10-3

Linear Systems Analysis in the Time Domain III

- Transient Response -

System Response with Additional Poles

Underdamped System with additional pole at $-\alpha$

$$p_1, p_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} +$$

$$y_{step}(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi) +$$

■ 극점 추가의 영향

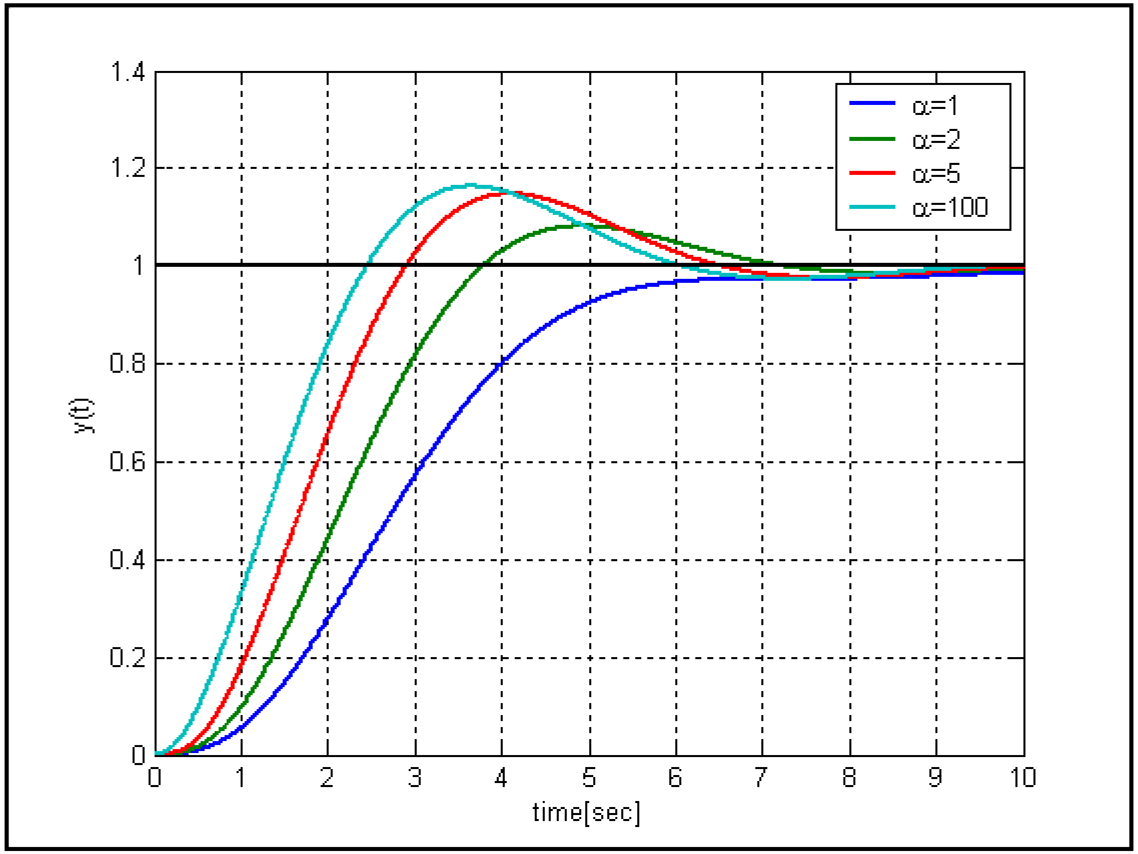
$$H(s) = \frac{1}{(s/\alpha\zeta + 1)[s^2 + 2\zeta s + 1]}, \qquad \zeta = 0.5$$

- 목적 : 제어시스템을 설계할 때 극점이 과도응답에 미치는 영향을 알아본다.

- 표준 2차 계단응답에 대한 부가 극점의 효과를 고려

■ 극점 추가의 영향

- Simulation Result $$H(s) = \frac{1}{(s/\alpha\zeta + 1)[s^2 + 2\zeta s + 1]}, \qquad \zeta = 0.5$$



■ 영점 추가의 영향(2)

$$H(s)=\frac{s/\alpha\zeta+1}{s^2+2\zeta s+1},\qquad \zeta=0.5,\quad \alpha=1,-1$$

$$H(s)=H_0(s)+H_d(s)=\frac{1}{s^2+2\zeta s+1}+\frac{1}{\alpha\zeta}\frac{s}{s^2+2\zeta s+1}$$

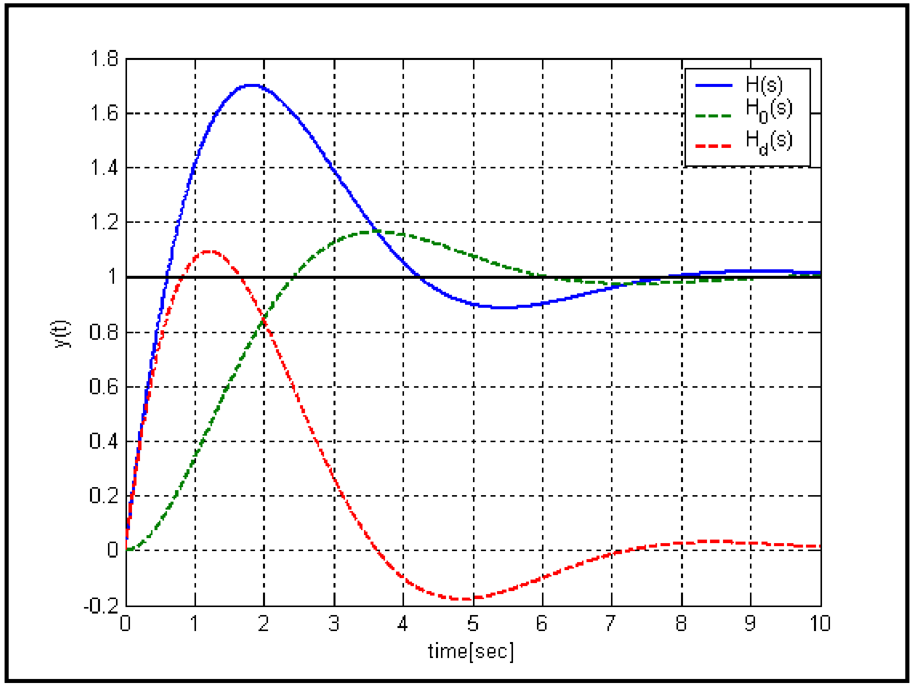
■ 영점 추가의 영향(2)

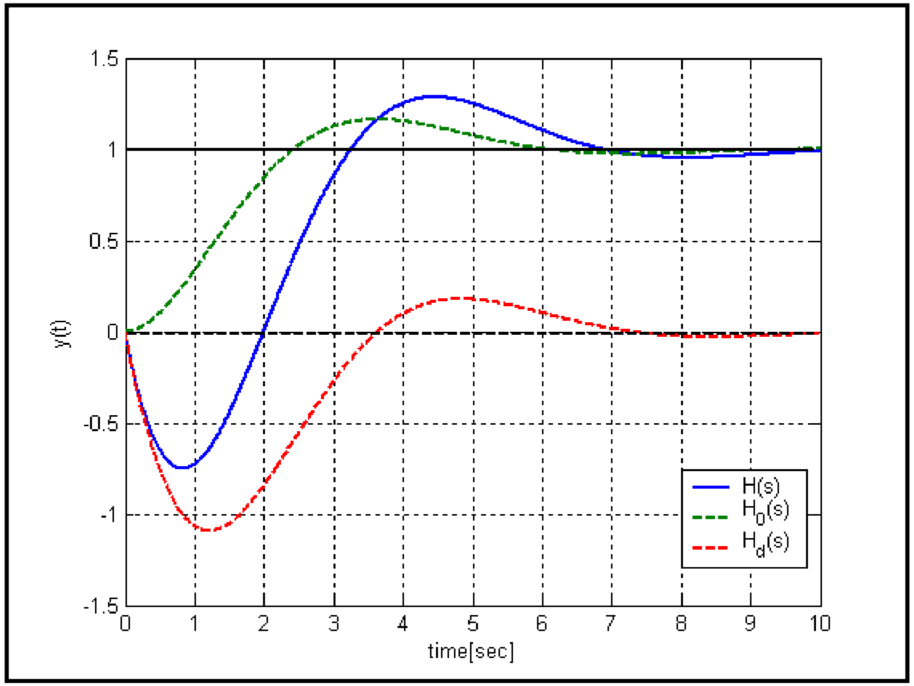
- Simulation Result

$$H(s)=\frac{s/\alpha\zeta+1}{s^2+2\zeta s+1},\quad H_0(s)=\frac{1}{s^2+2\zeta s+1},\quad H_d(s)=\frac{1}{\alpha\zeta}\frac{s}{s^2+2\zeta s+1}$$

$\zeta = 0.5, \quad \alpha = 1$

$\zeta = 0.5, \quad \alpha = -1$

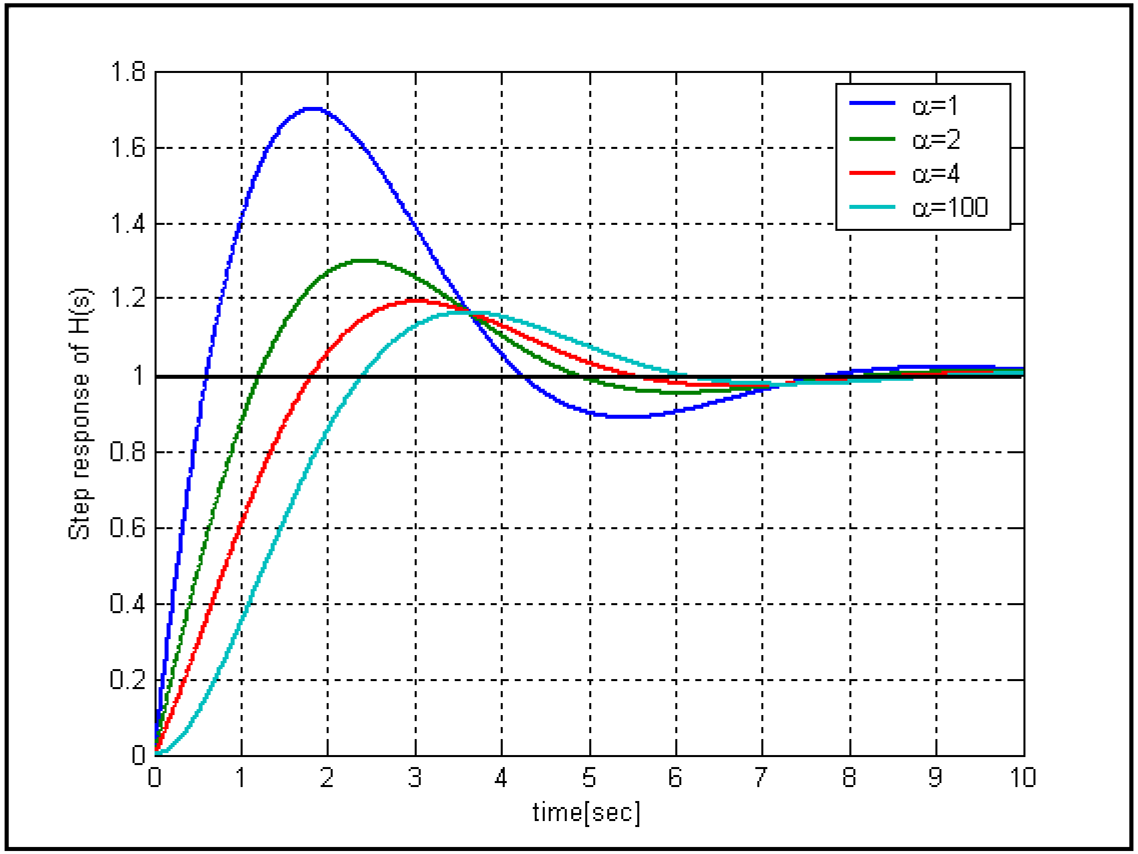




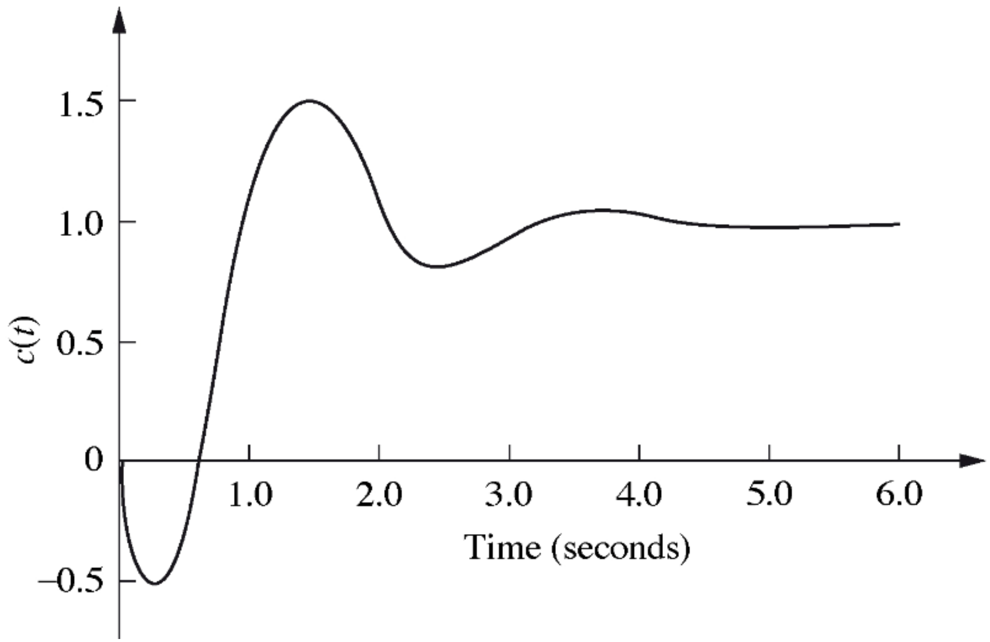
■ 영점 추가의 영향(1)

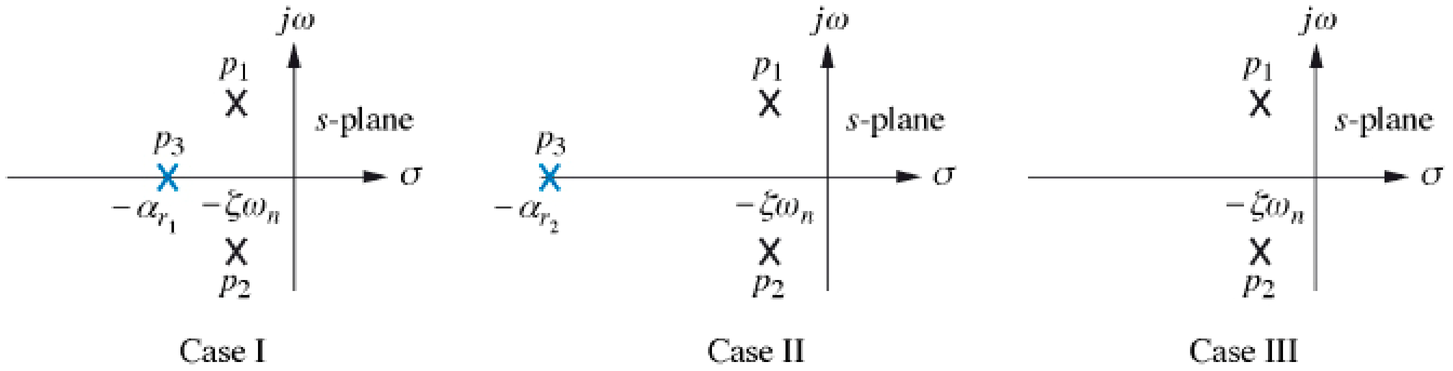
- Simulation Result

$$H(s) = \frac{s/\alpha\zeta + 1}{s^2 + 2\zeta s + 1}, \qquad \zeta = 0.5$$

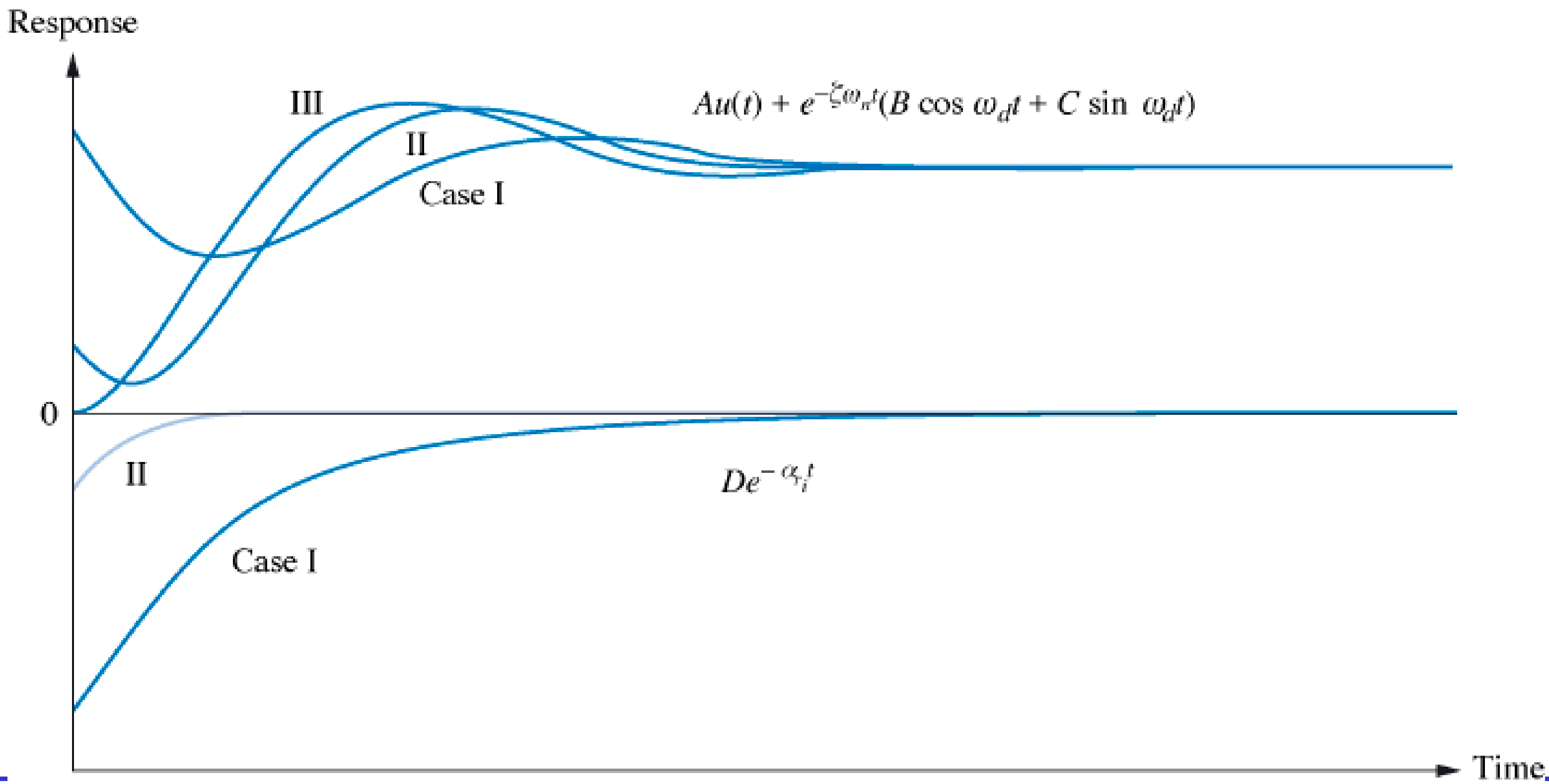


Non-minimum Phase System



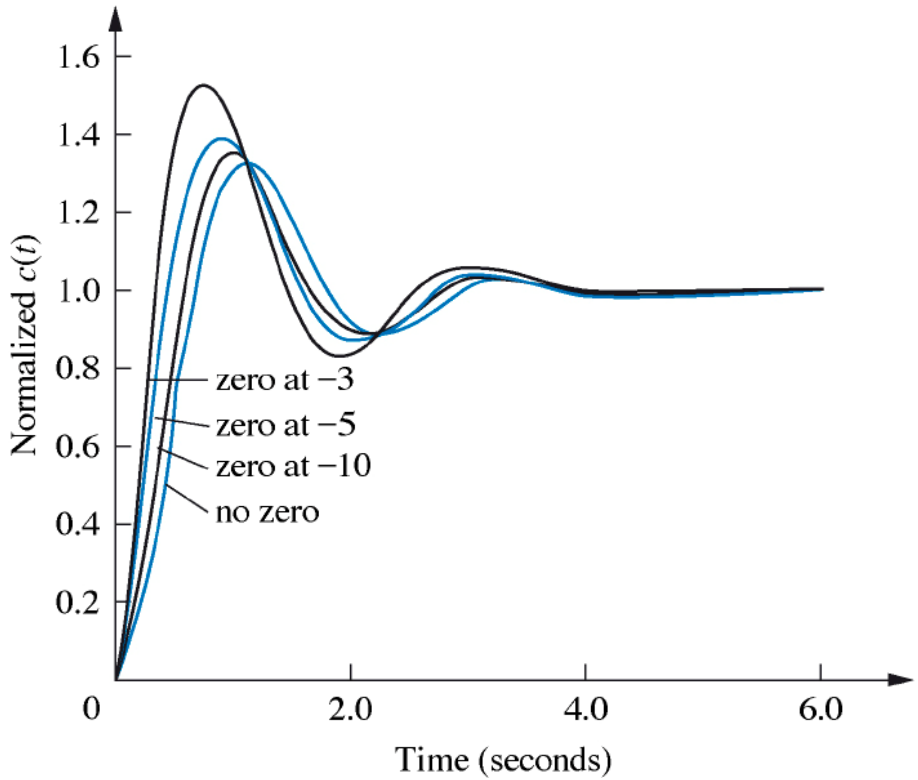

jω
s-plane
σ
p_1
p_2
p_3
$-\alpha_{r_1}$
$-\zeta\omega_n$
Case I
$-\alpha_{r_2}$
Case II
Case III

(*a*)


Response
III
II
Case I
$Au(t) + e^{-\zeta\omega_n t}(B \cos \omega_d t + C \sin \omega_d t)$
0
II
$De^{-\alpha_{r_i} t}$
Case I
Time

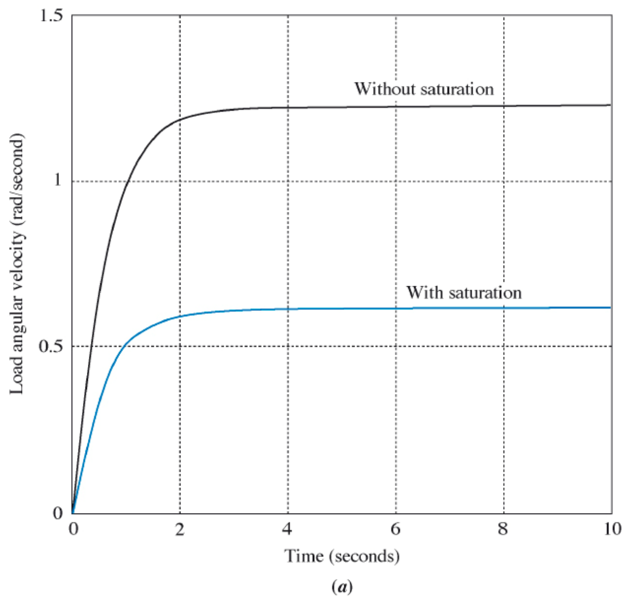
(*b*)

System Response with Zeros

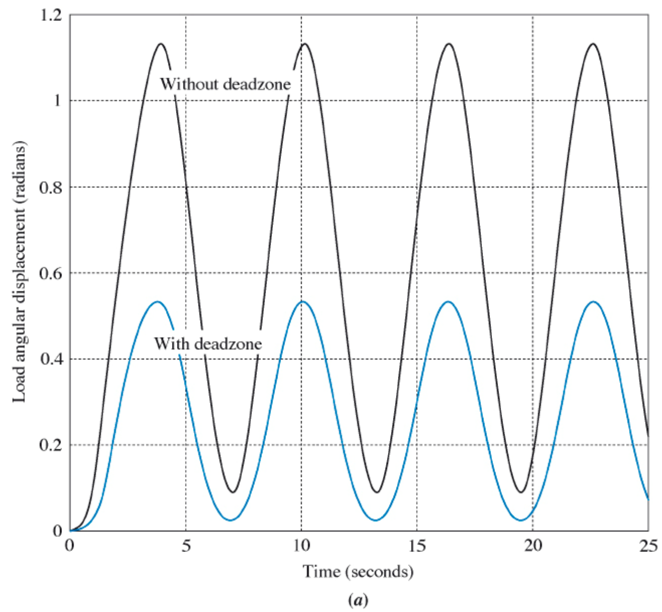


Effects of Nonlinearities

Saturation

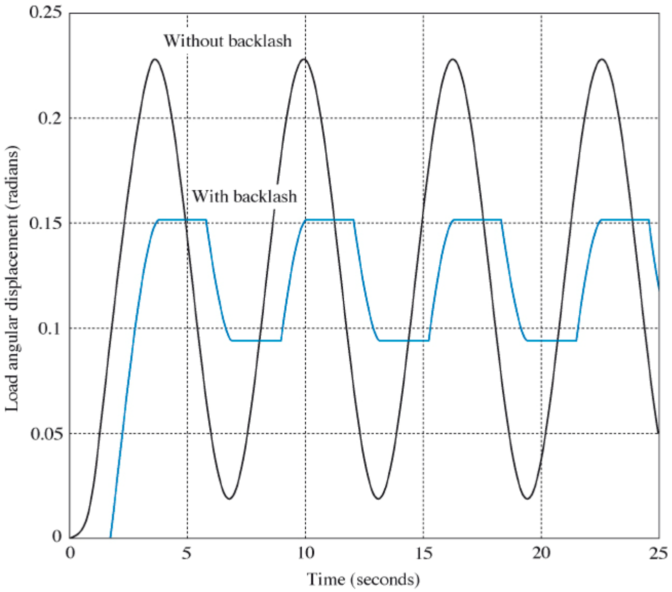


Deadzone



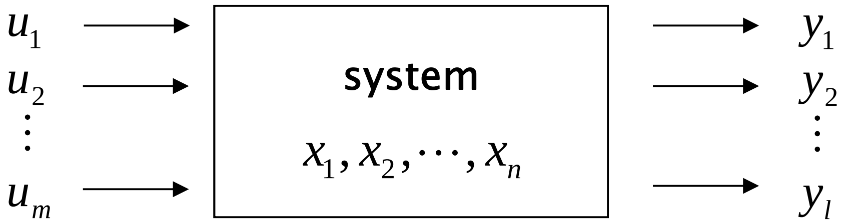
Effects of Nonlinearities

Backlash



(*a*)

Solution of Linear (Time Invariant) State Equation



$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$$x(t) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$$

Basic Matrix Linear Algebra -Homogeneous Solution

Scalar Function $u(t) = 0$

i) $\dot{x} = ax$

$x(t) = Ce^{at} \qquad t = 0, \quad x(0) = C$

$x(t) = x(0)e^{at}$

$x(t) = e^{a(t-t_0)}x(t_0), \qquad t = t_0, \;\; x(t_0)$

ii) $e^{at} = \exp(at) = 1 + \dfrac{at}{1!} + \dfrac{(at)^2}{2!} + \cdots + \dfrac{(at)^k}{k!} + \cdots$

Homogeneous Solution

i) $\dot{\mathrm{x}} = \mathrm{A}\,\mathrm{x} \qquad \mathrm{A} : n \times n, \quad \mathrm{x} : n \times 1$

$\mathrm{x}(t) = \exp\left[\mathrm{A}(t - t_0)\right]\mathrm{x}(t_0), \qquad \dfrac{d}{dt}(e^{\mathrm{A}t}) = \mathrm{A}\,e^{\mathrm{A}t}$

ii) *How to evaluate* $e^{\mathrm{A}t}$

State Transition Matrix

$$\mathrm{x}(t) = e^{\mathrm{A}(t-t_0)}\,\mathrm{x}(t_0)$$
$$= \Phi(t-t_0)\,\mathrm{x}(t_0)$$

$\Phi(t-t_0) = e^{A(t-t_0)} = \exp\left[\mathrm{A}(t-t_0)\right]$: State transition matrix (STM)

: fundamental matrix of the system

Properties of STM

1. $\Phi(t_2-t_1)\Phi(t_1-t_0) = \Phi(t_2-t_0)$ $\quad$ *for any* t_0, t_1, t_2
2. $\Phi(0) = I$
3. $\Phi(t)\Phi(t) = \Phi^2(t) = \Phi(2t)$ $\quad \left(\because \left(e^{\mathrm{A}t}\right)^2 = e^{\mathrm{A}\cdot 2t} = \Phi(2t)\right)$

   $$\Phi^g(t) = \Phi(gt)$$
4. $\Phi^{-1}(t) = \Phi(-t)$
5. $\Phi(t)$ is nonsingular for all finite values of t (inverse exists)

Complete Solution of the State Equation

$$\dot{x} = Ax + Bu, \qquad \dot{x} - Ax = Bu,$$

integrating factor e^{-At}

$$e^{-At}\dot{x} - Ae^{-At}x = e^{-At}Bu, \qquad \frac{d}{dt}\left[e^{-At}x\right] = e^{-At}Bu(t)$$

$$\Rightarrow e^{-At}\left(x(t) - x(0)\right) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$\Rightarrow x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$= \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

Complete Solution of the State Equation

$$\text{for } [t\ ,\ t_0],\qquad e^{-At}\,\mathrm{x}(t) - e^{-At_0}\,\mathrm{x}(t_0) = \int_{t_0}^{t} e^{-A\tau}\,\mathrm{B}u(\tau)d\tau$$

$$\Rightarrow \mathrm{x}(t) = \Phi(t-t_0)\,\mathrm{x}(t_0) + \int_{t_0}^{t} \Phi(t-\tau)\,\mathrm{B}u(\tau)d\tau$$

$$\Rightarrow \mathrm{x}(t) = \underline{\Phi(t)\ \mathrm{x}(0)} \quad + \quad \underline{\int_{0}^{t} \Phi(t-\tau)\,\mathrm{B}u(\tau)d\tau}$$

- Zero-input response
- free response

- Zero-state response
- forced response

$$\text{change of variable } \tau,\quad \text{let } \beta = t-\tau,\qquad \tau = 0 \rightarrow \beta = t$$

$$\tau = t \rightarrow \beta = 0$$

$$d\beta = -d\tau$$

$$\Rightarrow \mathrm{x}(t) = \Phi(t)\,\mathrm{x}(0) + \int_{0}^{t} \Phi(\beta)\,\mathrm{B}u(\beta)d\beta$$

Laplace Transformation Solution of State Equations

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \qquad \mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

Laplace transformation →

$$s\,\mathrm{X}(s) - x_0 = \mathrm{A}\,\mathrm{X}(s) + \mathrm{B}U(s)\,,$$

$$\mathrm{X}(s) = (s\,\mathrm{I} - \mathrm{A})^{-1} x_0 + (s\,\mathrm{I} - \mathrm{A})^{-1}\,\mathrm{B}U(s)$$

$$\mathrm{X}(s) = \frac{adj(s\,\mathrm{I} - \mathrm{A})}{\det(s\,\mathrm{I} - \mathrm{A})}[x(0) + \mathrm{B}U(s)]$$

$$Y(s) = C\,\mathrm{X}(s) + DU(s)$$

Transfer function form →

$$\frac{Y(s)}{U(s)} = C\frac{\mathrm{X}(s)}{U(s)} + D$$

$$\frac{Y(s)}{U(s)} = C\left[\frac{adj(s\,\mathrm{I} - \mathrm{A})}{\det(s\,\mathrm{I} - \mathrm{A})}\right]\mathrm{B} + D$$

$$= \frac{C\,adj(s\,\mathrm{I} - \mathrm{A})\,B + D\det(s\,\mathrm{I} - \mathrm{A})}{\det(s\,\mathrm{I} - \mathrm{A})}$$

System Poles→ Roots of the denominator / Solution of $\det(s\,\mathrm{I} - \mathrm{A}) = 0$

Eigenvalues of A matrix!!

Time Domain Solution of State Eqns. using Laplace Transform

$$\dot{x} = A x + B u$$

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

$$= \Phi(t) x(0) + \int_0^t \Phi(t-\tau) B u(\tau) d\tau$$

$$X(s) = (sI - A)^{-1} x_0 + (sI - A)^{-1} B u(s)$$

$$\therefore e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] = \mathcal{L}^{-1}\left[\frac{adj(sI - A)}{\det(sI - A)}\right] = \Phi(t)$$

Using Laplace transformation table

$$\mathcal{L}\left[t\right]=\frac{1}{s^2}, \quad \mathcal{L}\left[\frac{1}{(n-1)!}t^{n-1}\right]=\frac{1}{s^n}, \quad \mathcal{L}\left[t^n f(t)\right]=(-1)^n \frac{d^n}{ds^n}F(s)$$

$$\Rightarrow \mathcal{L}^{-1}\left[(s\mathrm{I}-\mathrm{A})^{-1}\right]=\mathcal{L}^{-1}\left[\frac{1}{s}\mathrm{I}+\frac{1}{s^2}\mathrm{A}+\frac{1}{s^3}\mathrm{A}^2+\frac{1}{s^4}\mathrm{A}^3\cdots\right]$$

$$=\mathrm{I}+\mathrm{A}t+\frac{1}{2!}\mathrm{A}^2t^2+\frac{1}{3!}\mathrm{A}^3t^3\cdots$$

$$\therefore e^{\mathrm{A}t}=\mathrm{I}+\mathrm{A}t+\frac{1}{2!}\mathrm{A}^2t^2+\frac{1}{3!}\mathrm{A}^3t^3+\cdots$$

Example1. Find the state transition matrix and solve for x(t). u(t) is a unit step.

$$\dot{\mathrm{x}}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \mathrm{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad \mathrm{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example2. Find the state transition matrix using $(sI-A)^{-1}$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$