{(\ln c_0)^2 + \alpha^2} \quad \text{and} \quad \tan\theta = -\frac{\alpha}{\ln c_0}$$

Hence r can be made arbitrarily small by adding multiples of 2π to α, leaving c unaltered.

Theorem 2. (Picard's theorem)
If $f(z)$ is analytic and has an isolated essential singularity at point z_0, it takes on every value, with at most one exceptional value, in an arbitrarily small neighborhood of z_0

Zeros of Analytic Functions.
- zero : $z = z_0$ such that $f(z_0) = 0$
- order n : $f' = f'' = \cdots = f^{(n-1)} = 0$ at $z = z_0$ but $f^{(n)}(z_0) \neq 0$
- 1^{st} order zero : simple zero.

Example 4. Zeros.

i) $f(z) = 1 + z^2 \quad f'(z) = 2z$
$\therefore$ simple zeros at $\pm i$.
ii) $f(z) = (1 - z^4)^2, \quad f'(z) = -4z^3(1 - z^4)\cdot 2 = (-4z^3 + 4z^7)\cdot 2$

$$f''(z) = (-12z^2 + 28z^6)\cdots 2 = -4z^2(3 - 7z^4)(2)$$

$$\therefore \text{ second-order zeros at } \pm 1 \text{ and } \pm i$$

iii) $f(z) = (z-a)^3$, $f'(z) = 3(z-a)^2$, $f''(z) = 6(z-a)$, $f''' = 6$

$$\therefore \text{third-order zero at } z = a$$

iv) $f(z) = e^z$: no zeros

v) $f(z) = \sin z$, $f'(z) = \cos z$

$$\therefore \text{ simple zeros at } 0, \pm\pi, \pm 2\pi, \cdots$$

vi) $f(z) = \sin^2 z$ $f'(z) = 2 \sin \cos z = \sin 2z$ $f''(z) = 2 \cos 2z$

$$\therefore \text{ second-order zeros at } 0, \pm\pi, \pm 2\pi, \cdots$$

vii) $f(z) = 1 - \cos z$, $f'(z) = \sin z$, $f''(z) = \cos z$

$$\therefore \text{ second-order zeros at } 0, \pm 2\pi, \pm 4\pi, \cdots$$

viii) $f(z) = (1 - \cos z)^2$, $f'(z) = 2 \sin z(1 - \cos z)$

$$\begin{aligned}
f''(z) &= 2 \cos z(1 - \cos z) + 2 \sin^2 z = 2 \cos z(1 - \cos z) + 2(1 - \cos^2 z) \\
&= 2(1 - \cos z)(2 \cos z + 1) \\
f'''(z) &= 2 \sin z(2 \cos z + 1) + 2(1 - \cos z)(-2 \sin z) \\
&= 2 \sin z(4 \cos z - 1) \\
f^{iv} &= 2 \cos z(4 \cos z - 1) + 2 \sin z(-4 \sin z) \\
&= 8 \cos^2 z - 2 \cos z - 8 \sin^2 z \\
&\quad \therefore \text{ fourth-order zeros at } 0, \pm 2\pi, \pm 4\pi, \cdots
\end{aligned}$$

Taylor Series at zero

$f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0$ at an n^{th} order. zero $z = z_0$

$$a_0 = a_1 = a_2 = \cdots = a_{n-1} = 0$$

(3)

$$\begin{aligned}
f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots \\
&= (z - z_0)^n [a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots] \quad (a_m \neq 0)
\end{aligned}$$

Theorem 3. (Zeros)
The zeros of an analytic function $f(z)(\neq 0)$ are isolated ; that is, each of them has a neighborhood that contains o further zeros of $f(z)$

Proof. In (3), the factor $(z - z_0)^n$ is zero only at $z = z_0$, The power series in the brackets $[\cdots]$ represents an analytic function (by Theorem 5 in Sec.14.3), call it $g(z)$. Now $g(z_0) = a_n \neq 0$, since analytic function is continuous, and because of this continuity, also $g(z) \neq 0$ in some neighborhood of $z = z_0$. Hence the same holds for $f(z)$.

Theorem 4. (Poles and zeros)
Let $f(z)$ be analytic at $z = z_0$ and have a zero of nth order at $z = z_0$. Then $1/f(z)$ has a pole of nth order at $z = z_0$. The same holds for $h(z)/f(z)$ if $h(z)$ is analytic at $z = z_0$ and $h(z_0) \neq 0$

30.2 Residue

$$\oint_c f(z)dz = ? \qquad C: \text{simple closed path}$$

-If $f(z)$ is analytic everywhere on C and inside C, answer$=0$ by Cauchy's integral theorem.
-If $f(z)$ has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C and inside C, then $f(z)$ has a Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots \tag{1}$$

This converges for all points near $z = z_0$ (except at $z = z_0$ itself), in some domain of the form $0 < |z - z_0| < R$.

30.2.1 Definition

From (1),

$$\boxed{b_1 = \tfrac{1}{2\pi i}\oint_c f(z)dz = \mathrm{Res}_{z=z_0} f(z) \quad : \quad \textbf{residue}}$$

$$2\pi i b_1 = \oint_c f(z)dz$$

Example 1. Evaluation of an integral by means of a residue

$$\oint_C \frac{\sin z}{z^4} dz \ = ? \quad C: |z| = 1 \text{ (ccw)}$$

Solution.

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$
$$|z| > 0$$
$$\therefore \ b_1 = -1/3! = -1/6$$
$$\therefore \ \oint_c \frac{\sin z}{z^4} dz = 2\pi i b_1 = -\frac{\pi i}{3} \qquad \blacksquare$$

Example 2. Be careful to use the right Laurent series !

$$f(z) = 1/(z^3 - z^4) \quad C: |z| = 1/2 \text{ (cw)}$$

Solution.

$$0 < |z| < 1 \qquad \frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots$$
$$b_1 = 1$$
$$\oint_c \frac{dz}{z^3 - z^4} = -2\pi i \mathrm{Res}_{z=0} f(z) = -2\pi i$$

30.2.2 Two Formulas for Residues at Simple Poles.

For a simple pole at $z = z_0$,

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (0 < |z - z_0| < R)$$

$$(z - z_0)f(z) = b_1 + (z - z_0)[a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots]$$

Now let $z \to z_0$, then

$$\boxed{\mathrm{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0}(z - z_0)f(z)} \tag{2}$$

Example 3. Residue at a simple pole.

$$\mathrm{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \lim_{z \to i}(z - i)\frac{9z + i}{z(z + i)(z - i)} = \left[\frac{9z + i}{z(z + i)}\right] = \frac{10i}{-2} = -5i.$$

If $f(z) = \frac{p(z)}{q(z)}$, p, q analytic, where $p(z_0) \neq 0$ and $q(z)$ has a simple zeros at z_0

$$q(z) = (z - z_0)q'(z_0) + \frac{(z - z_0)^2}{2!}q''(z_0) + \cdots$$

$$\mathrm{Res}_{z=z_0} f(z) = \lim_{z \to z_0}(z - z_0)\frac{p(z)}{q(z)} = \lim_{z \to z_0} \frac{(z - z_0)p(z)}{(z - z_0)[q'(z_0) + (z - z_0)q''/2 + \cdots]}$$

$$\boxed{\mathrm{Res}_{z=z_0} f(z) = \mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}} \tag{3}$$

Example 4. Residue at a simple pole calculated by formula (3)

$$\mathrm{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \left[\frac{9z + i}{3z^2 + 1}\right]_{z=i} = \frac{10i}{-2} = -5i.$$

30.2.3 Formula for the Residue at a pole of Any Order

If $f(z)$ has a pole of order $m > 1$ at $z = z_0$, its Laurent series converging near z_0 (except at a z_0 itself) is

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \cdots + \frac{b_2}{(z - z_0)^2} + \frac{b_1}{z - z_0} + a_0 + \\ + \cdots + a_1(z - z_0) + \cdots$$

where $b_m \neq 0$

$$(z - z_0)^m f(z) = b_m + b_{m-1}(z - z_0) + \cdots + b_2(z - z_0)^{m-2} + b_1(z - z_0)^{m-1} \\ + a_0(z - z_0)^m + a_1(z - z_0)^{m+1} + \cdots$$

b_1 of $f(z)$ at $z = z_0$ is now the coefficient of the power $(z - z_0)^{m-1}$ in the Taylor series of the function

$$g(z) = (z - z_0)^m f(z)$$

$$b_1 = \frac{1}{(m-1)!} g^{(m-1)}(z_0)$$

$$\boxed{\mathrm{Res}_{z=z_0} f(z) = \tfrac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \tfrac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right\}} \tag{4}$$

$$(m = 2) \qquad \mathrm{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \{[(z - z_0)^2 f(z)]'\}$$

Example 5. Residue at a pole of higher order

$$f(z) = \frac{50z}{(z+4)(z-1)^2}$$

$$\begin{aligned} \mathrm{Res}_{z=1} f(z) &= \lim_{z \to 1} \frac{d}{dz} [(z-1)^2 f(z)] = \lim_{z \to 1} \frac{d}{dz} \left(\frac{50z}{z+4} \right) = \lim_{z \to 1} \frac{50(z+4) - 50z}{(z+4)^2} \\ &= \frac{200}{5^2} = 8 \end{aligned}$$