

# **Chapter 1 Introduction:**

## **Vibration and the Free Response**

## Modeling and Degrees of Freedom

The examples on the previous slide many degrees of freedom and many parts, we will start with one degree of freedom and work towards many.

- **Recall from your study of statics and physics that a degree of freedom is the independent parameter needed to describe the configuration of a physical system**
- **So single degree of freedom system, which is where we start, is a system whose position in time and space can be defined by one coordinate, here a displacement or position.**

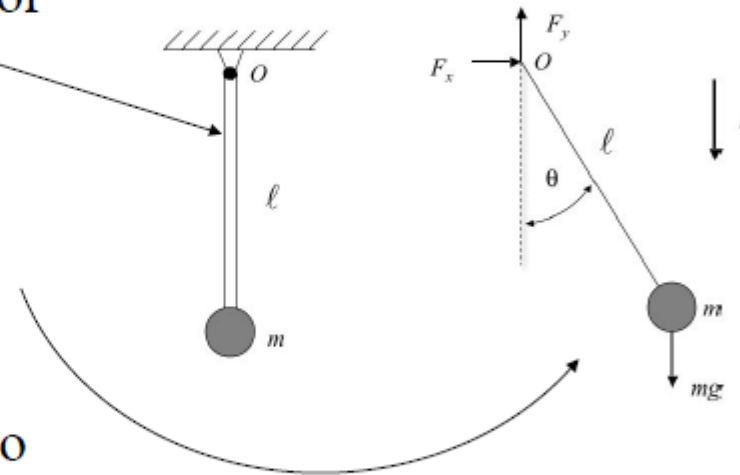
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## **Degrees of Freedom: The Minimum Number of coordinates to specify a configuration**

- For a single particle confined to a line, one coordinate suffices so it has one degree of freedom
- For a single particle in a plane two coordinates define its location so it has two degrees of freedom
- A single particle in space requires three coordinates so it has three degrees of freedom

## Example 1.1.1 The Pendulum

- Sketch the structure or part of interest
- Write down all the forces and make a “free body diagram”
- Use Newton’s Law and/or Euler’s Law to find the equations of motion



$$\sum \mathbf{M}_0 = J_0 \alpha, \quad J_0 = m\ell^2$$

**The problem is one dimensional,  
hence a scalar equation results**

$$J_0 \alpha(t) = -mgl \sin \theta(t) \Rightarrow m\ell^2 \ddot{\theta}(t) + \underbrace{mgl \sin \theta(t)}_{\substack{\text{restoring} \\ \text{force}}} = 0$$

Here the over dots denote differentiation with respect to time  $t$

This is a second order, nonlinear ordinary differential equation

We can *linearize* the equation by using the approximation  $\sin \theta \approx \theta$

$$\Rightarrow m\ell^2 \ddot{\theta}(t) + mgl \theta(t) = 0 \Rightarrow \ddot{\theta}(t) + \frac{g}{\ell} \theta(t) = 0$$

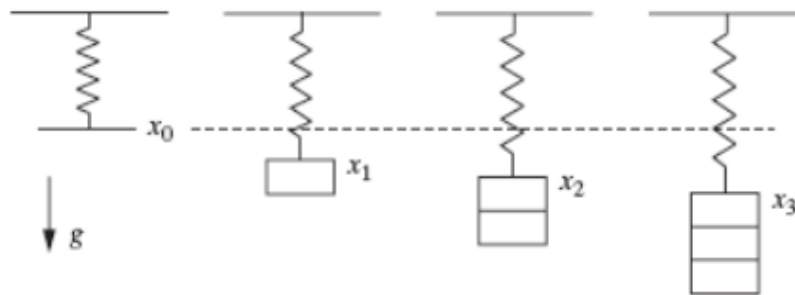
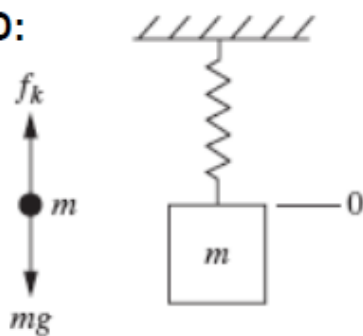
Requires knowledge of  $\theta(0)$  and  $\dot{\theta}(0)$

the initial position and velocity.

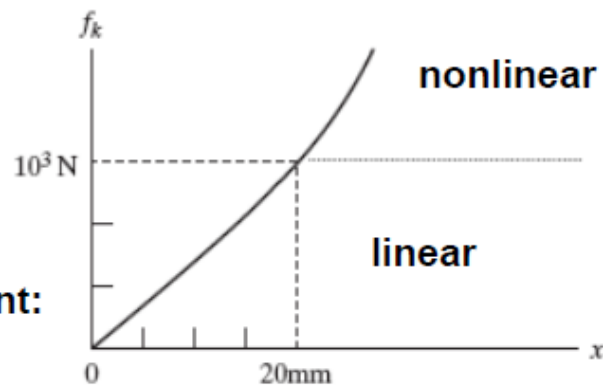
## Next consider a spring mass system and perform a static experiment:

- From strength of materials recall:

FBD:

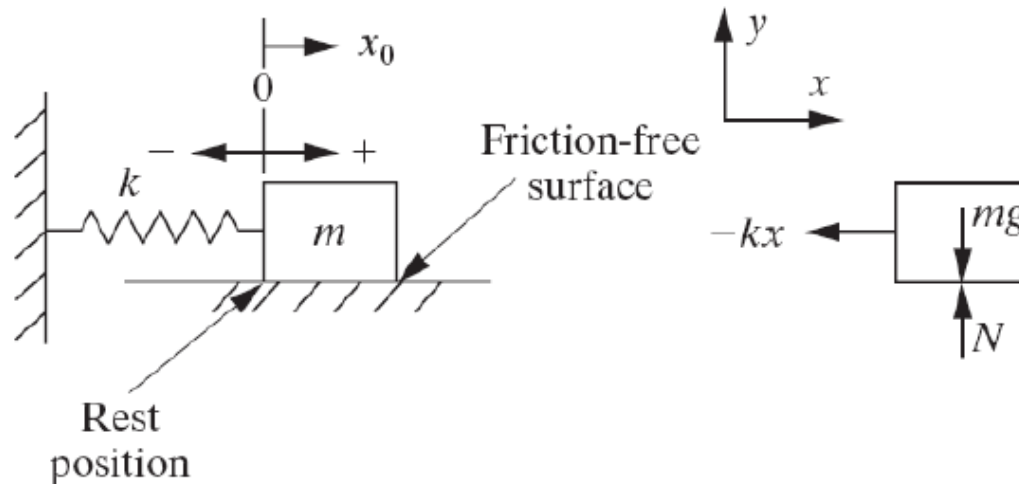


A plot of force versus displacement:



$$\text{experiment} \Rightarrow f_k = kx$$

## Free-body diagram and equation of motion



•Newton's Law:

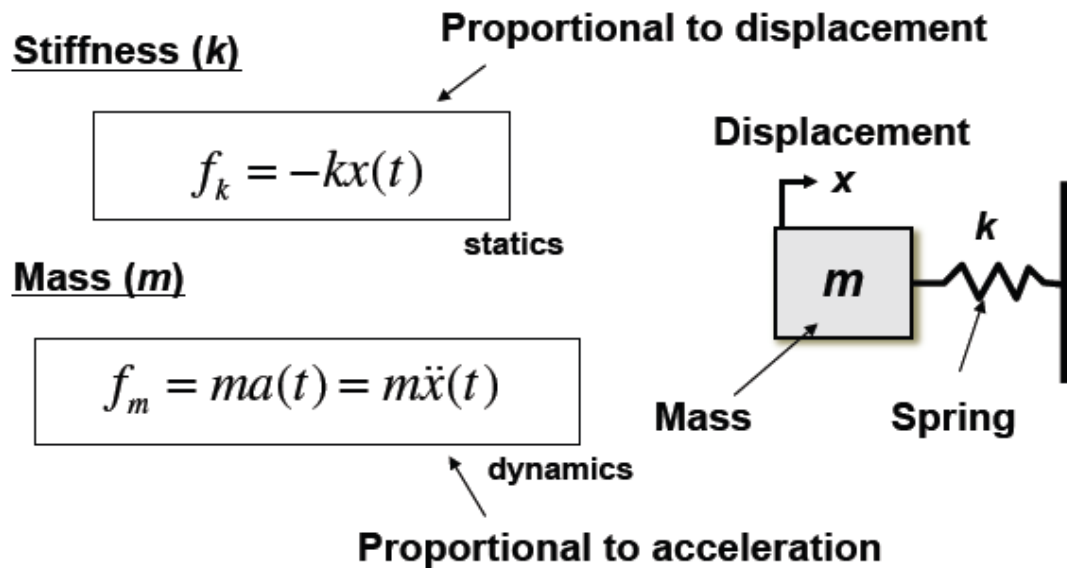
$$m\ddot{x}(t) = -kx(t) \Rightarrow m\ddot{x}(t) + kx(t) = 0$$

$$x(0) = x_0, \dot{x}(0) = v_0 \quad (1.2)$$

Again a 2nd order ordinary differential equation

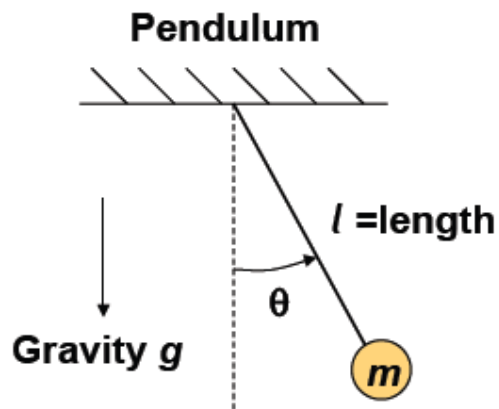
# Stiffness and Mass

Vibration is caused by the interaction of two different forces: one related to position (stiffness) and one related to acceleration (mass).

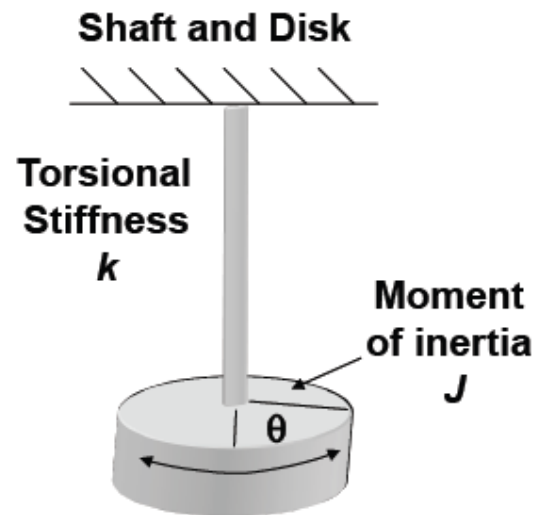




## Examples of Single-Degree-of-Freedom Systems



$$\ddot{\theta}(t) + \frac{g}{l}\theta(t) = 0$$

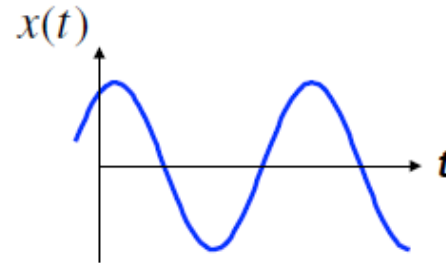


$$J\ddot{\theta}(t) + k\theta(t) = 0$$

## Solution of 2nd order DEs

Lets assume a solution:

$$x(t) = A \sin(\omega_n t + \phi) \quad (1.3)$$



Differentiating twice gives:

$$\dot{x}(t) = \omega_n A \cos(\omega_n t + \phi) \quad (1.4)$$

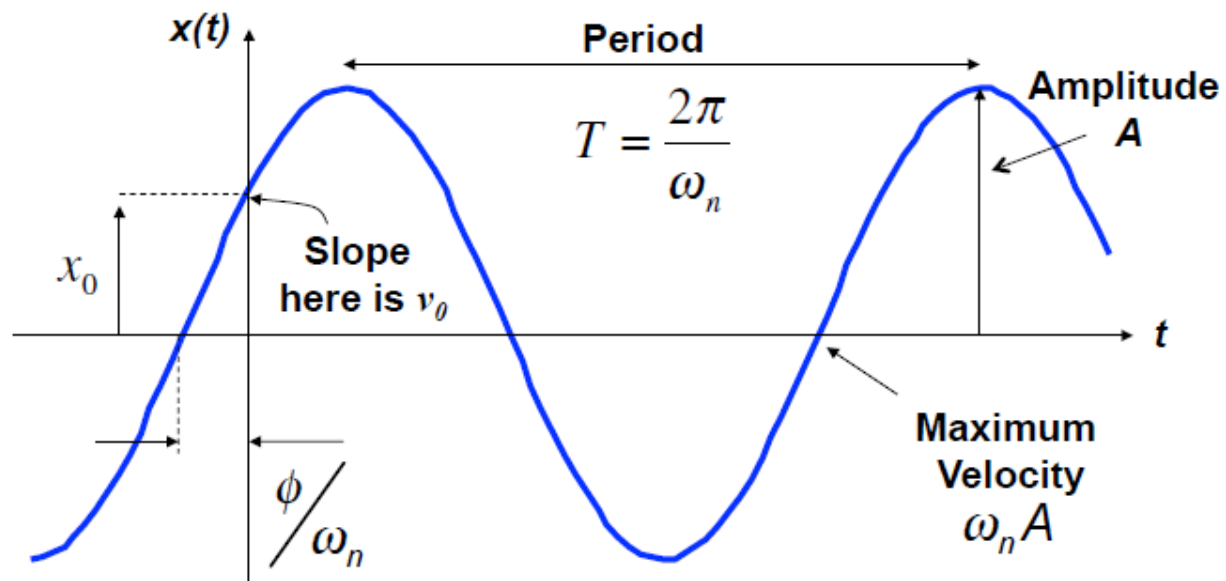
$$\ddot{x}(t) = -\omega_n^2 A \sin(\omega_n t + \phi) = -\omega_n^2 x(t) \quad (1.5)$$

Substituting back into the equations of motion gives:

$$-m\omega_n^2 A \sin(\omega_n t + \phi) + kA \sin(\omega_n t + \phi) = 0$$

$$-m\omega_n^2 + k = 0 \quad \text{or} \quad \omega_n = \sqrt{\frac{k}{m}} \quad \leftarrow \begin{array}{l} \text{Natural} \\ \text{frequency} \\ \text{rad/s} \end{array}$$

## Summary of simple harmonic motion



$$f_n = \frac{\omega_n \text{ rad/s}}{2\pi \text{ rad/cycle}} = \frac{\omega_n \text{ cycles}}{2\pi \text{ s}} = \frac{\omega_n}{2\pi} \text{ Hz}$$

## Initial Conditions

If a system is vibrating then we must assume that something must have (in the past) transferred energy into to the system and caused it to move. For example the mass could have been:

- moved a distance  $x_0$  and then released at  $t = 0$  (i.e. given Potential energy) or
- given an initial velocity  $v_0$  (i.e. given some kinetic energy) or
- Some combination of the two above cases

From our earlier solution we know that:

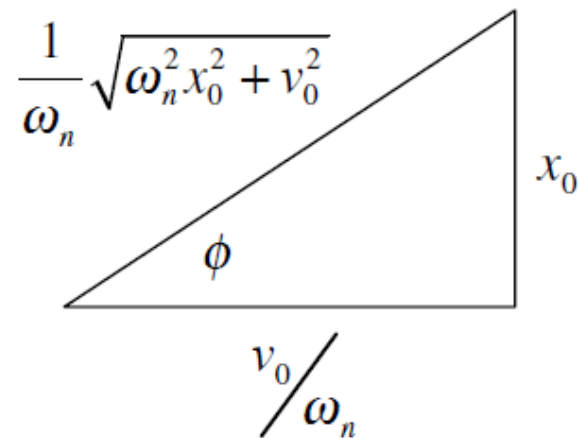
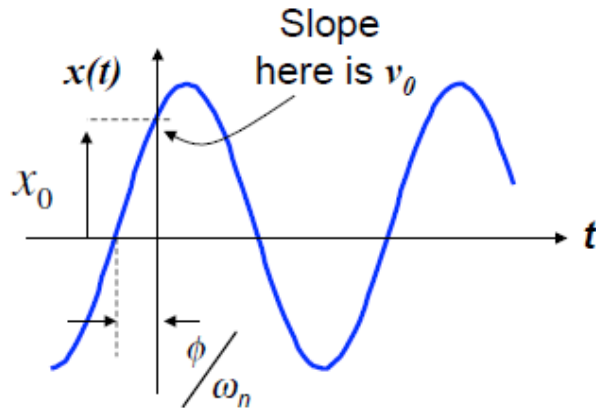
$$x_0 = x(0) = A \sin(\omega_n 0 + \phi) = A \sin(\phi)$$

$$v_0 = \dot{x}(0) = \omega_n A \cos(\omega_n 0 + \phi) = \omega_n A \cos(\phi)$$

# Initial Conditions Determine the Constants of Integration

Solving these two simultaneous equations for  $A$  and  $\phi$  gives:

$$\underbrace{A = \frac{1}{\omega_n} \sqrt{\omega_n^2 x_0^2 + v_0^2}}_{\text{Amplitude}}, \quad \underbrace{\phi = \tan^{-1} \left( \frac{\omega_n x_0}{v_0} \right)}_{\text{Phase}}$$



**Thus the total solution for the spring mass system becomes:**

$$x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \sin\left(\omega_n t + \tan^{-1} \frac{\omega_n x_0}{v_0}\right) \quad (1.10)$$

**Called the solution to a simple harmonic oscillator and describes oscillatory motion, or *simple harmonic motion*.**

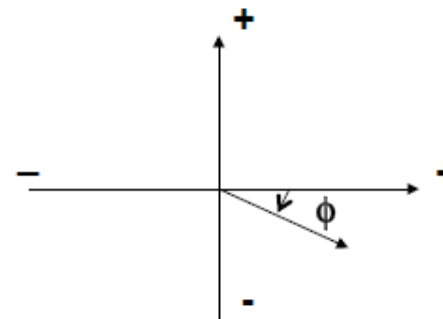
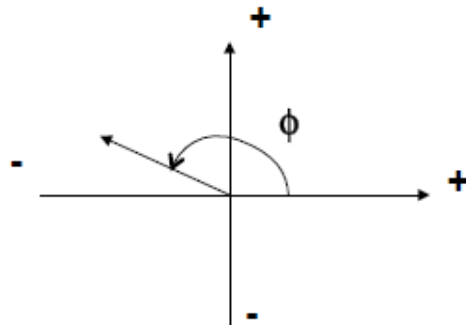
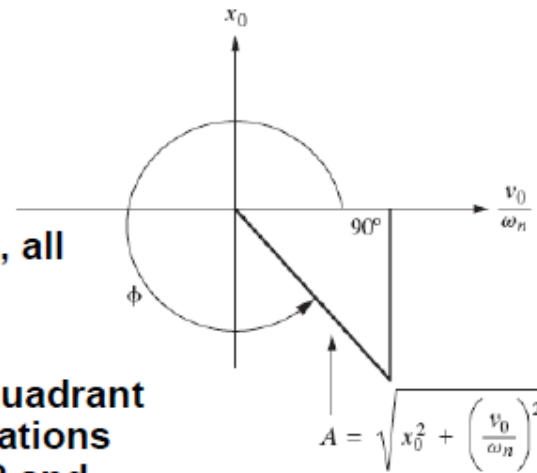
**Note (Example 1.1.2)**

$$x(0) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \frac{\omega_n x_0}{\sqrt{\omega_n^2 x_0^2 + v_0^2}} = x_0$$

**as it should**

## A note on arctangents

- Note that calculating arctangent from a calculator requires some attention. First, all machines work in *radians*.
- The argument  $\text{atan}(-/+)$  is in a different quadrant than  $\text{atan}(+/-)$ , and usual machine calculations will return an arctangent in between  $-\pi/2$  and  $+\pi/2$ , reading only the  $\text{atan}(-)$  for these two *different* cases.



In MATLAB, use the `atan2(x,y)` function to get the correct phase.

**Example 1.1.3 wheel, tire suspension**  
 **$m = 30 \text{ kg}$ ,  $f_n = 10 \text{ hz}$ , what is  $k$ ?**

$$k = m\omega_n^2 = (30 \text{ kg}) \left( 10 \frac{\text{cylce}}{\text{sec}} \cdot \frac{2\pi \text{ rad}}{\text{cylce}} \right) = 1.184 \times 10^5 \text{ N/m}$$

**There are of course more complex models of suspension systems and these appear latter in the course as our tools develop**



## Section 1.2 Harmonic Motion

The period is the time elapsed to complete one complete cycle

$$T = \frac{2\pi \text{ rad}}{\omega_n \text{ rad/s}} = \frac{2\pi}{\omega_n} \text{ s} \quad (1.11)$$

The natural frequency in the commonly used units of hertz:

$$f_n = \frac{\omega_n}{2\pi} = \frac{\omega_n \text{ rad/s}}{2\pi \text{ rad/cycle}} = \frac{\omega_n \text{ cycles}}{2\pi \text{ s}} = \frac{\omega_n}{2\pi} \text{ Hz} \quad (1.12)$$

For the pendulum:

$$\omega_n = \sqrt{\frac{g}{\ell}} \text{ rad/s}, \quad T = 2\pi \sqrt{\frac{\ell}{g}} \text{ s}$$

For the disk and shaft:

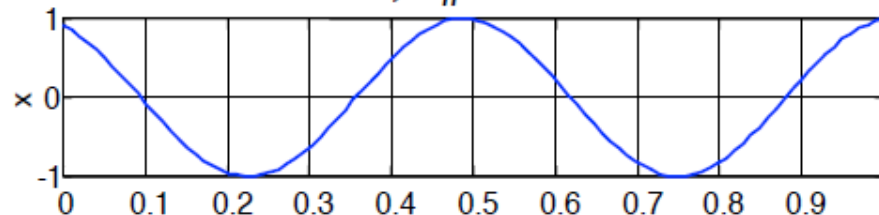
$$\omega_n = \sqrt{\frac{k}{J}} \text{ rad/s}, \quad T = 2\pi \sqrt{\frac{J}{k}} \text{ s}$$

# Relationship between Displacement, Velocity and Acceleration

$$A=1, \omega_n=12$$

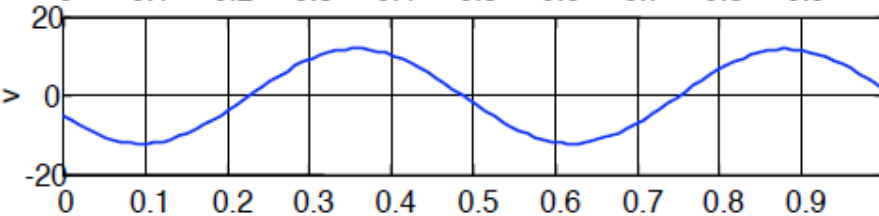
**Displacement**

$$x(t) = A \sin(\omega_n t + \phi)$$



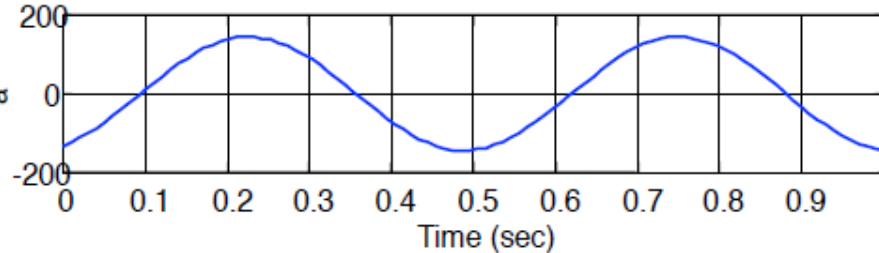
**Velocity**

$$\dot{x}(t) = \omega_n A \cos(\omega_n t + \phi)$$



**Acceleration**

$$\ddot{x}(t) = -\omega_n^2 A \sin(\omega_n t + \phi)$$



Note how the relative magnitude of each increases for  $\omega_n > 1$

**Example 1.2.1** Hardware store spring, bolt:  $m= 49.2 \times 10^{-3}$  kg,  $k=857.8$  N/m and  $x_0=10$  mm. Compute  $\omega_n$  and the max amplitude of vibration.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8 \text{ N/m}}{49.2 \times 10^{-3} \text{ kg}}} = 132 \text{ rad/s}$$

Note: common Units are Hertz

$$f_n = \frac{\omega_n}{2\pi} = 21 \text{ Hz}$$

To avoid Costly errors use  $f_n$  when working in Hertz and  $\omega_n$  when in rad/s

$$T = \frac{2\pi}{\omega_n} = \frac{1}{f_n} = \frac{1}{21 \text{ cycles/sec}} = 0.0476 \text{ s}$$

$$x(t)_{\max} = A = \frac{1}{\omega_n} \sqrt{\omega_n^2 x_0^2 + \cancel{v_0^2}} = x_0 = 10 \text{ mm}$$

Units depend on system

**Compute the solution and max velocity and acceleration:**

$$v(t)_{\max} = \omega_n A = 1320 \text{ mm/s} = 1.32 \text{ m/s} \quad 2.92 \text{ mph}$$

$$a(t)_{\max} = \omega_n^2 A = 174.24 \times 10^3 \text{ mm/s}^2 \\ = 174.24 \text{ m/s}^2 \approx 17.8g!$$

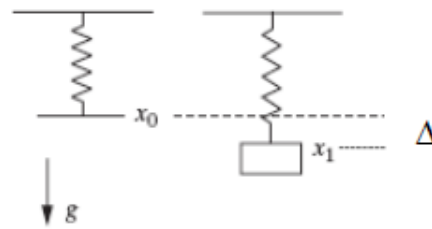
$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{0}\right) = \frac{\pi}{2} \text{ rad} \quad 90^\circ \quad g = 9.8 \text{ m/s}^2$$

$$x(t) = 10 \sin(132t + \pi/2) = 10 \cos(132t) \text{ mm}$$

~0.4 in max

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## Does gravity matter in spring problems?



Let  $\Delta$  be the deflection caused hanging a mass on a spring ( $\Delta = x_1 - x_0$  in the figure)

Then from static equilibrium:  $mg = k\Delta$

Next sum the forces in the vertical for some point  $x > x_1$  measured from  $\Delta$

$$m\ddot{x} = -k(x + \Delta) + mg = -kx + \underbrace{mg - k\Delta}_{=0}$$

$$\Rightarrow m\ddot{x}(t) + kx(t) = 0$$

So no, gravity does not have an effect on the vibration

(note that this is not the case if the spring is nonlinear)

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## Example 1.2.2 Pendulums and measuring $g$

- A 2 m pendulum swings with a period of 2.893 s

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\ell}{g}}$$

- What is the acceleration due to gravity at that location?

$$g = \frac{4\pi^2}{T^2} \ell = \frac{4\pi^2}{2.893^2 \text{ s}^2} 2 \text{ m}$$

$$\Rightarrow g = 9.796 \text{ m/s}^2$$

This is  $g$  in Denver, CO USA, at 1638m and a latitude of  $40^\circ$

## Review of Complex Numbers and Complex Exponential (See Appendix A)

A complex number can be written with a real and imaginary part or as a complex exponential

$$c = a + jb = Ae^{j\theta}$$

Where

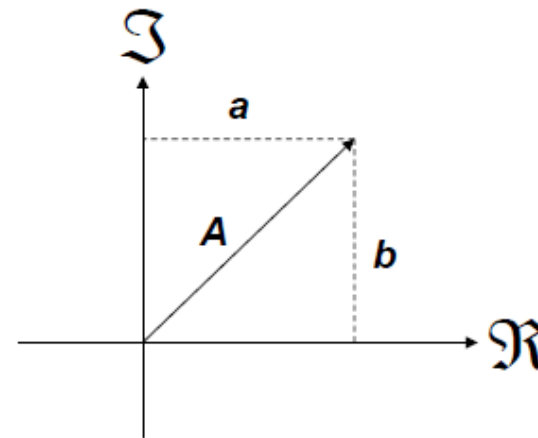
$$a = A \cos \theta, b = A \sin \theta$$

Multiplying two complex numbers:

$$c_1 c_2 = A_1 A_2 e^{j(\theta_1 + \theta_2)}$$

Dividing two complex numbers:

$$\frac{c_1}{c_2} = \frac{A_1}{A_2} e^{j(\theta_1 - \theta_2)}$$



## Equivalent Solutions to 2nd order Differential Equations (see Window 1.4)

All of the following solutions are equivalent:

$$x(t) = A \sin(\omega_n t + \phi) \quad \text{Called the magnitude and phase form}$$

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t \quad \text{Sometimes called the Cartesian form}$$

$$x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t} \quad \text{Called the polar form}$$

The relationships between  $A$  and  $\phi$ ,  $A_1$  and  $A_2$ , and  $a_1$  and  $a_2$  can be found in Window 1.4 of the text, page 19..

- Each is useful in different situations
- Each represents the same information
- Each solves the equation of motion



## Derivation of the solution

Substitute  $x(t) = ae^{\lambda t}$  into  $m\ddot{x} + kx = 0 \Rightarrow$

$$m\lambda^2 ae^{\lambda t} + kae^{\lambda t} = 0 \Rightarrow$$

$$m\lambda^2 + k = 0 \Rightarrow$$

$$\lambda = \pm \sqrt{-\frac{k}{m}} = \pm \sqrt{\frac{k}{m}} j = \pm \omega_n j \Rightarrow$$

$$x(t) = a_1 e^{\omega_n j t} \quad \text{and} \quad x(t) = a_2 e^{-\omega_n j t} \Rightarrow$$

$$x(t) = a_1 e^{\omega_n j t} + a_2 e^{-\omega_n j t} \quad (1.18)$$

This approach will be used again for more complicated problems

## Is frequency always positive?

From the preceding analysis,  $\lambda = \pm \omega_n$  then

$$x(t) = a_1 e^{\omega_n j t} + a_2 e^{-\omega_n j t}$$

Using the Euler relations\* for trigonometric functions, the above solution can be written as (recall Window 1.4)

$$x(t) = A \sin(\omega_n t + \phi) \quad (1.19)$$

It is in this form that we identify as the natural frequency  $\omega_n$  and this is positive, because the  $\pm$  sign being used up in the transformation from exponentials to the sine function.

\* [http://en.wikipedia.org/wiki/Euler's\\_formula](http://en.wikipedia.org/wiki/Euler's_formula)

$$e^{ix} = \cos x + i \sin x$$

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## Calculating root mean square (*RMS*) values

May need to be limited due  
to physical constraints

$A =$  peak value

Not very useful since for  
a sine function the  
average value is zero

$$\bar{x} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t) dt = \text{average value}$$

$$\bar{x}^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x^2(t) dt = \text{mean-square value} \quad (1.21)$$

$$x_{rms} = \sqrt{\bar{x}^2} = \text{root mean square value}$$

Proportional  
to energy

Also useful when the vibration is random

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## The Decibel or dB scale

It is often useful to use a logarithmic scale to plot vibration levels (or noise levels). One such scale is called the *decibel* or **dB** scale. The **dB** scale is always relative to some reference value  $x_0$ . It is define as:

$$dB = 10 \log_{10} \left( \frac{x}{x_0} \right)^2 = 20 \log_{10} \left( \frac{x}{x_0} \right) \quad (1.22)$$

For example: if an acceleration value was  $19.6\text{m/s}^2$  then relative to  $1\text{g}$  (or  $9.8\text{m/s}^2$ ) the level would be **6dB**,

$$10 \log_{10} \left( \frac{19.6}{9.8} \right)^2 = 20 \log_{10} (2) = 6dB$$

Or for **Example 1.2.1: The Acceleration Magnitude is  $20 \log_{10}(17.8) = 25dB$  relative to  $1\text{g}$ .**

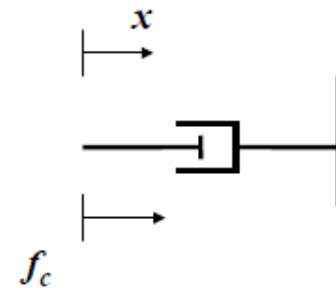
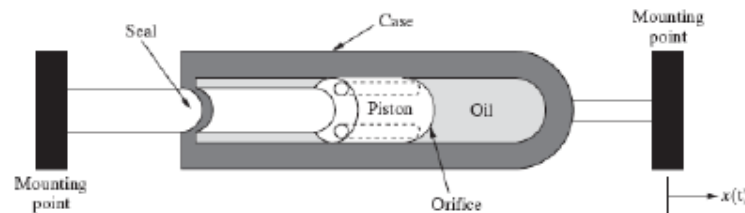
## 1.3 Viscous Damping

All real systems dissipate energy when they vibrate. To account for this we must consider damping. The most simple form of damping (from a mathematical point of view) is called viscous damping. A viscous damper (or **dashpot**) produces a force that is proportional to velocity.

Mostly a mathematically motivated form, allowing a solution to the resulting equations of motion that predicts reasonable (observed) amounts of energy dissipation.

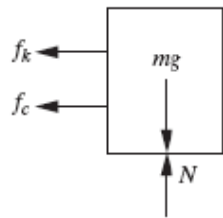
### Damper (c)

$$f_c = -cv(t) = -c\dot{x}(t)$$



# Differential Equation Including Damping

For this damped single degree of freedom system the force acting on the mass is due to the spring and the dashpot i.e.  $f_m = -f_k - f_c$

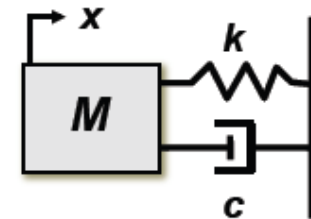


$$m\ddot{x}(t) = -kx(t) - c\dot{x}(t)$$

or

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \quad (1.25)$$

Displacement



To solve this for of the equation it is useful to assume a solution of the form (again):

$$x(t) = ae^{\lambda t}$$

## Solution to DE with damping included (dates to 1743 by Euler)

The velocity and acceleration can then be calculated as:

$$\dot{x}(t) = \lambda a e^{\lambda t}$$

$$\ddot{x}(t) = \lambda^2 a e^{\lambda t}$$

If this is substituted into the equation of motion we get:

$$a e^{\lambda t} (m \lambda^2 + c \lambda + k) = 0 \quad (1.26)$$

Divide equation by  $m$ , substitute for natural frequency and assume a non-trivial solution

$$a e^{\lambda t} \neq 0 \quad \Rightarrow \quad \left( \lambda^2 + \frac{c}{m} \lambda + \omega_n^2 \right) = 0$$

## Solution to our differential equation with damping included:

For convenience we will define a term known as the *damping ratio* as:

$$\zeta = \frac{c}{2\sqrt{km}} \quad (1.30)$$

Lower case Greek zeta  
NOT  $\xi$  as some like to use

The equation of motion then becomes:

$$(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2) = 0$$

Solving for  $\lambda$  then gives,

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \quad (1.31)$$



## Possibility 1. Critically damped motion

Critical damping occurs when  $\zeta = 1$ . The damping coefficient  $c$  in this case is given by:

$$\zeta = 1 \Rightarrow c = \underbrace{c_{cr} = 2\sqrt{km}}_{\text{definition of critical damping coefficient}} = 2m\omega_n$$

Solving for  $\lambda$  then gives,

$$\lambda_{1,2} = -1\omega_n \pm \omega_n \sqrt{1^2 - 1} = -\omega_n$$

A repeated, real root

The solution then takes the form

$$x(t) = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t}$$

Needs two independent solutions, hence the  $t$  in the second term

## Critically damped motion

$a_1$  and  $a_2$  can be calculated from initial conditions ( $t=0$ ),

$$x = (a_1 + a_2 t)e^{-\omega_n t}$$

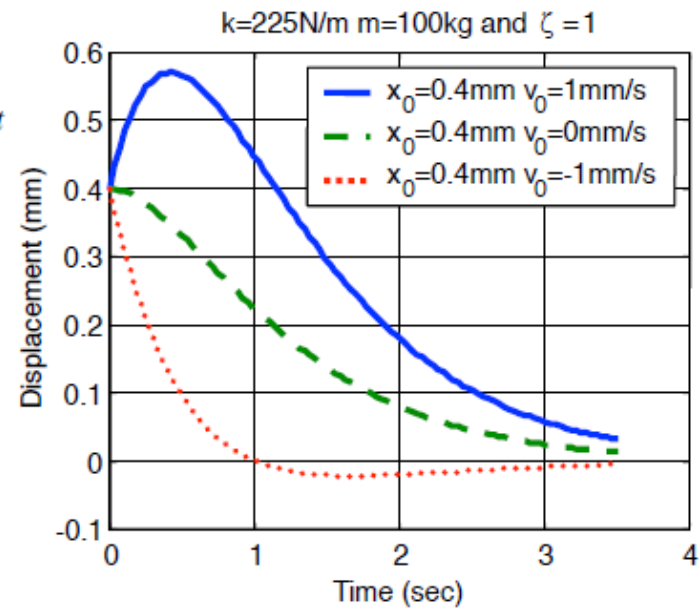
$$\Rightarrow a_1 = x_0$$

$$v = (-\omega_n a_1 - \omega_n a_2 t + a_2)e^{-\omega_n t}$$

$$v_0 = -\omega_n a_1 + a_2$$

$$\Rightarrow a_2 = v_0 + \omega_n x_0$$

- No oscillation occurs
- Useful in door mechanisms, analog gauges



## Possibility 2: Overdamped motion

An overdamped case occurs when  $\zeta > 1$ . Both of the roots of the equation are again real.

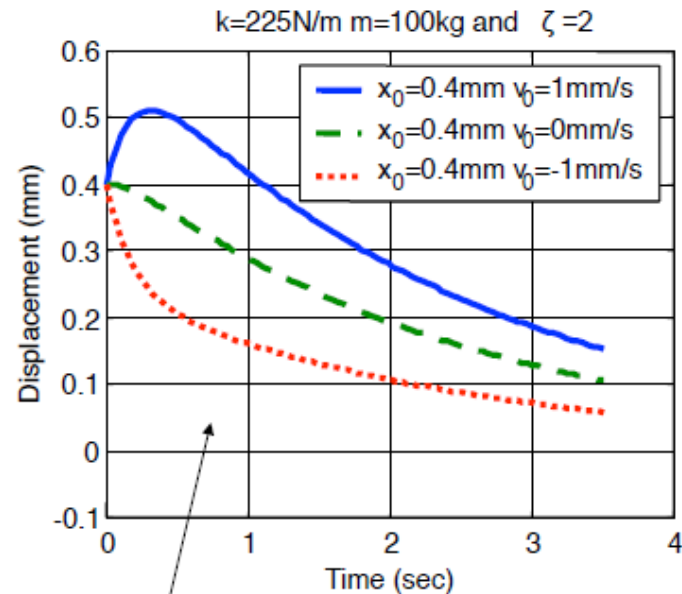
$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$x(t) = e^{-\zeta\omega_n t} (a_1 e^{-\omega_n t\sqrt{\zeta^2 - 1}} + a_2 e^{\omega_n t\sqrt{\zeta^2 - 1}})$$

$a_1$  and  $a_2$  can again be calculated from initial conditions ( $t=0$ ),

$$a_1 = \frac{-v_0 + (-\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$a_2 = \frac{v_0 + (\zeta + \sqrt{\zeta^2 - 1})\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$



Slower to respond than critically damped case

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## Possibility 3: Underdamped motion

An underdamped case occurs when  $\zeta < 1$ . The roots of the equation are complex conjugate pairs. This is the most common case and the only one that yields oscillation.

$$\begin{aligned}\lambda_{1,2} &= -\zeta\omega_n \pm \omega_n j\sqrt{1-\zeta^2} \\ x(t) &= e^{-\zeta\omega_n t} (a_1 e^{j\omega_n t\sqrt{1-\zeta^2}} + a_2 e^{-j\omega_n t\sqrt{1-\zeta^2}}) \\ &= A e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)\end{aligned}$$

The frequency of oscillation  $\omega_d$  is called the *damped natural frequency* is given by.

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \quad (1.37)$$

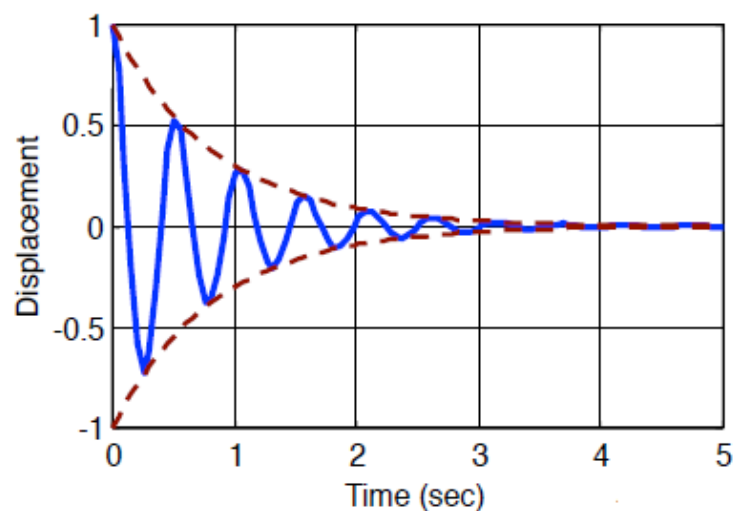
## Constants of integration for the underdamped motion case

As before  $A$  and  $\phi$  can be calculated from initial conditions ( $t=0$ ),

$$A = \frac{1}{\omega_d} \sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}$$

$$\phi = \tan^{-1} \left( \frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} \right)$$

- Gives an oscillating response with exponential decay
- Most natural systems vibrate with an underdamped response
- See Window 1.5 for details and other representations



**Example 1.3.1:** consider the spring of 1.2.1, if  $c = 0.11$  kg/s, determine the damping ratio of the spring-bolt system.

$$m = 49.2 \times 10^{-3} \text{ kg}, \quad k = 857.8 \text{ N/m}$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{49.2 \times 10^{-3} \times 857.8} \\ = 12.993 \text{ kg/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{0.11 \text{ kg/s}}{12.993 \text{ kg/s}} = 0.0085 \Rightarrow$$

the motion is *underdamped*  
and the bolt will oscillate

## Example 1.3.2

The human leg has a *measured* natural frequency of around 20 Hz when in its rigid (knee locked) position, in the longitudinal direction (i.e., along the length of the bone) with a damping ratio of  $\zeta = 0.224$ . Calculate the response of the tip if the leg bone to an initial velocity of  $v_0 = 0.6$  m/s and zero initial displacement (this would correspond to the vibration induced while landing on your feet, with your knees locked from a height of 18 mm) and plot the response. What is the maximum acceleration experienced by the leg assuming no damping?

## Solution:

$$\omega_n = \frac{20 \text{ cycles}}{1 \text{ s}} \frac{2\pi \text{ rad}}{\text{cycles}} = 125.66 \text{ rad/s}$$

$$\omega_d = 125.66 \sqrt{1 - (.224)^2} = 122.467 \text{ rad/s}$$

$$A = \frac{\sqrt{(0.6 + (0.224)(125.66)(0))^2 + (0)(122.467)^2}}{122.467} = 0.005 \text{ m}$$

$$\phi = \tan^{-1} \left( \frac{(0)(\omega_d)}{v_0 + \zeta \omega_n(0)} \right) = 0$$

$$\Rightarrow \underline{x(t) = 0.005 e^{-28.148t} \sin(122.467t)}$$



Use the *undamped* formula to get maximum acceleration:

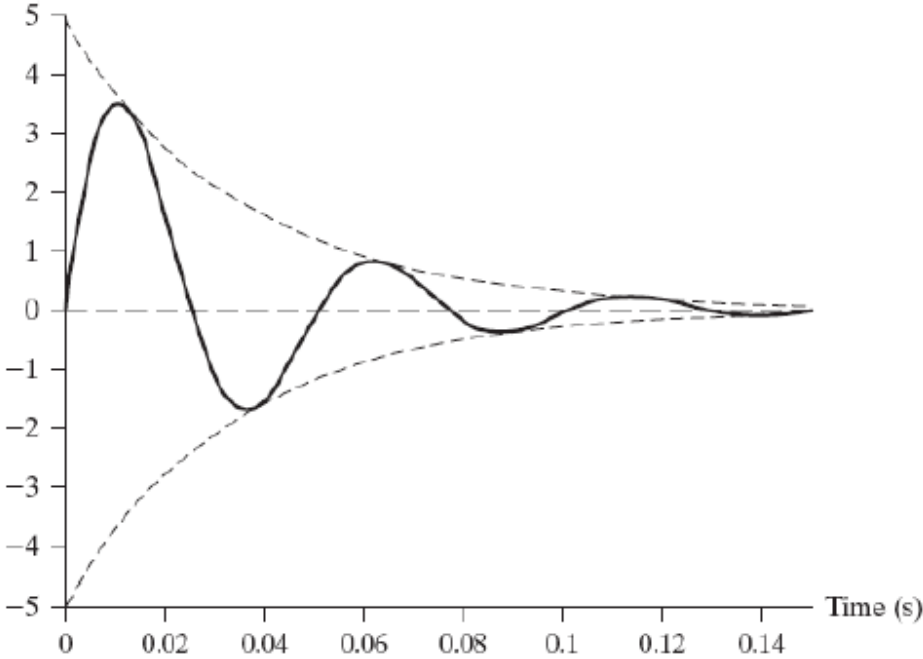
$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}, \quad \omega_n = 125.66, \quad v_0 = 0.6, \quad x_0 = 0$$

$$A = \frac{v_0}{\omega_n} \text{ m} = \frac{0.6}{\omega_n} \text{ m}$$

$$\max(\ddot{x}) = \left| -\omega_n^2 A \right| = \left| -\omega_n^2 \left( \frac{0.6}{\omega_n} \right) \right| = (0.6)(125.66 \text{ m/s}^2) = \underline{75.396 \text{ m/s}^2}$$

$$\text{maximum acceleration} = \frac{75.396 \text{ m/s}^2}{9.81 \text{ m/s}^2} \text{ g} = 7.68 \text{ g's}$$

**Here is a plot of the displacement response versus time**



**Example 1.3.3** Compute the form of the response of an underdamped system using the Cartesian form of the solution given in Window 1.5.

$$\sin(x + y) = \sin x \sin y + \cos x \cos y \Rightarrow$$

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi) = e^{-\zeta\omega_n t} (A_1 \sin \omega_d t + A_2 \cos \omega_d t)$$

$$x(0) = x_0 = e^0 (A_1 \sin(0) + A_2 \cos(0)) \Rightarrow \underline{A_2 = x_0}$$

$$\begin{aligned} \dot{x} = & -\zeta\omega_n e^{-\zeta\omega_n t} (A_1 \sin \omega_d t + A_2 \cos \omega_d t) \\ & + \omega_d e^{-\zeta\omega_n t} (A_1 \cos \omega_d t - A_2 \sin \omega_d t) \end{aligned}$$

$$v_0 = -\zeta\omega_n (A_1 \sin 0 + x_0 \cos 0) + \omega_d (A_1 \cos 0 - x_0 \sin 0)$$

$$\Rightarrow \underline{A_1 = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}} \Rightarrow$$

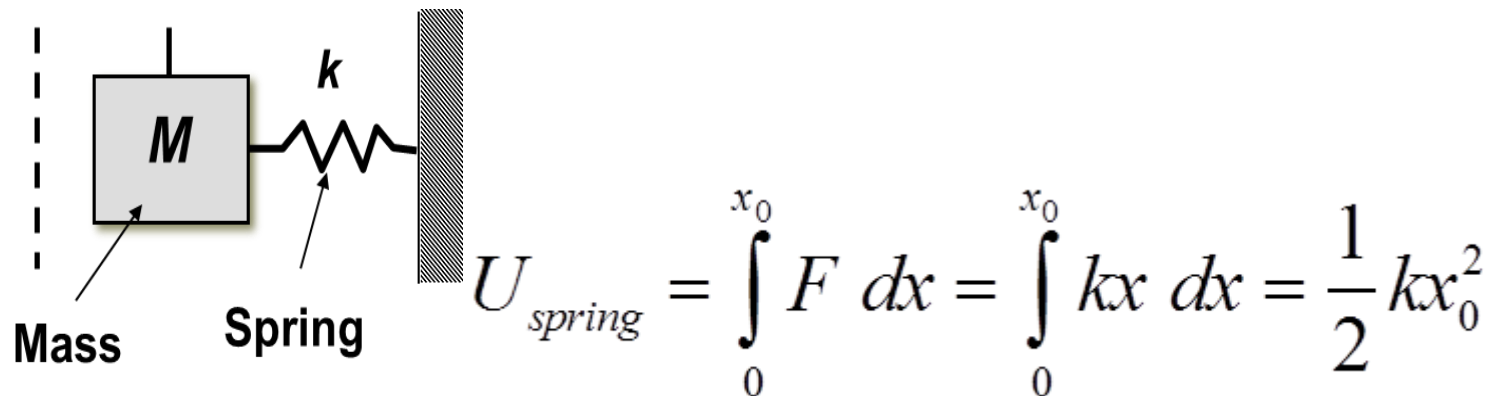
$$x(t) = e^{-\zeta\omega_n t} \left( \frac{v_0 + \zeta\omega_n x_0}{\omega_d} \sin \omega_d t + x_0 \cos \omega_d t \right)$$

## Section 1.4 Modeling and Energy Methods

- **Provides an alternative way to determine the equation of motion, and an alternative way to calculate the natural frequency of a system**
- **Useful if the forces or torques acting on the object or mechanical part are difficult to determine**
- **Very useful for more complicated systems later (MDOF and distributed mass systems)**

## Potential and Kinetic Energy

The potential energy of mechanical systems  $U$  is often stored in “springs” (remember that for a spring  $F = kx$ )



The kinetic energy of mechanical systems  $T$  is due to the motion of the “mass” in the system

$$T_{trans} = \frac{1}{2} m\dot{x}^2, \quad T_{rot} = \frac{1}{2} J\dot{\theta}^2$$

## Conservation of Energy

**For a simple, conservative (i.e. no damper), mass spring system the energy must be conserved:**

$$T + U = \text{constant}$$

$$\text{or} \quad \frac{d}{dt}(T + U) = 0$$

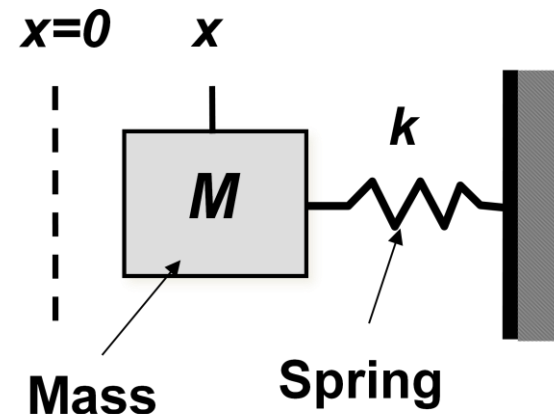
**At two different times  $t_1$  and  $t_2$  the increase in potential energy must be equal to a decrease in kinetic energy (or visa-versa).**

$$U_1 - U_2 = T_2 - T_1$$

and

$$U_{\max} = T_{\max}$$

# Deriving the equation of motion from the energy approach



$$\frac{d}{dt}(T + U) = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

$$\Rightarrow \dot{x}(m\ddot{x} + kx) = 0$$

Since  $\dot{x}$  cannot be zero for all time, then

$$m\ddot{x} + kx = 0$$

## Determining the Natural frequency directly from the energy

If the solution is given by  $x(t) = A \sin(\omega t + \phi)$  then the maximum potential and kinetic energies can be used to calculate the natural frequency of the system

$$U_{\max} = \frac{1}{2} k A^2 \quad T_{\max} = \frac{1}{2} m (\omega_n A)^2$$

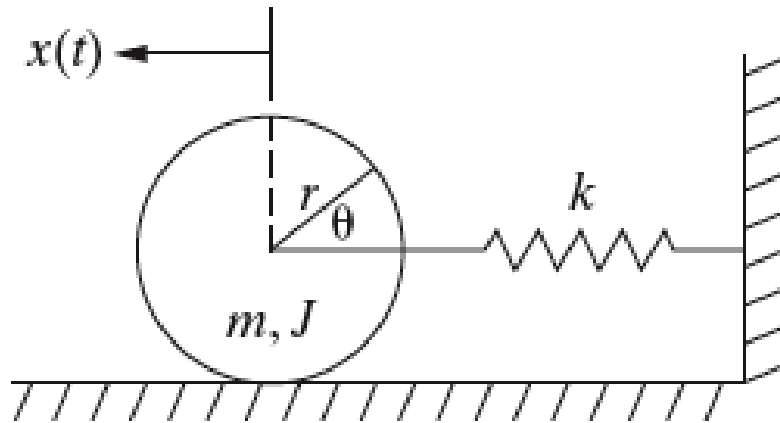
Since these two values must be equal

$$\frac{1}{2} k A^2 = \frac{1}{2} m (\omega_n A)^2$$

$$\Rightarrow k = m \omega_n^2 \Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$



## Example 1.4.1



**Compute the natural frequency of this roller fixed in place by a spring. Assume it is a conservative system (i.e. no losses) and rolls without slipping.**

$$T_{\text{rot}} = \frac{1}{2} J \dot{\theta}^2 \quad \text{and} \quad T_{\text{trans}} = \frac{1}{2} m \dot{x}^2$$

## Solution continued

$$x = r\theta \Rightarrow \dot{x} = r\dot{\theta} \Rightarrow T_{\text{Rot}} = \frac{1}{2} J \frac{\dot{x}^2}{r^2}$$

The max value of  $T$  happens at  $v_{\text{max}} = \omega_n A$

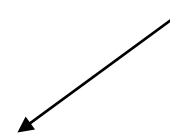
$$\Rightarrow T_{\text{max}} = \frac{1}{2} J \frac{(\omega_n A)^2}{r^2} + \frac{1}{2} m (\omega_n A)^2 = \frac{1}{2} \left( m + \frac{J}{r^2} \right) \omega_n^2 A^2$$

The max value of  $U$  happens at  $x_{\text{max}} = A$

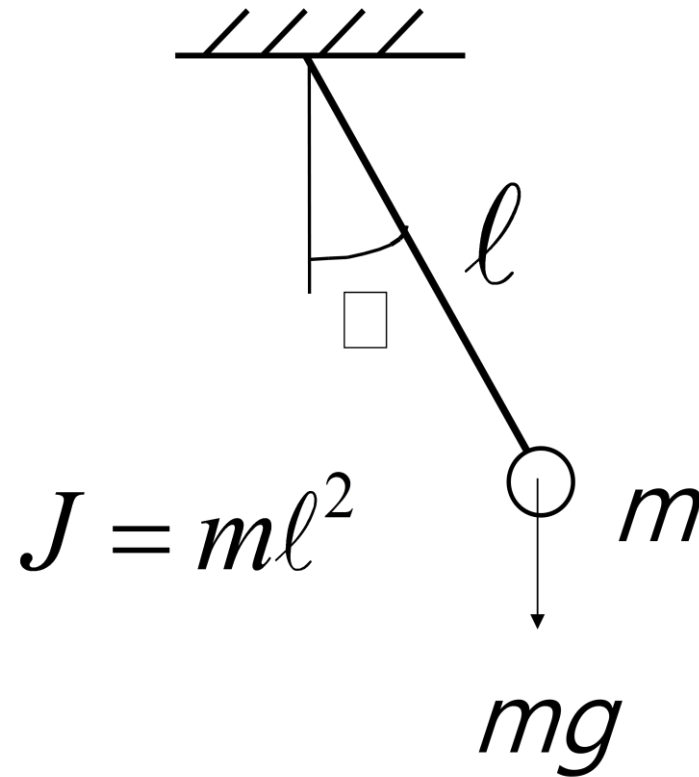
$$\Rightarrow U_{\text{max}} = \frac{1}{2} k A^2 \quad \text{Thus } T_{\text{max}} = U_{\text{max}} \Rightarrow$$

$$\frac{1}{2} \left( m + \frac{J}{r^2} \right) \omega_n^2 A^2 = \frac{1}{2} k A^2 \Rightarrow \omega_n = \sqrt{\frac{k}{\left( m + \frac{J}{r^2} \right)}}$$

**Effective mass**



**Example 1.4.2 Determine the equation of motion of the pendulum using energy**



**Now write down the energy**

$$T = \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} m \ell^2 \dot{\theta}^2$$

$U = mg\ell(1 - \cos \theta)$ , the change in elevation  
is  $\ell(1 - \cos \theta)$

$$\frac{d}{dt}(T + U) = \frac{d}{dt} \left( \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell(1 - \cos \theta) \right) = 0$$

$$ml^2 \ddot{\theta} + mgl(\sin \theta) \dot{\theta} = 0$$

$$\Rightarrow \dot{\theta} (ml^2 \ddot{\theta} + mgl(\sin \theta)) = 0$$

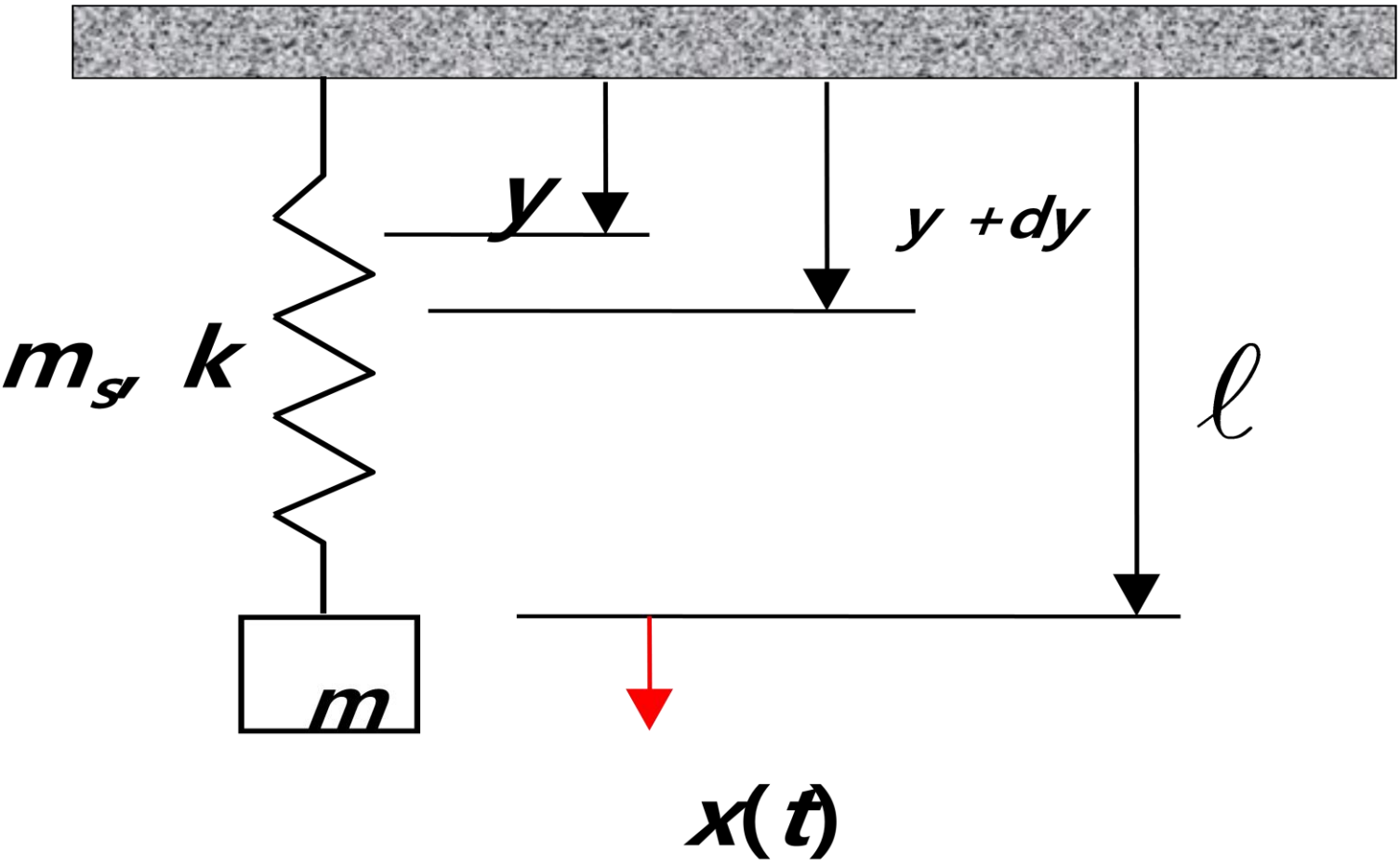
$$\Rightarrow ml^2 \ddot{\theta} + mgl(\sin \theta) = 0$$

$$\Rightarrow \ddot{\theta}(t) + \frac{g}{l} \sin \theta(t) = 0$$

Using the small angle approximation for sine:

$$\Rightarrow \ddot{\theta}(t) + \frac{g}{l} \theta(t) = 0 \quad \Rightarrow \omega_n = \sqrt{\frac{g}{l}}$$

**Example 1.4.4** The effect of including the mass of the spring on the value of the frequency.



$$\left. \begin{array}{l} \text{mass of element } dy : \frac{m_s}{\ell} dy \\ \text{velocity of element } dy : v_{dy} = \frac{y}{\ell} \dot{x}(t), \end{array} \right\} \text{assumptions}$$

$$\begin{aligned} T_{spring} &= \frac{1}{2} \int_0^{\ell} \frac{m_s}{\ell} \left[ \frac{y}{\ell} \dot{x} \right]^2 dy \quad (\text{adds up the KE of each element}) \\ &= \frac{1}{2} \left( \frac{m_s}{3} \right) \dot{x}^2 \end{aligned}$$

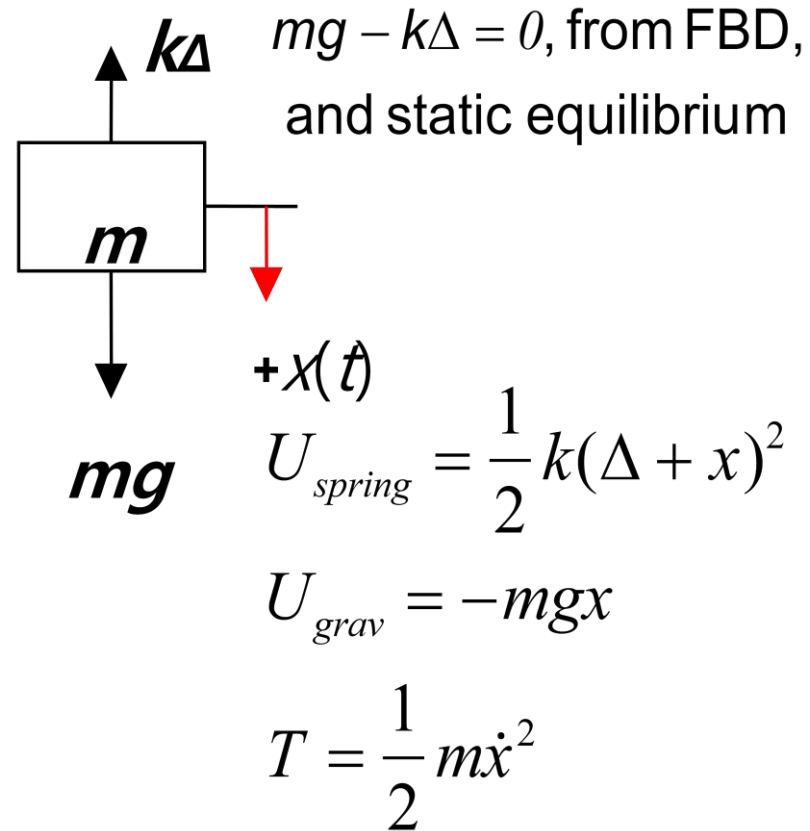
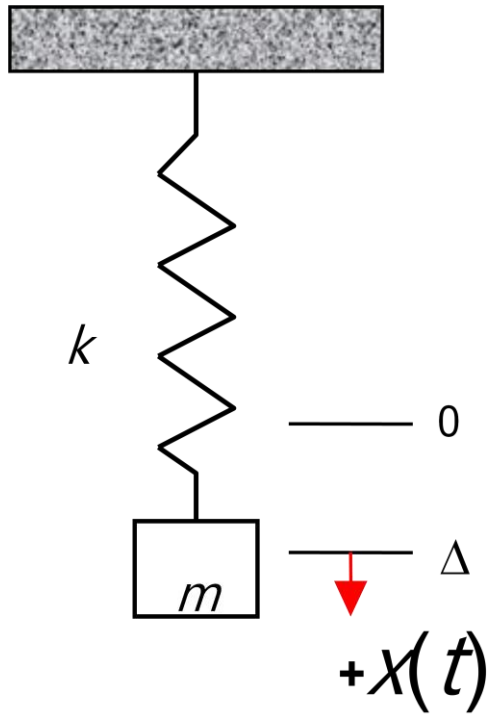
$$T_{mass} = \frac{1}{2} m \dot{x}^2 \Rightarrow T_{tot} = \left[ \frac{1}{2} \left( \frac{m_s}{3} \right) + \frac{1}{2} m \right] \dot{x}^2 \Rightarrow T_{max} = \frac{1}{2} \left( m + \frac{m_s}{3} \right) \omega_n^2 A^2$$

$$U_{max} = \frac{1}{2} k A^2$$

$$\Rightarrow \omega_n = \sqrt{\frac{k}{m + \frac{m_s}{3}}}$$

- This provides some simple design and modeling guides

## What about gravity?





Now use  $\frac{d}{dt}(T + U) = 0$

$$\Rightarrow \frac{d}{dt} \left[ \frac{1}{2} m \dot{x}^2 - mgx + \frac{1}{2} k(\Delta + x)^2 \right] = 0$$

$$\Rightarrow m\ddot{x} - mg + k(\Delta + x)\dot{x}$$

$$\Rightarrow \dot{x}(m\ddot{x} + kx) + \underbrace{\dot{x}(k\Delta - mg)}_{\substack{0 \text{ from static} \\ \text{equilibrium}}} = 0$$

$$\Rightarrow m\ddot{x} + kx = 0$$

- Gravity does not effect the equation of motion or the natural frequency of the system for a linear system as shown previously with a force balance.

## Lagrange's Method for deriving equations of motion.

Again consider a conservative system and its energy.

It can be shown that if the Lagrangian  $L$  is defined as

$$L = T - U$$

Then the equations of motion can be calculated from

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad (1.62)$$

Which becomes

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = 0 \quad (1.63)$$

Here  $q$  is a generalized coordinate

## Example 1.4.7 Derive the equation of motion of a spring mass system via the Lagrangian

$$T = \frac{1}{2}m\dot{x}^2 \quad \text{and} \quad U = \frac{1}{2}kx^2$$

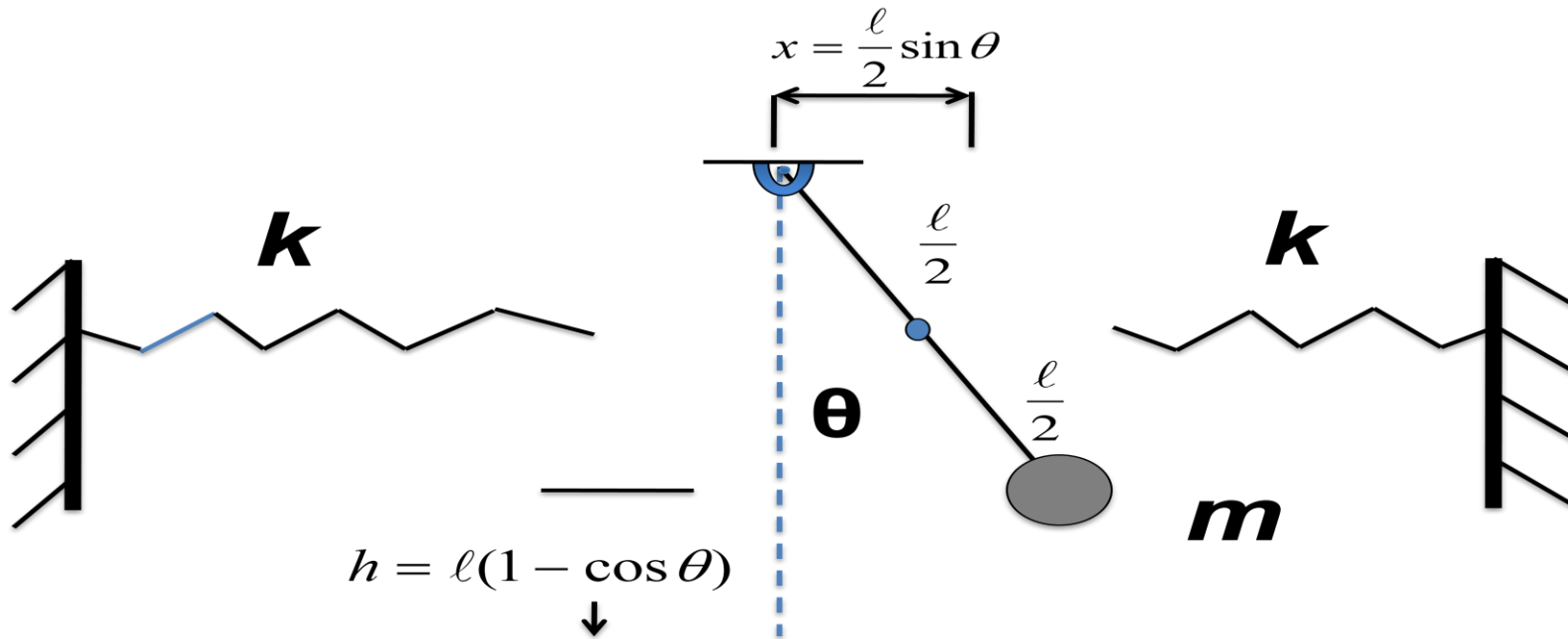
Here  $q = x$ , and the Lagrangian becomes

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Equation (1.64) becomes

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} &= \frac{d}{dt} (m\dot{x}) - 0 + kx = 0 \\ \Rightarrow m\ddot{x} + kx &= 0 \end{aligned}$$

## Example



$$U = \frac{1}{2} kx^2 + \frac{1}{2} kx^2 + mg\ell(1 - \cos \theta)$$
$$= \frac{k\ell^2}{4} \sin^2 \theta + mg\ell(1 - \cos \theta)$$

**The Kinetic energy term is :**  $T = \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} m \ell^2 \dot{\theta}^2$

**Compute the terms in Lagrange's equation:**

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) = \frac{d}{dt} (m \ell^2 \dot{\theta}) = m \ell^2 \ddot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial U}{\partial \theta} = \frac{\partial}{\partial \theta} \left( \frac{k \ell^2}{4} \sin^2 \theta + m g \ell (1 - \cos \theta) \right) = \frac{k \ell^2}{2} \sin \theta \cos \theta + m g \ell \sin \theta$$

**Lagrange's equation (1.64) yields:**

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = m \ell^2 \ddot{\theta} + \frac{k \ell^2}{2} \sin \theta \cos \theta + m g \ell \sin \theta = 0$$

**Does it make sense:**

$$ml^2\ddot{\theta} + \underbrace{\frac{kl^2}{2} \sin \theta \cos \theta}_{0 \text{ if } k=0} + mgl \sin \theta = 0$$

**Linearize to get small angle case:**

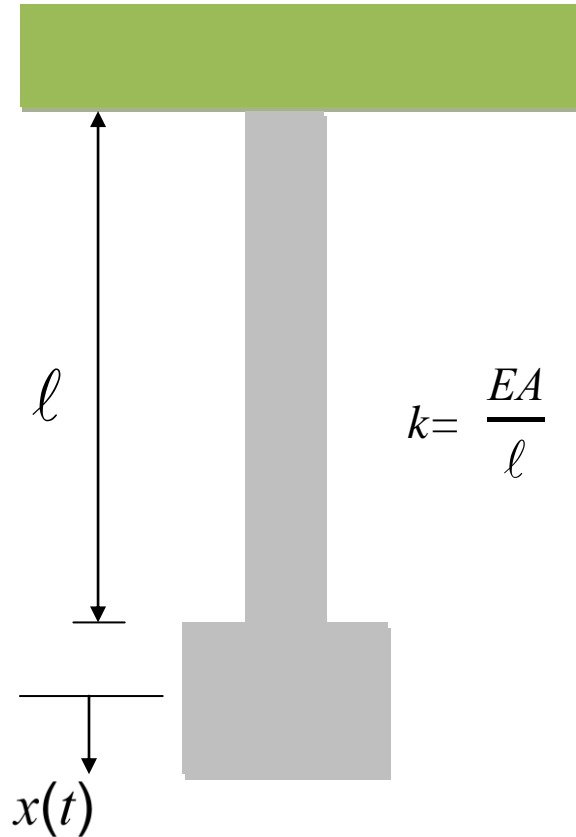
$$ml^2\ddot{\theta} + \frac{kl^2}{2}\theta + mgl\theta = 0$$

$$\Rightarrow \ddot{\theta} + \left( \frac{kl + 2mg}{2ml} \right) \theta = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{kl + 2mg}{2ml}}$$

**What happens if you linearize first?**

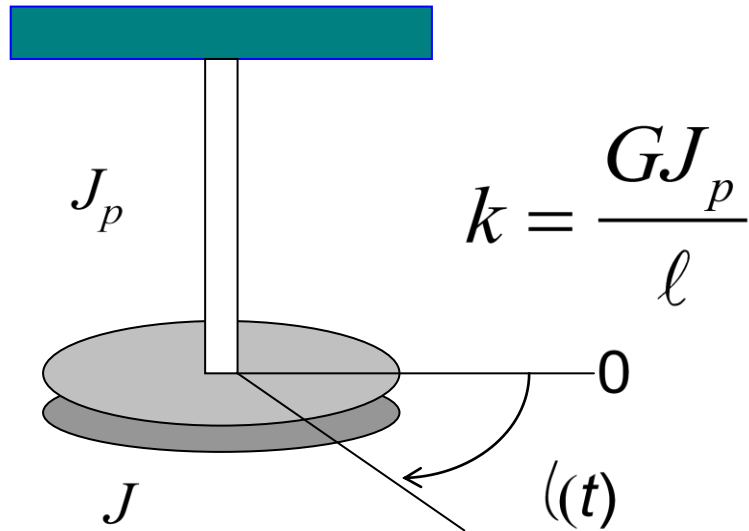
## 1.5 More on springs and stiffness



$$k = \frac{EA}{\ell}$$

- Longitudinal motion
- $A$  is the cross sectional area ( $\text{m}^2$ )
- $E$  is the elastic modulus ( $\text{Pa} = \text{N}/\text{m}^2$ )
- $\ell$  is the length (m)
- $k$  is the stiffness ( $\text{N}/\text{m}$ )

## Figure 1.21 Torsional Stiffness



- $J_p$  is the polar moment of inertia of the rod
- $J$  is the mass moment of inertia of the disk
- $G$  is the shear modulus,  $l$  is the length



## Example 1.5.1 compute the frequency of a shaft/mass system

$$\{J = 0.5 \text{ kg} \cdot \text{m}^2\}$$

From Equation (1.50)

$$\sum M = J\ddot{\theta} \Rightarrow J\ddot{\theta}(t) + k\theta(t) = 0$$

$$\Rightarrow \ddot{\theta}(t) + \frac{k}{J}\theta(t) = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{GJ_p}{\ell J}}, \quad J_p = \frac{\pi d^4}{32}$$

For a 2 m steel shaft, diameter of 0.5 cm  $\Rightarrow$

$$\begin{aligned} \omega_n &= \sqrt{\frac{GJ_p}{\ell J}} = \sqrt{\frac{(8 \times 10^{10} \text{ N/m}^2)[\pi(0.5 \times 10^{-2} \text{ m})^4 / 32]}{(2 \text{ m})(0.5 \text{ kg} \cdot \text{m}^2)}} \\ &= 2.2 \text{ rad/s} \end{aligned}$$

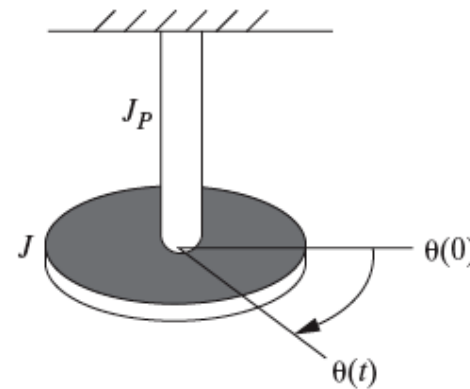
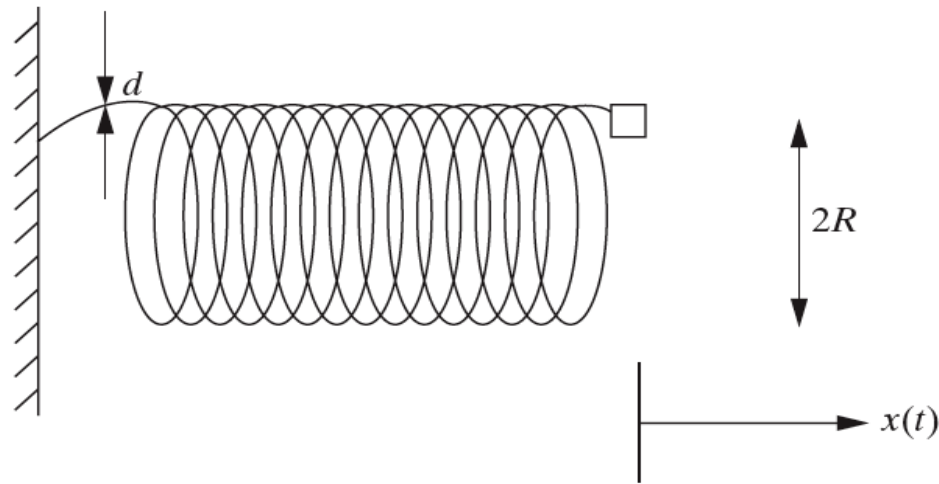


Figure 1.22

## Fig. 1.22 Helical Spring



$d$  = diameter of spring material  
 $2R$  = diameter of turns  
 $n$  = number of turns  
 $x(t)$  = deflection

$$k = \frac{Gd^4}{64nR^3}$$

$d$  = diameter of wire

$2R$  = diameter of turns

$n$  = number of turns

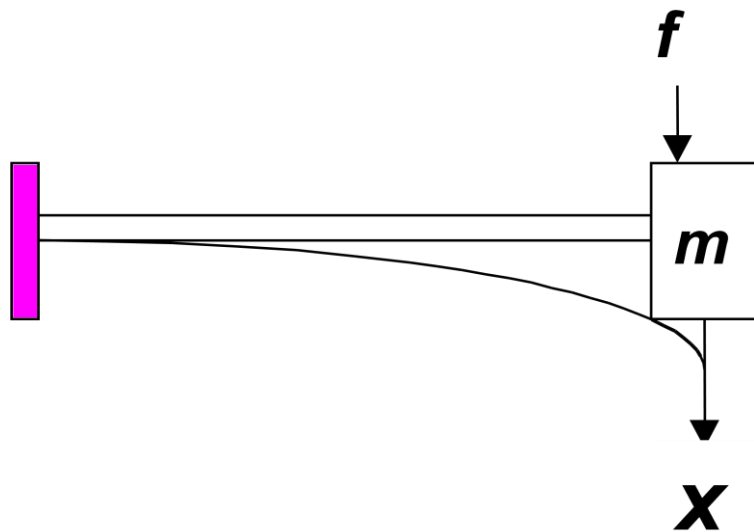
$x(t)$  = end deflection

$G$  = shear modulus of  
spring material

$$k = \frac{Gd^4}{64nR^3}$$

**Allows the design of springs  
to have specific stiffness**

## Fig 1.23 Transverse beam stiffness



- Strength of materials and experiments yield:

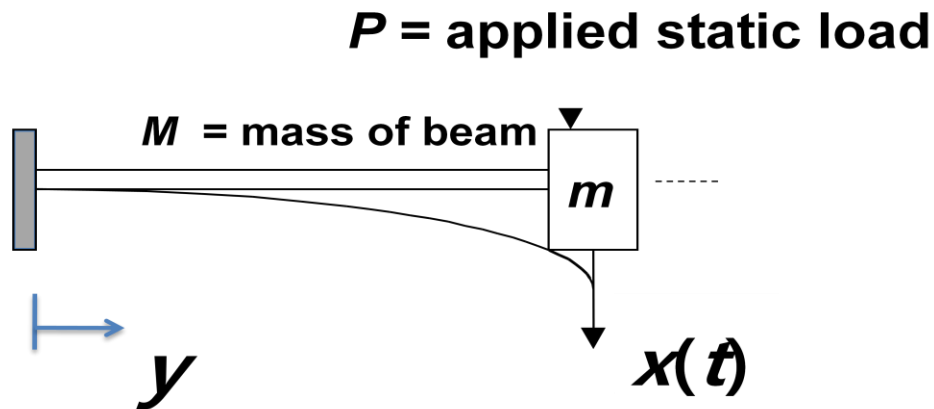
$$k = \frac{3EI}{l^3}$$

With a mass at the tip:

$$\omega_n = \sqrt{\frac{3EI}{ml^3}}$$

## Example for a Heavy Beam

Consider the beam of Figure 1.28 and what happens if the mass of the beam is considered.



Much like example 1.4.4 where the mass of a spring was considered, the procedure is to calculate the kinetic energy of the beam itself, by looking at a differential element of the beam and integrating over the beam length

From strength of materials the static deflection of a cantilever beam of length  $l$  is:

$$x(y) = \frac{Py^2}{6EI} (3l - y) \text{ Which has maximum value of (at } y = l \text{ )}: x_{\max} = \frac{Pl^3}{3EI}$$

**Next integrate along the beam to compute the beam's kinetic energy contribution**

$$\begin{aligned}
 T_{\max} &= \frac{1}{2} \int_0^{\ell} (\text{mass of differential element}) \cdot (\text{velocity of differential})^2 \\
 &= \frac{1}{2} \int_0^{\ell} [\dot{x}(y)]^2 \frac{M}{\ell} dy = \frac{1}{2} \frac{M}{\ell} \frac{\dot{x}_{\max}^2}{4\ell^6} \int_0^{\ell} (3\ell y^2 - y^3) dy
 \end{aligned}$$

Mass of  
element dy

\*\*

$$= \frac{1}{2} \left( \frac{33}{140} M \right) \dot{x}_{\max}^2$$

**Thus the equivalent mass of the beam is:**  $M_{\text{eq}} = \frac{33}{140} M$

**And the equivalent mass of the beam- mass system is:**

$$m_{\text{system}} = \frac{33}{140} M + m$$

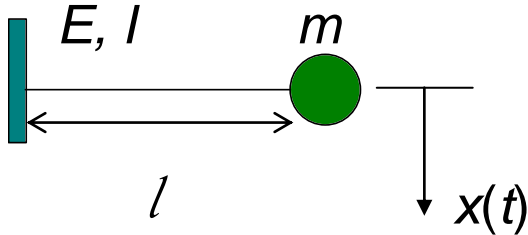
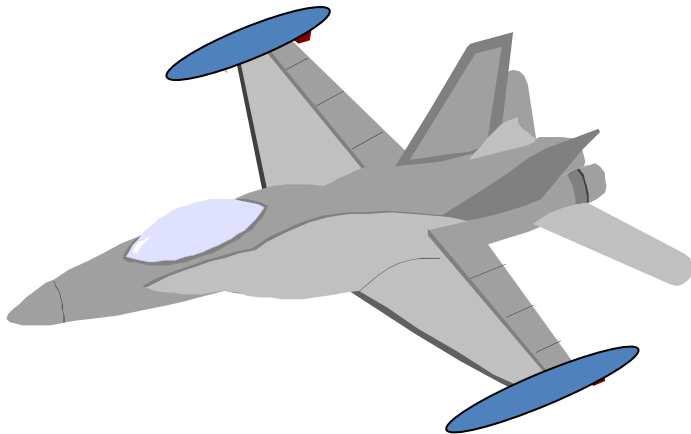
With the equivalent mass known the frequency adjustment for including the mass of the beam becomes

$$\begin{aligned}\omega_n &= \sqrt{\frac{k}{m_{\text{eq}}}} = \sqrt{\frac{\frac{3EI}{l^3}}{m + \frac{33}{140}M}} \\ &= \sqrt{\frac{3EI}{l^3 \left( m + \frac{33}{140}M \right)}} \text{ rad/s}\end{aligned}$$

## **Samples of Vibrating Systems**

- **Deflection of continuum (beams, plates, bars, etc) such as airplane wings, truck chassis, disc drives, circuit boards...**
- **Shaft rotation**
- **Rolling ships**
- **See text for more examples.**

## Example 1.5.2 Effect of fuel on frequency of an airplane wing



- **Model wing as transverse beam**
- **Model fuel as tip mass**
- **Ignore the mass of the wing and see how the frequency of the system changes as the fuel is used up**



**Mass of pod 10 kg empty 1000 kg full**

**$I = 5.2 \times 10^{-5} \text{ m}^4$ ,  $E = 6.9 \times 10^9 \text{ N/m}$ ,  $l = 2 \text{ m}$**

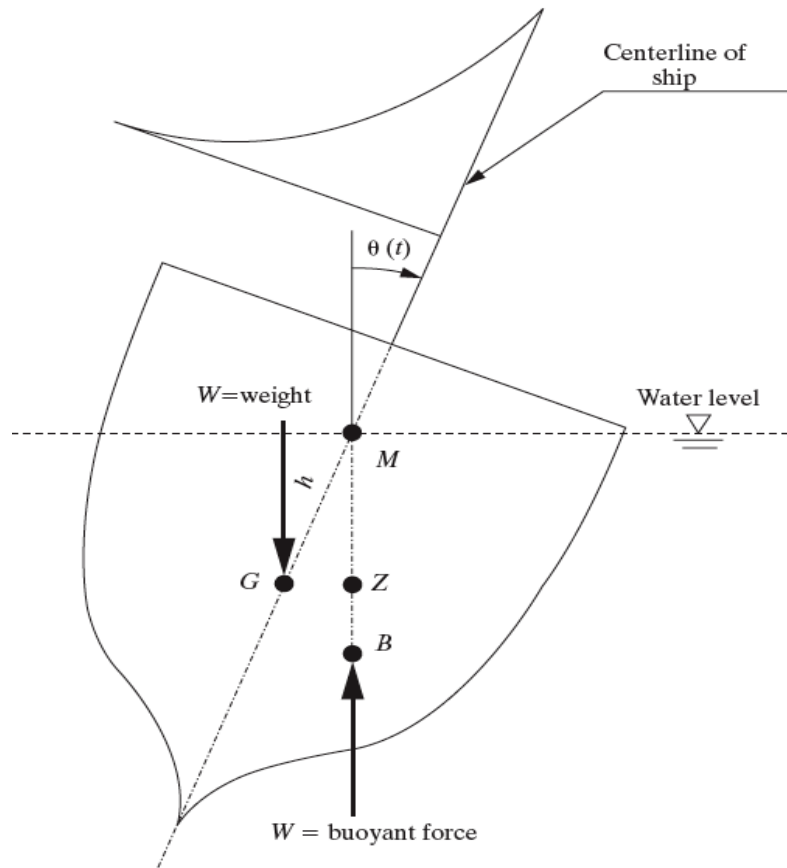
- Hence the natural frequency changes by an order of magnitude while it empties out fuel.**

$$\begin{aligned}\omega_{\text{full}} &= \sqrt{\frac{3EI}{ml^3}} = \sqrt{\frac{3(6.9 \times 10^9)(5.2 \times 10^{-5})}{1000 \cdot 2^3}} \\ &= 11.6 \text{ rad/s} = 1.8 \text{ Hz}\end{aligned}$$

$$\begin{aligned}\omega_{\text{empty}} &= \sqrt{\frac{3EI}{ml^3}} = \sqrt{\frac{3(6.9 \times 10^9)(5.2 \times 10^{-5})}{10 \cdot 2^3}} \\ &= 115 \text{ rad/s} = 18.5 \text{ Hz}\end{aligned}$$

**This ignores the mass of the wing**

## Example 1.5.3 Rolling motion of a ship



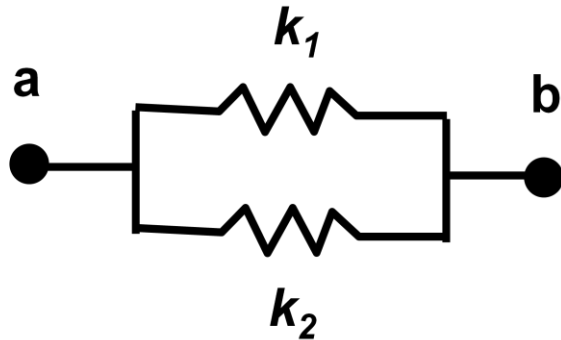
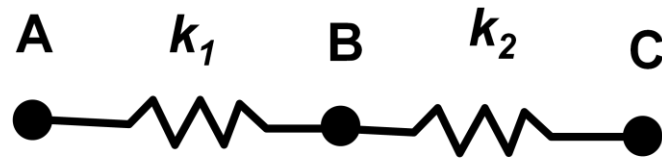
$$J\ddot{\theta}(t) = -W\overline{GZ} = -Wh \sin \theta(t)$$

For small angles this becomes

$$J\ddot{\theta}(t) + Wh\theta(t) = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{hW}{J}}$$

**Combining Springs: Springs are usually only available in limited stiffness values. Combining them allows other values to be obtained**



• Equivalent Spring

$$\text{series: } k_{AC} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

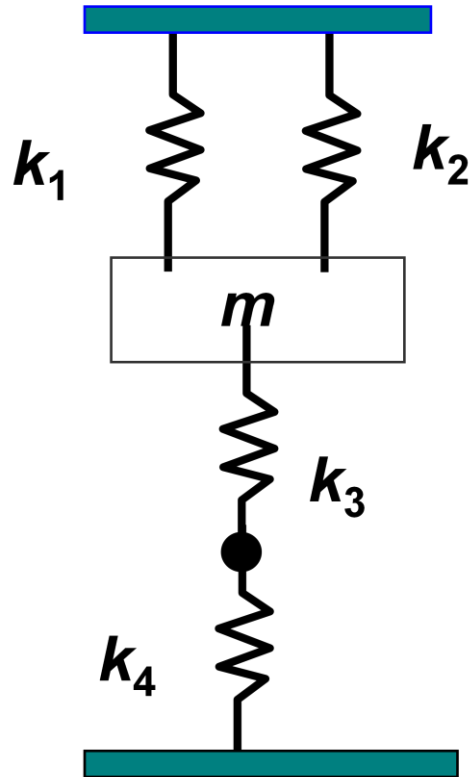
$$\text{parallel: } k_{ab} = k_1 + k_2$$

**This is identical to the combination of capacitors in electrical circuits**

## Use these to design from available parts

- **Discrete springs available in standard values**
- **Dynamic requirements require specific frequencies**
- **Mass is often fixed or  $\pm$  small amount**
- **Use spring combinations to adjust  $\omega_n$**
- **Check static deflection**

## Example 1.5.5 Design of a spring mass system using available springs: series vs parallel



- Let  $m = 10$  kg
- Compare a series and parallel combination
- a)  $k_1 = 1000$  N/m,  $k_2 = 3000$  N/m,  $k_3 = k_4 = 0$
- b)  $k_3 = 1000$  N/m,  $k_4 = 3000$  N/m,  $k_1 = k_2 = 0$

Case a) parallel connection:

$$k_3 = k_4 = 0, k_{eq} = k_1 + k_2 = 1000 + 3000 = 4000 \text{ N/m}$$

$$\Rightarrow \omega_{parallel} = \sqrt{\frac{k_{eg}}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

Case b) series connection:

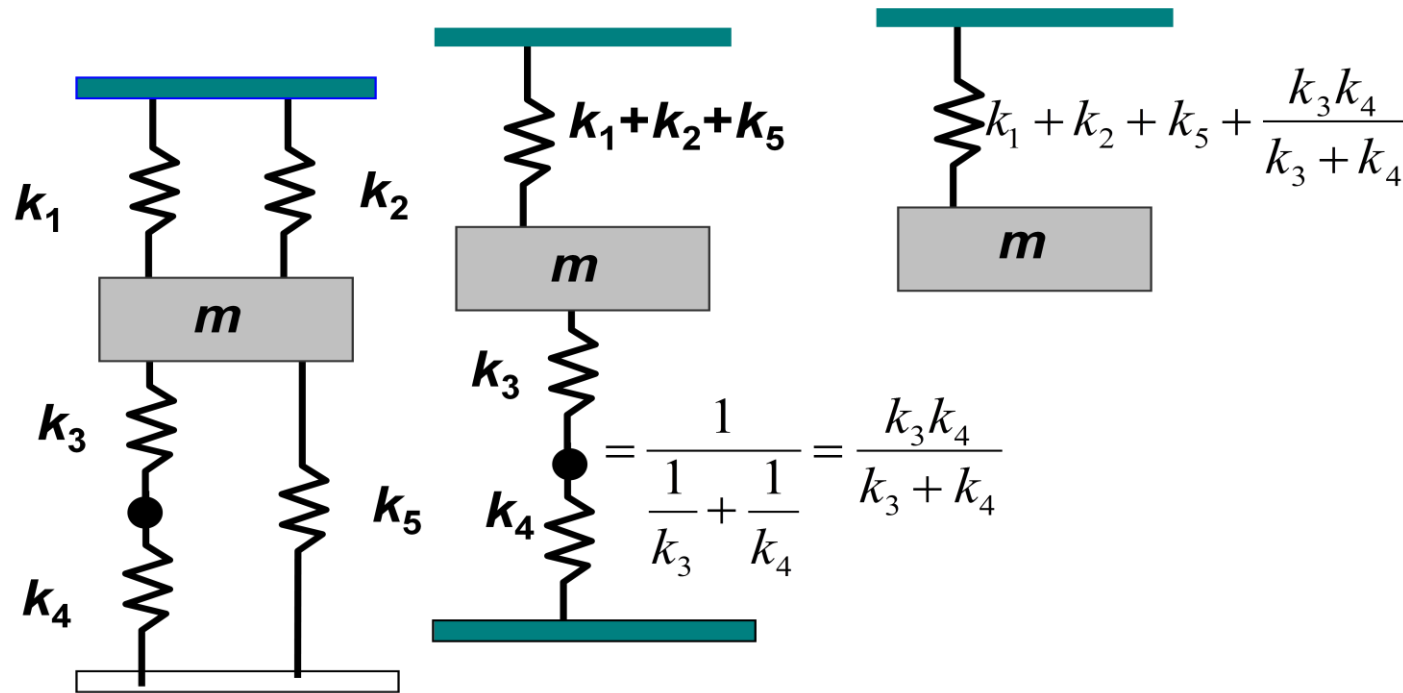
$$k_1 = k_2 = 0, k_{eq} = \frac{1}{(1/k_3) + (1/k_4)} = \frac{3000}{3+1} = 750 \text{ N/m}$$

$$\Rightarrow \omega_{series} = \sqrt{\frac{k_{eg}}{m}} = \sqrt{\frac{750}{10}} = 8.66 \text{ rad/s}$$

**Same physical components, very different frequency**

**Allows some design flexibility in using off the shelf components**

**Example: Find the equivalent stiffness  $k$  of the following system (Fig 1.26, page 47)**



$$\omega_n = \sqrt{\frac{k_1 k_3 + k_2 k_3 + k_5 k_3 + k_1 k_4 + k_2 k_4 + k_5 k_4 + k_3 k_4}{m(k_3 + k_4)}}$$

## Example 1.5.5 Compare the natural frequency of two springs connected to a mass in parallel with two in series

**A series connect of  $k_1 = 1000$  N/m and  $k_2 = 3000$  N/m with  $m = 10$  kg yields:**

$$k_{eq} = \frac{1}{1/1000 + 1/3000} = 750 \text{ N/m} \Rightarrow \omega_{series} = \sqrt{\frac{750 \text{ N/m}}{10 \text{ kg}}} = 8.66 \text{ rad/s}$$

**A parallel connect of  $k_1 = 1000$  N/m and  $k_2 = 3000$  N/m with  $m = 10$  kg yields:**

$$k_{eg} = 1000 \text{ N/m} + 3000 \text{ N/m} = 4000 \text{ N/m} \Rightarrow \omega_{par} = \sqrt{\frac{4000 \text{ N/m}}{10 \text{ kg}}} = 20 \text{ rad/s}$$

**Same components, very different frequency**



## Static Deflection

**Another important consideration in designing with springs is the static deflection**

$$\Delta k = mg \Rightarrow \Delta = \frac{mg}{k}$$

**This determines how much a spring compresses or sags due to the static mass (you can see this when you jack your car up**

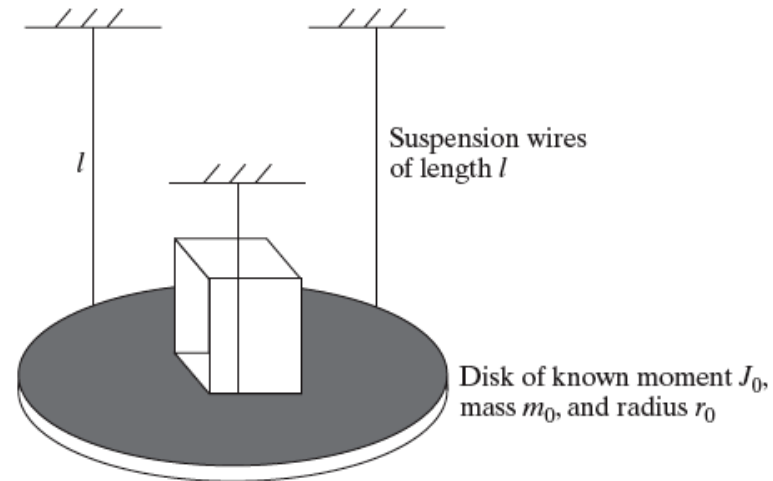
**The other concern is “rattle space” which is the maximum deflection  $A$**

## Section 1.6 Measurement

- **Mass:** usually pretty easy to measure using a balance- a static experiment
- **Stiffness:** again can be measured statically by using a simple displacement measurement and knowing the applied force
- **Damping:** can only be measured dynamically

# Measuring moments of inertia using a Trifilar suspension system

$$J = \frac{gT^2 r_0^2 (m_0 + m)}{4\pi^2 \ell} - J_0$$



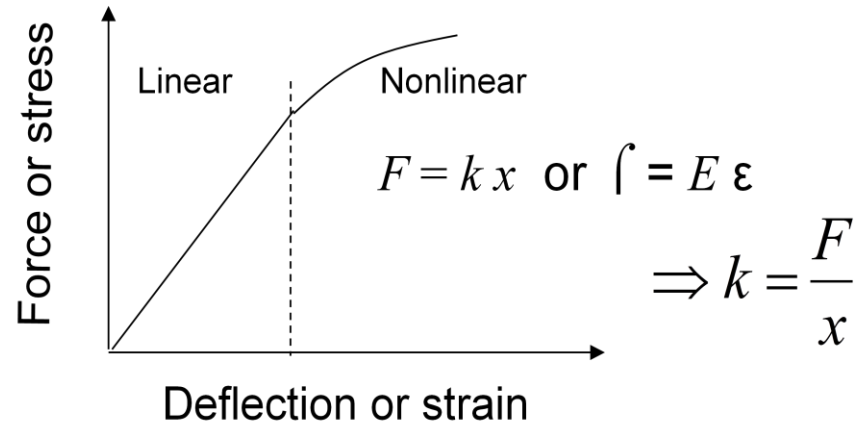
**Figure 1.33**

**$T$  is the measured period**

**$g$  is the acceleration due to gravity**

# Stiffness Measurements

**From Static Deflection:**

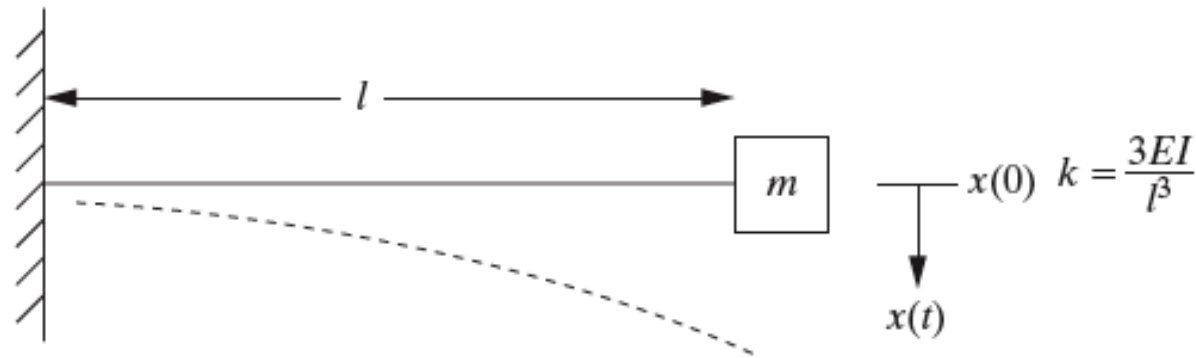


**From Dynamic Frequency:**

**Figure 1.34**

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_n^2$$

## Example 1.6.1 Use the beam stiffness equation to compute the modulus of a material



$E$  = elastic modulus

$l$  = length of beam

$I$  = moment of inertia of cross-sectional area about the neutral axis

**Figure 1.24**  $l = 1 \text{ m}$ ,  $m = 6 \text{ kg}$ ,  $I = 10^{-9} \text{ m}^4$ , and measured  $T = 0.62 \text{ s}$

$$T = 2\pi \sqrt{\frac{ml^3}{3EI}} = 0.62 \text{ s}$$

$$\Rightarrow E = \frac{4\pi^2 ml^3}{3T^2 I} = \frac{4\pi^2 (6 \text{ kg})(1 \text{ m})^3}{3(0.62 \text{ s})^2 (10^{-9} \text{ m}^4)} = 2.05 \times 10^{11} \text{ N/m}^2$$

## Damping Measurement (Dynamic only)

Define the Logarithmic Decrement:

$$\delta = \ln \frac{x(t)}{x(t+T)} \quad (1.79)$$

$$\delta = \ln \frac{Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi)}{Ae^{-\zeta\omega_n(t+T)} \sin(\omega_d t + \omega_d T + \phi)} \quad (1.80)$$

$$\delta = \zeta\omega_n T$$

$$\zeta = \frac{c}{c_{cr}} = \frac{\delta}{\omega_n T} = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \quad (1.83)$$

## Section 1.7: Design Considerations

**Using the analysis so far to guide the selection of components.**

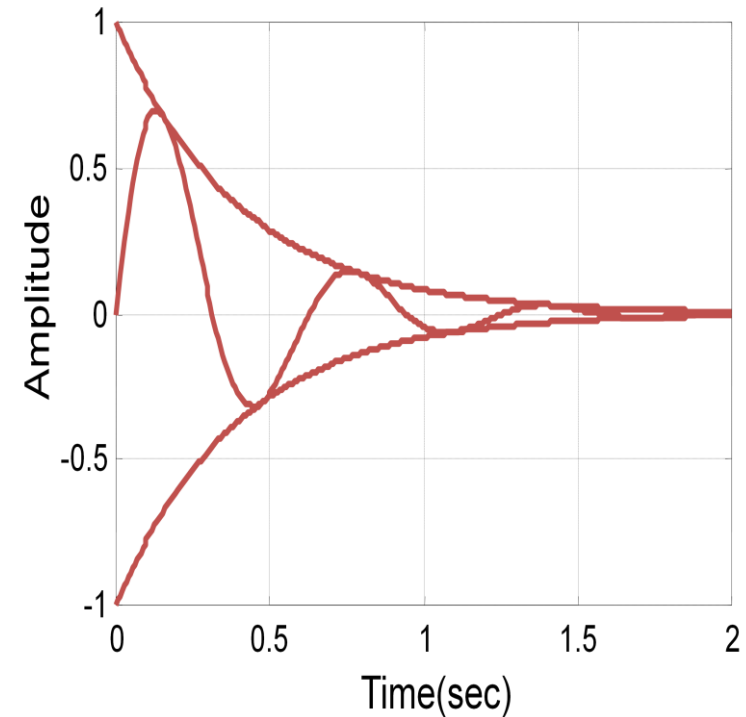
## Example 1.7.1

- Mass  $2 \text{ kg} \leq m \leq 3 \text{ kg}$  and  $k \geq 200 \text{ N/m}$
- For a possible frequency range of  $8.16 \text{ rad/s} \leq \omega_n \leq 10 \text{ rad/s}$
- For initial conditions:  $x_0 = 0$ ,  $v_0 < 300 \text{ mm/s}$
- Choose a  $c$  so response is always  $\leq 25 \text{ mm}$



## Solution

- Write down  $x(t)$  for 0 initial displacement
- Look for max amplitude
- Occurs at time of first peak ( $T_{\max}$ )
- Compute the amplitude at  $T_{\max}$
- Compute  $\zeta$  for  $A(T_{\max})=0.025$



$$x(t) = \underbrace{\frac{v_0}{\omega_d}}_{\text{Amplitude}} e^{-\zeta\omega_n t} \sin(\omega_d t)$$

$\Rightarrow$  worst case happens at smallest  $\omega_d \Rightarrow \omega_n = 8.16$  rad/s

$\Rightarrow$  worst case happens at max  $v_0 = 300$  mm/s

With  $\omega_n$  and  $v_0$  fixed at these values, investigate how varies with  $\zeta$

First peak is highest and occurs at

$$\frac{d}{dt}(x(t)) = 0 \Rightarrow \omega_d e^{-\zeta\omega_n t} \cos(\omega_d t) - \zeta\omega_n e^{-\zeta\omega_n t} \sin(\omega_d t) = 0$$

$$\text{Solve for } t = T_{\max} \Rightarrow T_m = \frac{1}{\omega_d} \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right) = \frac{1}{\omega_d} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\text{Sub } T_{\max} \text{ into } x(t) : A_m(\zeta) = x(T_m) = \frac{v_0}{\omega_n \sqrt{1-\zeta^2}} e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)} \sin\left(\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$$

$$A_m(\zeta) = \frac{v_0}{\omega_n} e^{-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)}$$

To keep the max value less than 0.025 m solve

$$A_{\max}(\zeta) = 0.025 \Rightarrow \zeta = 0.281$$

Using the upper limit on the mass ( $m = 3$  kg)  
yields

$$c = 2m\omega_n\zeta = 2 \cdot 3 \cdot 8.16 \cdot 0.281 = \underline{14.15 \text{ kg/s}}$$

FYI,  $\zeta = 0$  yields  $A_{\max} = \frac{v_0}{\omega_n} = 37 \text{ mm}$

**Example 1.7.3** What happens to a good design when some one changes the parameters? (Car suspension system). How does  $\zeta$  change with mass, ie when the car more passengers and luggage?

Given  $\zeta = 0.3$ ,  $m = 1361$  kg,  $\Delta = 0.05$  m, compute  $c$ ,  $k$  .

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = 1361\omega_n^2, \quad mg = k\Delta \Rightarrow k = \frac{mg}{\Delta}$$

$$\Rightarrow \omega_n = \sqrt{\frac{mg}{m\Delta}} = \sqrt{\frac{9.81}{0.05}} = 14 \text{ rad/s} \Rightarrow$$

$$k = 1361(14)^2 = \underline{2.668 \times 10^5 \text{ N/m}}$$

$$\zeta = 0.3 \Rightarrow c = 2(0.3)m\omega_n = 0.6(1361)(14) = \underline{1.143 \times 10^4 \text{ kg/s}}$$

**Now add 290 kg of passengers and luggage. What happens?**

$$m = 1361 + 290 = 1651 \text{ kg}$$

$$\Rightarrow \Delta = \frac{mg}{k} = \frac{1651 \cdot 9.8}{2.668 \times 10^5} \approx 0.06 \text{ m}$$

$$\Rightarrow \omega_n = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{9.8}{0.06}} = \underline{12.7 \text{ rad/s}}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{1.143 \times 10^4}{2m\omega_n} = \underline{0.27}$$

**A lower frequency and damping ratio results meaning the car takes longer to damp out**

$$e^{-\zeta\omega_n t}$$

## Section 1.8 Stability

**Stability is defined for the solution of free response case:**

**Stable:**  $|x(t)| < M, \quad \forall t > 0$

**Asymptotically Stable:**  $\lim_{t \rightarrow \infty} x(t) = 0$

**Unstable:**

**if it is not stable or asymptotically stable**

# Examples of the types of stability

Stable

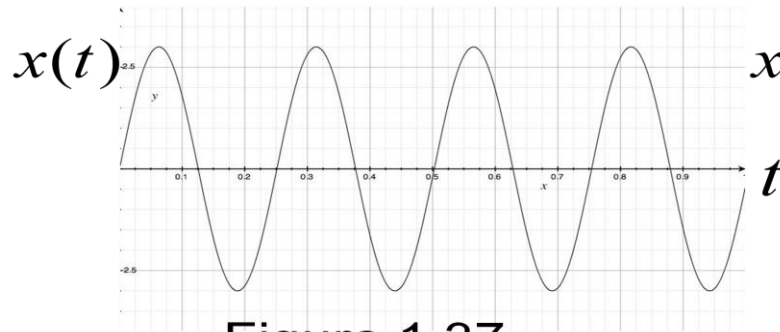
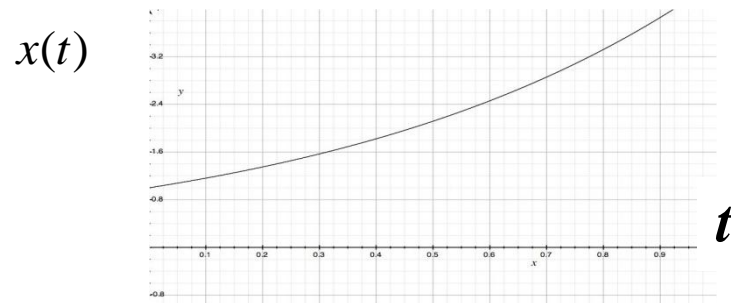
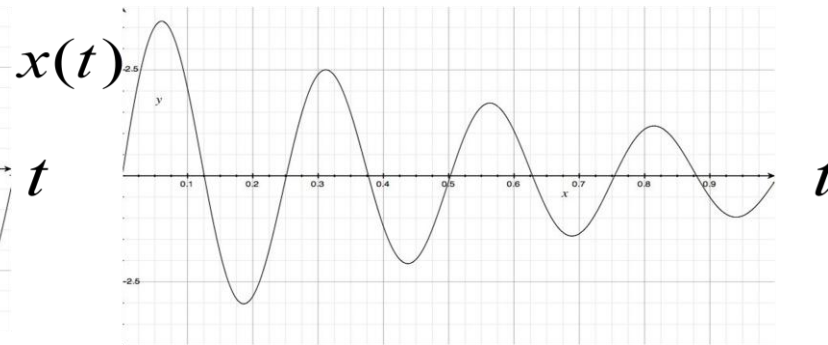


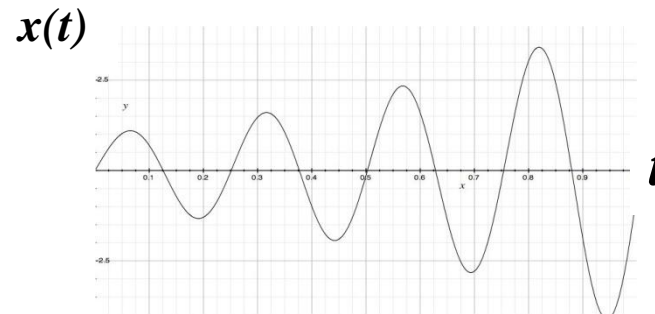
Figure 1.37

Asymptotically



Divergent instability

Figure 1.38

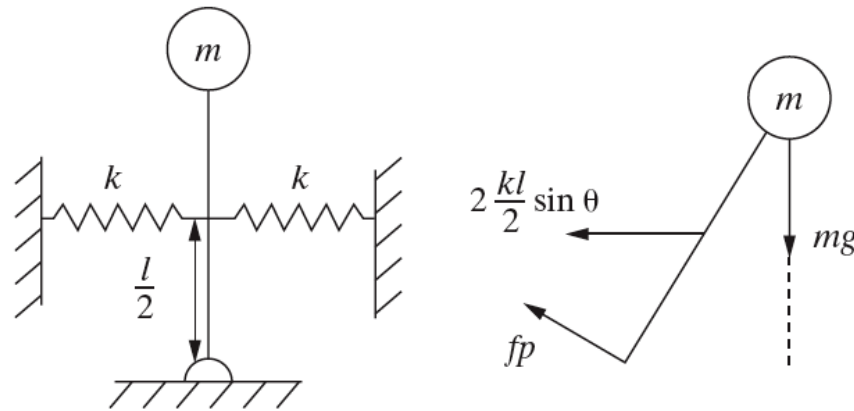


Flutter instability

Figure 1.39

# Example: 1.8.1: For what values of the spring constant will the response be stable?

Figure 1.37



$$ml^2\ddot{\theta} + \left( \frac{kl^2}{2} \sin \theta \right) \cos \theta - mgl \sin \theta = 0 \Rightarrow ml^2\ddot{\theta} + \frac{kl^2}{2} \theta - mgl \theta = 0$$

$$\Rightarrow 2ml\ddot{\theta} + (kl - 2mg)\theta = 0 \quad (\text{for small } \theta)$$

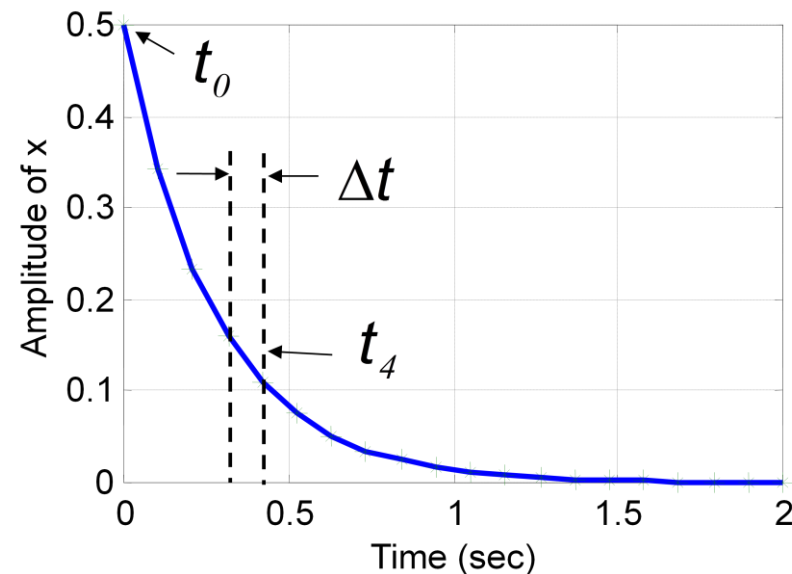


## 1.9 Numerical Simulation

- Solving differential equations by numerical integration
- Euler, Runge-Kutta, etc.
- Available in Mathcad, Matlab, Mathematica and Maple (or in FORTRAN)
- Or use Engineering Vibration Toolbox
- Will use these to examine nonlinear vibration problems that do not have analytical expressions for solutions

$$\frac{dx(t_i)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{x(t_{i+1}) - x(t_i)}{\Delta t}$$

$$\Delta t = t_{i+1} - t_i$$



## First order differential equation

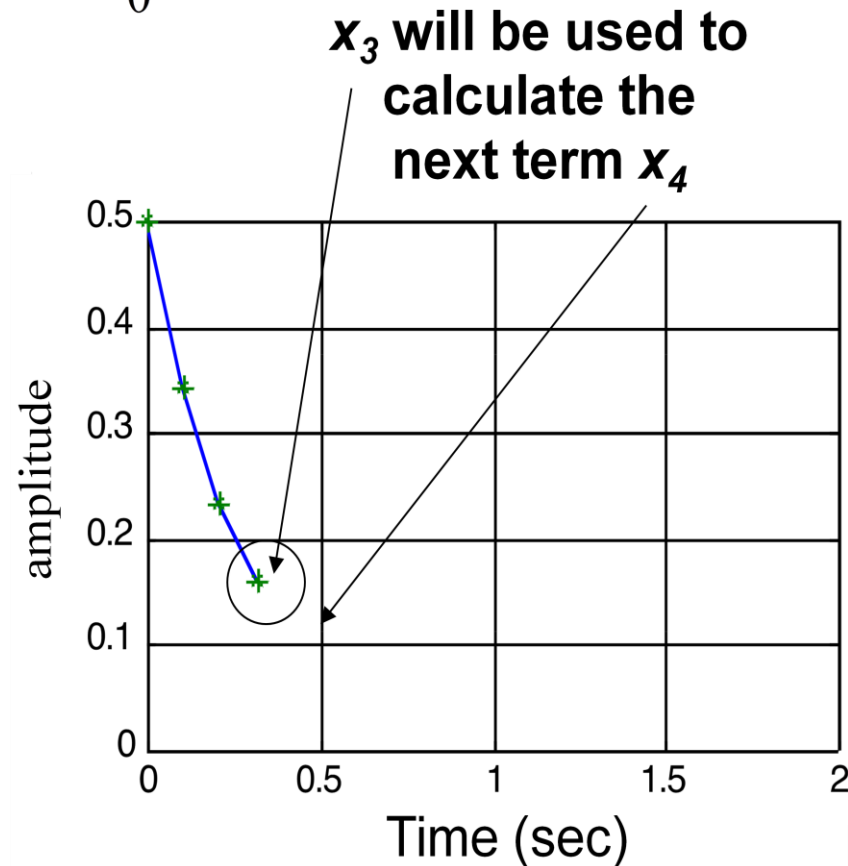
solve  $\dot{x}(t) = ax(t)$ ,  $x(0) = x_0$

$$\frac{x_{i+1} - x_i}{\Delta t} = ax_i, x_0$$

$$x_i = x(t_i), \Delta t = t_{i+1} - t_i$$

$$x_{i+1} = x_i [1 + a\Delta t]$$

The new value of  $x$  is calculated from the old value of  $x$ .



## Example 1.9.1 solve $dx/dt = -3x, x(0)=1$

$$a = -3, \text{ take } \Delta t = 0.5$$

$$x_0 = 1$$

$$x_1 = x_0 + (0.5)(-3)(x_0) = -0.5$$

$$x_2 = x_1 + (0.5)(-3)(x_1) = 0.25$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad = \quad \vdots$$

### **Numerical solution**

Note that the numerical  
Solution is different for  
Each choice of  $\Delta t$

$$x(t) = Ae^{\lambda t}$$

$$\dot{x}(t) = -3x(t) \Rightarrow -\lambda Ae^{\lambda t} = -3Ae^{\lambda t}$$
$$\Rightarrow \lambda = 3,$$

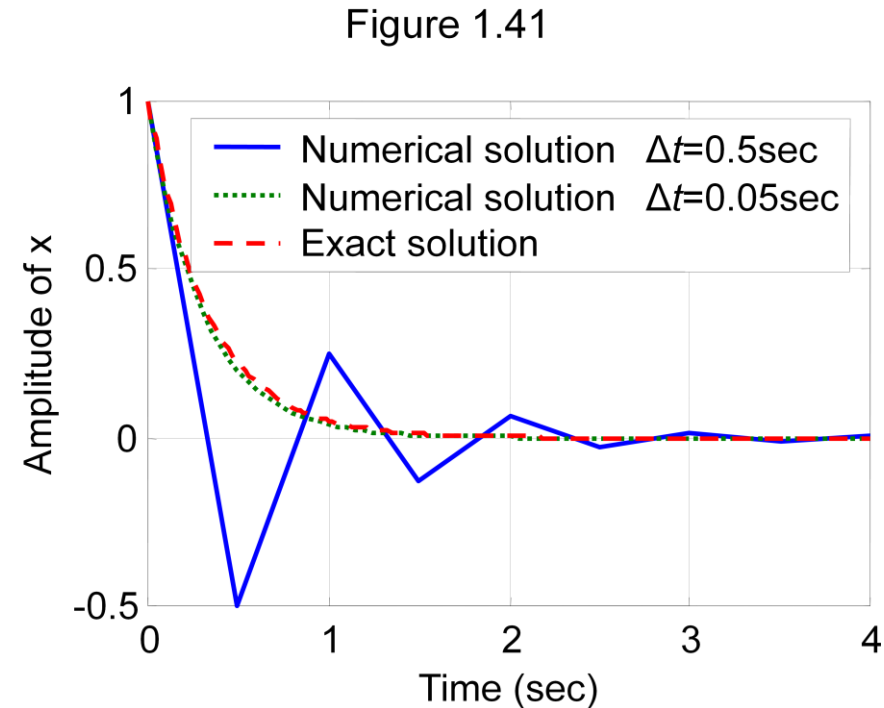
$$x(0) = x_0 = 1 = Ae^0 \Rightarrow A = 1 \text{ so that}$$

$$\underline{x(t) = e^{-3t}}$$

### **Analytical solution**

## Time step

- With time step at **0.5 sec** the numerical solution oscillates about the exact solution
- Large errors can be caused by choosing the time step to be too small
- Small time steps require more computation



## Numerical solution of the 2nd order equation of vibration:

It is necessary to convert the second order equation into two first order equations. To achieve this two new variables  $x_1$  and  $x_2$  are defined as follows.

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\text{Let } x_1 = x, \quad x_2 = \dot{x}$$

From this two first order differential equations can be written.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{c}{m}x_2 - \frac{k}{m}x_1 \text{ Called state space}$$

**The matrix form is**

**Combining these first order DEs in matrix form gives.**

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}}$$

State  
vector

State  
matrix

**The Euler numerical method can then be applied to the matrix form to give.**

$$\mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + \Delta t A \mathbf{x}(t_i)$$
$$\mathbf{x}_{i+1} = [I + \Delta t A] \mathbf{x}_i$$

## Matlab Solutions 'ode23' and 'ode45'

- **Use Runge-Kutta. More sophisticated than the Euler method but more accurate**
- **Often picks  $\Delta t$  (i.e. if solution  $x(t)$  is rapidly changing  $\Delta t$  is chosen to be small and visa-versa**
- **Works for nonlinear equations too**

### Create Matlab function

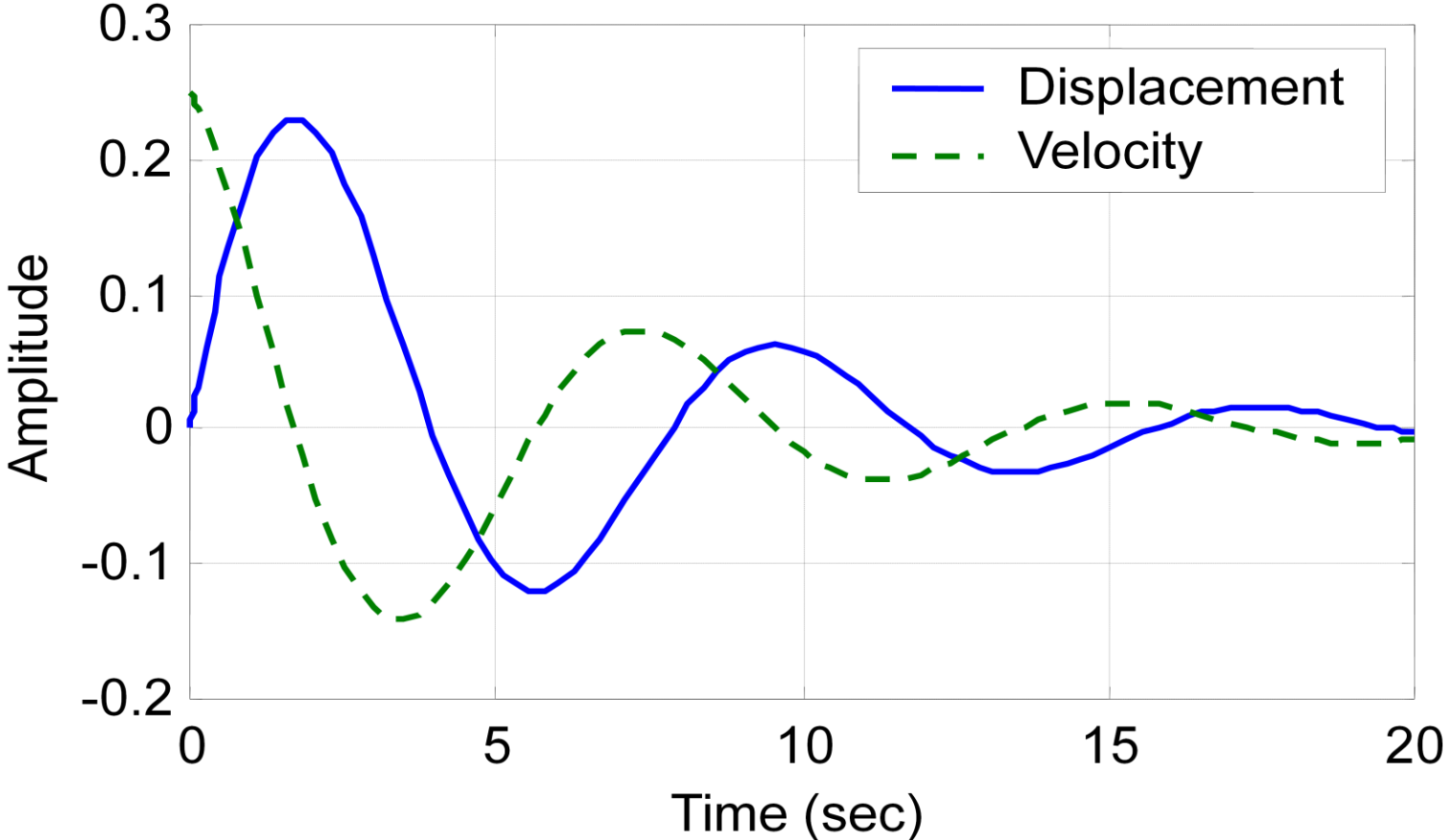
```
function xdot=sdof(t,x)
k=2;c=1;m=3;
A=[0 1;-k/m -c/m];
xdot=A*x;
```

### In the command window

```
» t0=0;tf=20;
» x0=[0 ; 0.25];
» [t,x]=ode45('sdof',[t0 tf],x0);
» plot(t,x)
```

# Resulting solution

Figure 1.42





## Why use numerical simulation when we can compute the analytical solution and plot it?

- **To have a tool that we are confident with that will allow us to solve for the response when an analytical solution cannot be found**
- **Nonlinear systems do not have analytical solutions, but can be simulated numerically**

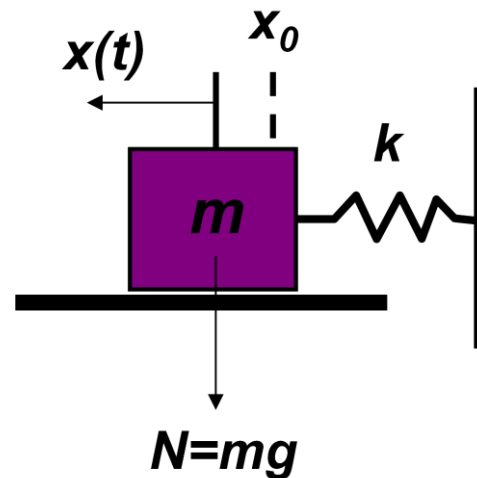
## **Section 1.10 Coulomb Friction and the Pendulum**

Nonlinear phenomenon in vibration analysis

## Vibration of Nonlinear Systems

Sliding or Coulomb Friction

$$f_c(\dot{x}) = \begin{cases} -\mu N & \dot{x}(t) > 0 \\ 0 & \dot{x}(t) = 0 \\ \mu N & \dot{x}(t) < 0 \end{cases}$$



**The force due to Coulomb friction opposes motion, hence the ‘sgn’ function is used. The force is proportional to the normal force and independent of the velocity of the mass.**

# The free body diagram split depending on the direction of motion:

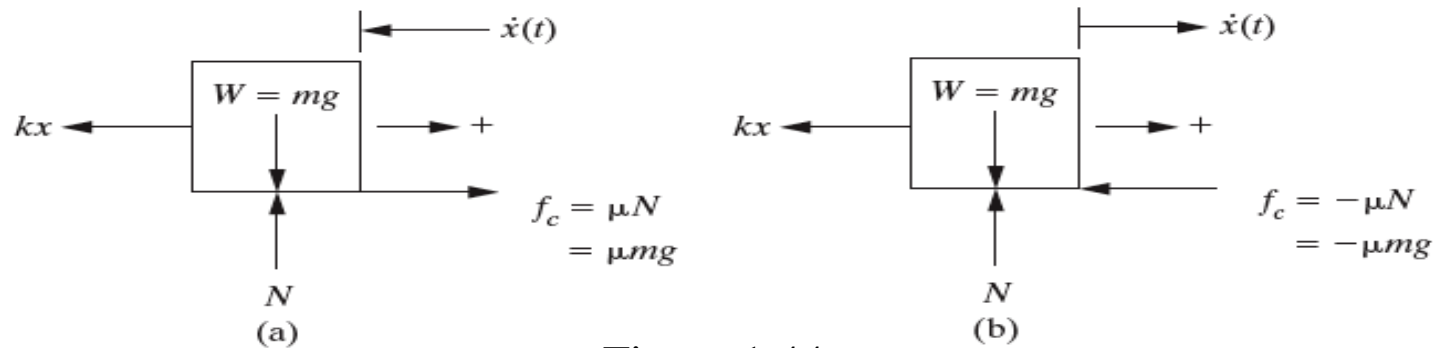


Figure 1.44

mass moving right

$$\dot{x}(t) > 0$$

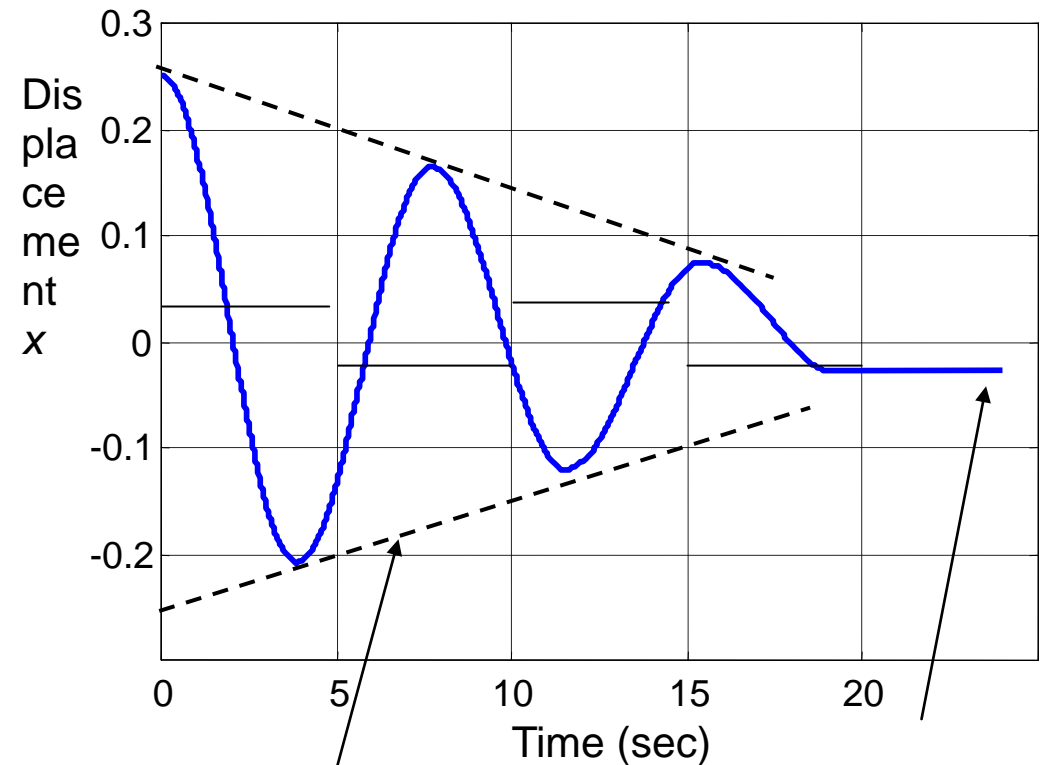
mass moving left

$$\dot{x}(t) < 0$$

$$\Rightarrow m\ddot{x} + \mu mg \operatorname{sgn}(\dot{x}) + kx = 0 \quad (1.100)$$

## The 'sgn' function is nonlinear

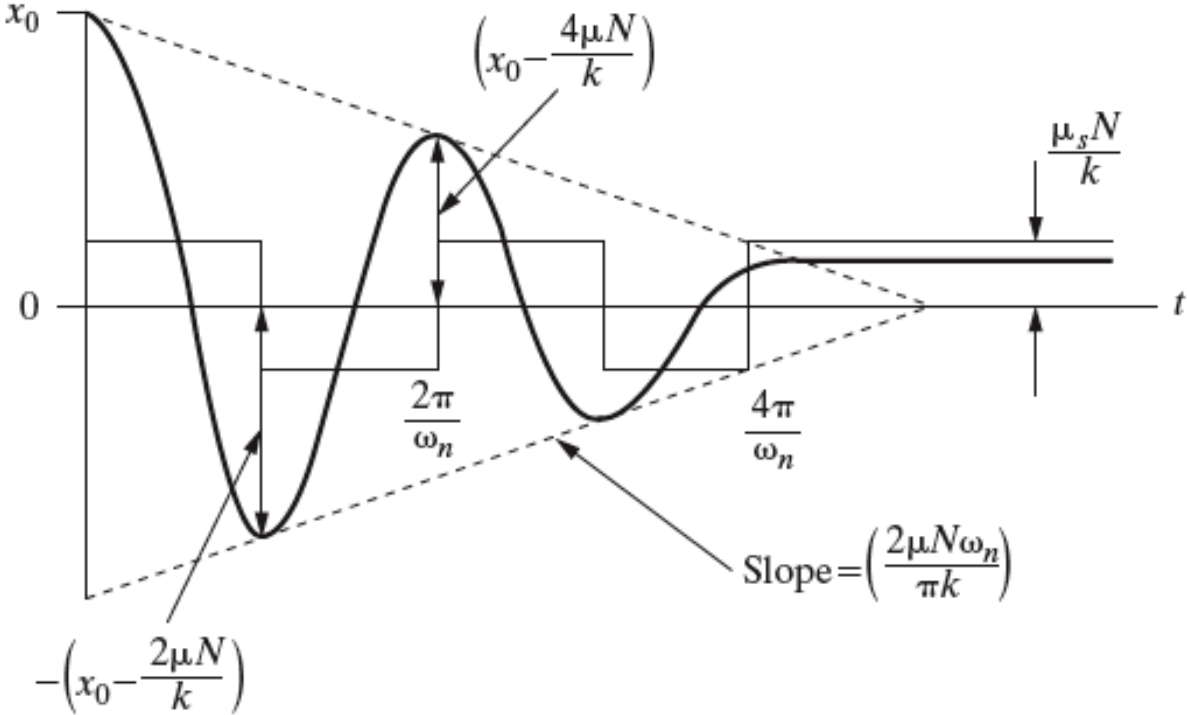
- Causes equation of motion to be nonlinear
- Can solve as piecewise linear (see text)
- Can solve numerically
- Has more than one equilibrium position
- Decay is linear rather than exponential
- Comes to rest when spring cannot overcome friction at the



Linear  
decay

Does not  
settle at  $x=0$

Figure 1.45 shows the details of the free response of a system with Coulomb damping



**A General second order system can be written as a single first order equation**

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -f(x_1, x_2) \end{bmatrix} = \mathbf{F}(\mathbf{x})$$

**The equilibrium position is defined:**

$$\mathbf{F}(\mathbf{x}_e) = 0$$

**For Coulomb friction this is defined as:**

$$x_2 = 0 \text{ and } -\frac{\mu_s mg}{k} < x_1 < \frac{\mu_s mg}{k}$$

**i.e. the positions where the force due to the spring can no longer overcome the sliding friction force**

## Example 1.10.2: Calculating the equilibrium position for nonlinear DEs

**Equation of motion:**

$$\ddot{x} + x - \beta^2 x^3 = 0$$

**State space form:**

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_1(\beta^2 x_1^2 - 1)$$

**Equilibrium positions:**

$$x_2 = 0$$

$$x_1(\beta^2 x_1^2 - 1) = 0 \Rightarrow$$

$$x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\beta} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{\beta} \\ 0 \end{bmatrix}$$

Multiple equilibrium positions possible



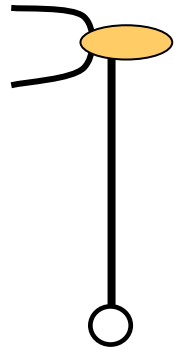


# The pendulum

**Stable**

**Equilibrium**

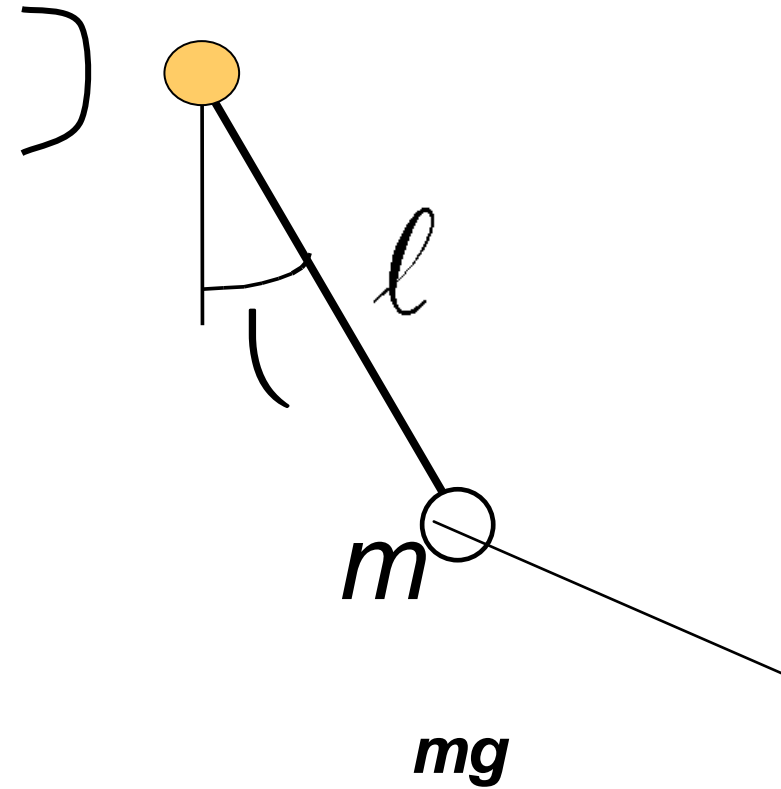
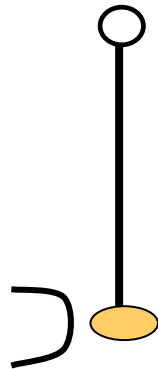
$$\theta = 0, 2\pi, 4\pi \dots$$



**Unstable**

**Equilibrium**

$$\theta = 0, \pi, 3\pi \dots$$



# Example 1.10.2 Equilibrium of a Pendulum

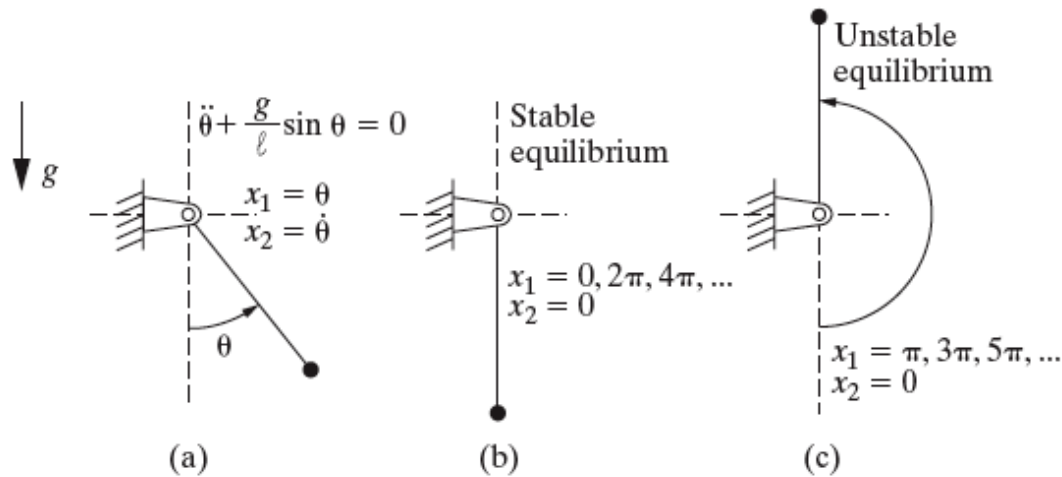


Figure 1.44

$$\ddot{\theta}(t) + \frac{g}{\ell} \sin \theta(t) = 0$$

$$x_1 = \theta, \quad x_2 = \dot{\theta}$$

$\Rightarrow$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{\ell} \sin x_1$$

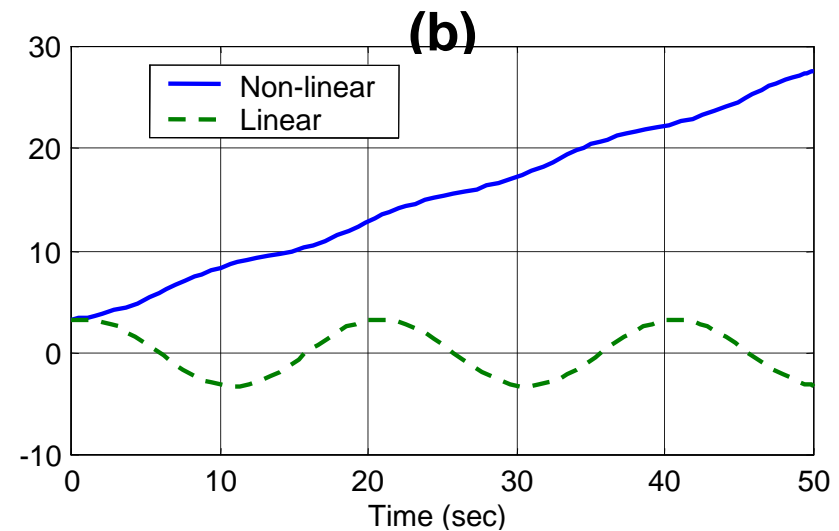
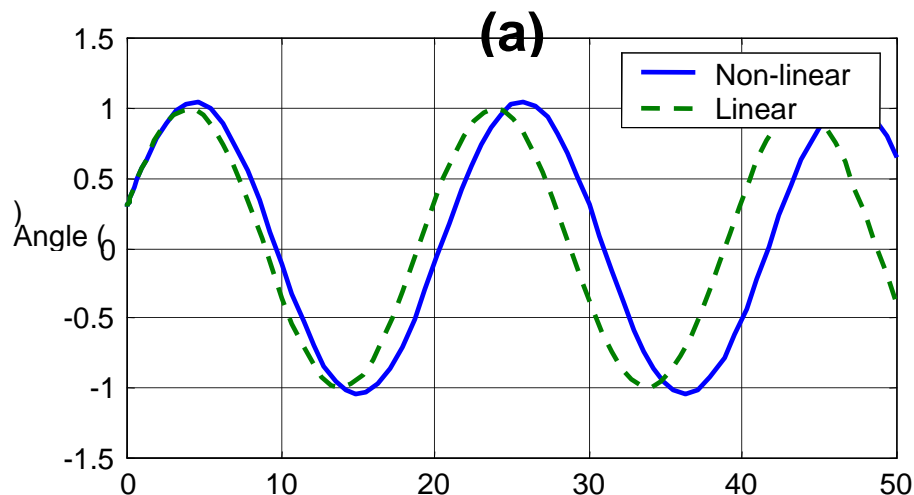
$$\Rightarrow \mathbf{F}(\mathbf{x}) = \begin{bmatrix} x_2 \\ -\frac{g}{\ell} \sin x_1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} x_2 = 0 \\ \sin x_1 = 0 \end{cases}$$

$$\Rightarrow x_2 = 0 \text{ and } x_1 = n\pi, \quad n = 0, 1, 2, \dots$$

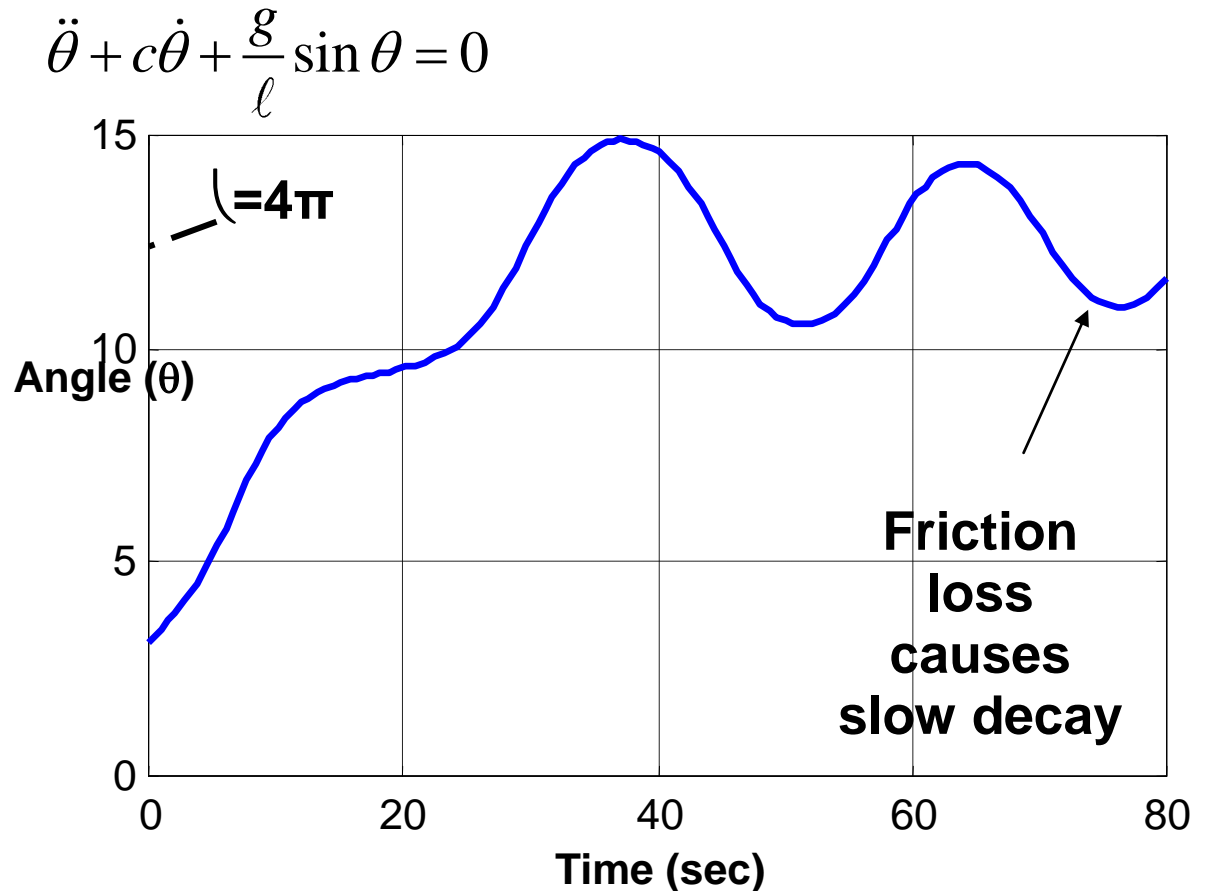
## Solution to the pendulum

- Can use numerical simulation to examine both linear and nonlinear response
- Let  $(g/L)=(0.1)^2$  so that  $\omega_n = 0.1$
- a) use  $\theta(0)=0.3$  rad & initial vel: 0.3 rad/s
- b) change the initial position to:  $\theta(0)=\pi$  rad which is near the unstable equilibrium



## Pendulum with friction added

After making a single loop the pendulum cannot make a second rotation and settles to the stable equilibrium position of  $\theta=4\pi$



# Summary of Nonlinear Vibrations

- **Additional phenomena over linear case**
- **Multiple equilibrium**
- **Instabilities possible with positive coefficients**
- **Form of response dependent on initial conditions**
- **Closed form solutions usually not available**
- **Can simulate numerically**
- **Linear model has tremendous advantages**
- **Linear combination of inputs yields linear combination of outputs**
- **Linear ode techniques very powerful**
- **But don't make a design error by ignoring important nonlinear situations**
- **All systems have nonlinear ranges of operation**
  - **Need to sort out when nonlinearity is important to consider and when to ignore it**