Chap. 3-2. Methods to design $F^{n}_{i+1/2}$: II. Flux Difference Splitting

+ **Higher order extension: *i* → *L*, *i+1* → *R* by monotonic interpolations**

– **Three-wave approximation with $\tilde{u}$ and $\tilde{c}$**

- No built-in mechanism to distinguish expansion shock and compression shock (violation of entropy condition)

$$\xrightarrow[\text{with a tunable parameter}]{\text{Entropy fix}} \left|\lambda_i^{'}\right| = \begin{cases} \left|\lambda_i\right| & \text{if } \left|\lambda_i\right| \ge \varepsilon \\ \dfrac{\lambda_i^2 + \varepsilon^2}{2\varepsilon} & \text{if } \left|\lambda_i\right| < \varepsilon \end{cases}$$

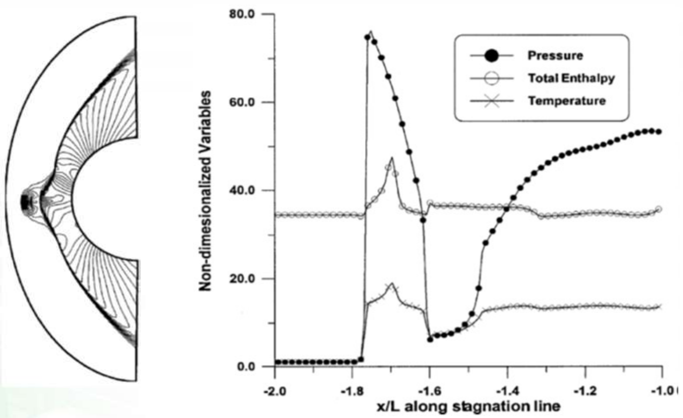
but more rigorous remedy is necessary.

- Do not satisfy the positivity condition (or failure of linearization)

  → problems in high-expansion region

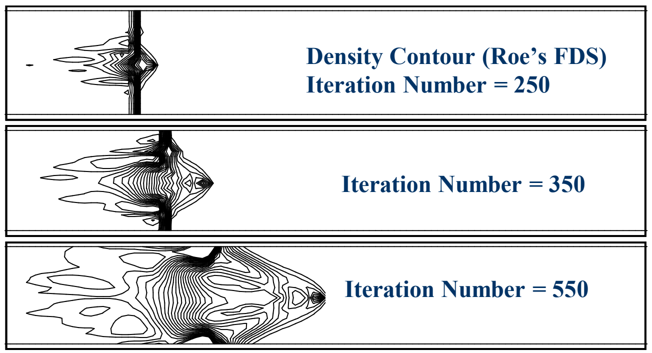
- Suffering from shock instability known as carbuncle phenomenon

Supersonic flow with *M = 8.0* around cylinder



Moving shock wave with M_s = *6.0*

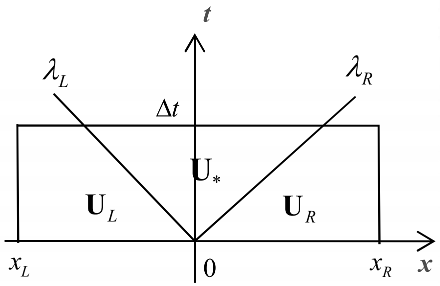
· Grid perturbation $(x_{i,j}, y_{i,j})$ with $y_{i,j} = \begin{cases} y_{j,mid} + 10^{-4} & \text{for } i \text{ even}, \\ y_{j,mid} - 10^{-4} & \text{for } i \text{ odd} \end{cases}$



Chap. 3-2. Methods to design $\mathbf{F}^n_{i+1/2}$: II. Flux Difference Splitting

- ***Two-wave Approximate Riemann Solver***
  - **A two-wave approximation to increase the robustness of Roe's approximate Riemann solver**
    - See the works by Harten, Lax and van Leer(1983), and others
    - Assume three Riemann-states divided by minimum and maximum wave (λ_L, λ_R) speeds and obtain a cell-interface flux by integrating conservation laws directly.

• Three Riemann-states (or two-wave) approximation



By integrating $\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}$ over $(x_L \sim x_R)\times(0\sim\Delta t)$,

$$\int_{x_L}^{x_R} \mathbf{U}(x,\Delta t)\,dx = \int_{x_L}^{x_R} \mathbf{U}(x,0)\,dx - \left(\int_0^{\Delta t} \mathbf{F}\left[\mathbf{U}(x_R,t)\right]dt - \int_0^{\Delta t} \mathbf{F}\left[\mathbf{U}(x_L,t)\right]dt\right)$$

Using cell-averaged values $(\mathbf{U}_L, \mathbf{U}_R)$ and interface fluxes $(\mathbf{F}_L, \mathbf{F}_R)$

$$\left.\begin{aligned}\int_{x_L}^{x_R} \mathbf{U}(x,\Delta t)\,dx &= x_R\mathbf{U}_R - x_L\mathbf{U}_L - \Delta t(\mathbf{F}_R - \mathbf{F}_L)\\ \int_{x_L}^{x_R} \mathbf{U}(x,\Delta t)\,dx &= \int_{\lambda_L\Delta t}^{\lambda_R\Delta t} \mathbf{U}(x,\Delta t)\,dx + (\lambda_L\Delta t - x_L)\mathbf{U}_L + (x_R - \lambda_R\Delta t)\mathbf{U}_R\end{aligned}\right\} \rightarrow \int_{\lambda_L\Delta t}^{\lambda_R\Delta t} \mathbf{U}(x,\Delta t)\,dx = \Delta t(\lambda_R\mathbf{U}_R - \lambda_L\mathbf{U}_L + \mathbf{F}_L - \mathbf{F}_R)$$

Thus, we can obtain the averaged-state in the star region: $\mathbf{U}_* = (\lambda_R\mathbf{U}_R - \lambda_L\mathbf{U}_L + \mathbf{F}_L - \mathbf{F}_R)/(\lambda_R - \lambda_L)$

• Flux at a cell-interface using $\mathbf{U}_*$

Integration over $(x_L \sim 0)\times(0\sim\Delta t)$ to obtain

$$-\lambda_L\Delta t(\mathbf{U}_* - \mathbf{U}_L) + \Delta t\left(\mathbf{F}_{0L} - \mathbf{F}_L\right) = 0 \rightarrow \mathbf{F}_{0L} = \mathbf{F}_L + \lambda_L(\mathbf{U}_* - \mathbf{U}_L)$$

Similarly, integration over $(0\sim x_R)\times(0\sim\Delta t)$ gives $\mathbf{F}_{0R} = \mathbf{F}_R + \lambda_R(\mathbf{U}_* - \mathbf{U}_R)$.

Chap. 3-2. Methods to design $\mathbf{F}_{i+1/2}^{n}$: II. Flux Difference Splitting

From conservative requirement $(\mathbf{F}_{0L} = \mathbf{F}_{0R} \equiv \mathbf{F}_{HLL})$, we have

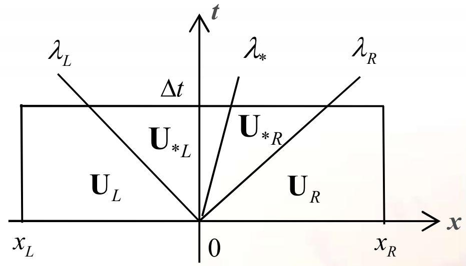
$$\mathbf{F}_{HLL} = \frac{\lambda_R \mathbf{F}_L - \lambda_L \mathbf{F}_R + \lambda_L \lambda_R (\mathbf{U}_R - \mathbf{U}_L)}{\lambda_R - \lambda_L}.$$

Thus, the final flux form is given by

$$\mathbf{F}_{i+1/2} = \begin{cases} \mathbf{F}_L & \text{if } 0 \le \lambda_L, \\ \mathbf{F}_{HLL} & \text{if } \lambda_L \le 0 \le \lambda_R, \\ \mathbf{F}_R & \text{if } \lambda_R \le 0. \end{cases}$$

- Thee-state approximation improves robustness (ex: shock stability and positivity condition) significantly but contact discontinuity cannot be captured accurately.
  - Estimation of (λ_L, λ_R) crucial to determine the accuracy and robustness: $\lambda_L = \min(0, u_i - c_i, \tilde{u} - \tilde{c})$, $\lambda_R = \max(0, u_{i+1} + c_{i+1}, \tilde{u} + \tilde{c})$ (Einfeldt *et al.*(1991))

- **Modified HLL scheme**
  - Four Riemann-states by adding contact discontinuity into HLL scheme (HLLC scheme)
    - See the works by Toro *et al.*(1994)
  - Four Riemann-states approximation



Assuming that $\int_{\lambda_L \Delta t}^{\lambda_R \Delta t} \mathbf{U}(x,\Delta t)\,dx = \int_{\lambda_L \Delta t}^{\lambda_* \Delta t} \mathbf{U}(x,\Delta t)\,dx + \int_{\lambda_* \Delta t}^{\lambda_R \Delta t} \mathbf{U}(x,\Delta t)\,dx$,

$$(\lambda_R - \lambda_L)\Delta t \mathbf{U}_* = (\lambda_* - \lambda_L)\Delta t \mathbf{U}_{*L} + (\lambda_R - \lambda_*)\Delta t \mathbf{U}_{*R}$$

$\mathbf{U}_* = \dfrac{\lambda_* - \lambda_L}{\lambda_R - \lambda_L}\mathbf{U}_{*L} + \dfrac{\lambda_R - \lambda_*}{\lambda_R - \lambda_L}\mathbf{U}_{*R}$. From the Rankine-Hugoniot relation, $[\mathbf{F}] = S[\mathbf{U}]$, across $\lambda_{L/*/R}$,

$\mathbf{F}_{*L} = \mathbf{F}_L + \lambda_L(\mathbf{U}_{*L} - \mathbf{U}_L)$, $\mathbf{F}_{*R} = \mathbf{F}_{*L} + \lambda_*(\mathbf{U}_{*R} - \mathbf{U}_{*L})$ and $\mathbf{F}_{*R} = \mathbf{F}_R - \lambda_R(\mathbf{U}_R - \mathbf{U}_{*R})$.

Chap. 3-2. Methods to design $\mathbf{F}_{i+1/2}^{n}$: II. Flux Difference Splitting

By imposing $u_{*L} = u_{*R} = u_*$, $p_{*L} = p_{*R} = p_*$ and $\lambda_* = u_*$ on $\lambda_L \mathbf{U}_{*L} - \mathbf{F}_{*L} = \lambda_L \mathbf{U}_L - \mathbf{F}_L$, $\lambda_R \mathbf{U}_{*R} - \mathbf{F}_{*R} = \lambda_R \mathbf{U}_R - \mathbf{F}_R$, we have

$$\mathbf{U}_{*L/*R} = \rho_{L/R} \frac{(\lambda_{L/R} - u_{L/R})}{(\lambda_{L/R} - \lambda_*)} \begin{bmatrix} 1 \\ \lambda_* \\ E_{L/R}/\rho_{L/R} + (\lambda_* - u_{L/R})\{\lambda_* + p_{L/R}/[\rho_{L/R}(\lambda_{L/R} - u_{L/R})]\} \end{bmatrix}.$$

The final flux form is then given by

$$\mathbf{F}_{i+1/2} = \begin{cases} \mathbf{F}_L & \text{if } 0 \le \lambda_L, \\ \mathbf{F}_{*L} = \mathbf{F}_L + \lambda_L(\mathbf{U}_{*L} - \mathbf{U}_L) & \text{if } \lambda_L \le 0 \le \lambda_*, \\ \mathbf{F}_{*R} = \mathbf{F}_R + \lambda_R(\mathbf{U}_{*R} - \mathbf{U}_R) & \text{if } \lambda_* \le 0 \le \lambda_R, \\ \mathbf{F}_R & \text{if } \lambda_R \le 0. \end{cases}$$

- Estimation of λ_L, λ_R and λ_*

  From $\lambda_L \mathbf{U}_{*L} - \mathbf{F}_{*L} = \lambda_L \mathbf{U}_L - \mathbf{F}_L$, $\lambda_R \mathbf{U}_{*R} - \mathbf{F}_{*R} = \lambda_R \mathbf{U}_R - \mathbf{F}_R$, we have

  $$p_{*L} = p_L + \rho_L(\lambda_L - u_L)(\lambda_* - u_L), \quad p_{*R} = p_R + \rho_R(\lambda_R - u_R)(\lambda_* - u_R).$$

  Also from $p_{*L} = p_{*R}$, $\lambda_* = \dfrac{p_R - p_L + \rho_L u_L(\lambda_L - u_L) - \rho_R u_R(\lambda_R - u_R)}{\rho_L(\lambda_L - u_L) - \rho_R(\lambda_R - u_R)}$.

Chap. 3-2. Methods to design $\mathbf{F}^n_{i+1/2}$: II. Flux Difference Splitting

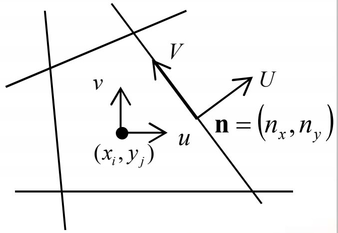
- ***RoeM***
  - **Cure the shock instability of Roe's FDS while maintaining the accuracy**
    - See the work by Kim *et al.*(2003), and others
    - Introduce a multi-dimensional dissipation term controlled by Mach number-based weighting functions to cure shock instability
  - **Behavior of numerical mass flux and its connection to shock instability**

$$\mathbf{F}_{i+1/2,j} = 0.5\left(\mathbf{F}_{i,j} + \mathbf{F}_{i+1,j}\right) - \mathbf{D}_{i+1/2,j}$$

$$\mathbf{D}_{i+1/2,j\_Roe} = 0.5\left|\tilde{A}_{i+1/2,j}\right|\Delta\mathbf{U}_{i+1/2,j} = 0.5\left[\hat{M}\Delta\mathbf{F}_{i+1/2,j} - \tilde{c}\left(\hat{M}^2 - 1\right)\Delta\mathbf{U}_{i+1/2,j} - \tilde{c}\left(1 - \left|\hat{M}\right|\right)\mathbf{B}\Delta\mathbf{U}_{i+1/2,j}\right]$$

$$\mathbf{D}_{i+1/2,j\_HLL} = 0.5\left[\hat{M}\Delta\mathbf{F}_{i+1/2,j} - \tilde{c}\left(\hat{M}^2 - 1\right)\Delta\mathbf{U}_{i+1/2,j}\right] \quad \text{with } \hat{M} = sign\left(\tilde{M}\right)\times\min\left(1, \left|\tilde{M}\right|\right),$$

$$\tilde{M} = \tilde{U}/\tilde{c}, \text{ and } \mathbf{B}\Delta\mathbf{U}_{i+1/2,j} = \left(\Delta\rho - \frac{\Delta p}{\tilde{c}^2}\right)\begin{bmatrix} 1 \\ \tilde{u} \\ \tilde{v} \\ 0.5(\tilde{u}^2 + \tilde{v}^2) \end{bmatrix} + \tilde{\rho}\begin{bmatrix} 0 \\ \Delta u - n_x\Delta U \\ \Delta v - n_y\Delta U \\ \tilde{u}\Delta u + \tilde{v}\Delta v - \tilde{U}\Delta U \end{bmatrix}.$$



Compare the mass flux of $\mathbf{F}^{(\rho)}_{i+1/2} = 0.5\left[(\rho U)_i + (\rho U)_{i+1}\right] - 0.5\left[D^{(\rho)}\Delta\rho + D^{(U)}\Delta U + \frac{D^{(p)}}{\tilde{c}^2}\Delta p\right],$

- Roe's FDS: $D^{(\rho)}_{Roe} = \tilde{c}\left|\tilde{M}\right|,\ D^{(P)}_{Roe} = \tilde{c}\left(1 - \left|\tilde{M}\right|\right) \rightarrow$ shock instability but exact capturing of CD
- HLL: $D^{(\rho)}_{HLL} = \tilde{c},\ D^{(P)}_{HLL} = 0 \rightarrow$ shock stability but no exact capturing of CD

Chap. 3-2. Methods to design $\mathbf{F}^n_{i+1/2}$: II. Flux Difference Splitting

- Linear stability analysis with $\begin{cases} \rho_i^n = \rho + \tilde{\rho}^n,\ p_i^n = p + \tilde{p}^n,\ u_i^n = u^0 = 0,\ v_i^n = v^0 = 0, & \text{if } i \text{ is even} \\ \rho_i^n = \rho - \tilde{\rho}^n,\ p_i^n = p - \tilde{p}^n,\ u_i^n = u^0 = 0,\ v_i^n = v^0 = 0, & \text{if } i \text{ is odd} \end{cases}$

  • HLL: $\tilde{\rho}^{n+1} = \left(1 - 2\nu_y\right)\tilde{\rho}^n,\ \ \tilde{p}^{n+1} = \left(1 - 2\nu_y\right)\tilde{p}^n$ with $\nu_y = \dfrac{c\Delta t}{\Delta y}$

  → pressure field and density field are not coupled and simultaneously damped out.

  → no shock instability

  • Roe's FDS: $\tilde{\rho}^{n+1} = \tilde{\rho}^n - \dfrac{2\nu_y}{\tilde{c}^2}\tilde{p}^n,\ \ \tilde{p}^{n+1} = \left(1 - 2\nu_y\right)\tilde{p}^n$

  → pressure field and density field are coupled, and they are out-of-phase.

  → pressure perturbation feeding into density field to amplify density perturbation

  → shock instability

- **Mach-number-based weighting functions *f* and *g* to control the feeding rate of pressure field and the damping rate of density field.**

  • $D_{Roe}^{(P)} = \tilde{c}\left(1 - \left|\tilde{M}\right|\right) \rightarrow \tilde{c}f\left(1 - \left|\tilde{M}\right|\right)$: $\tilde{\rho}^{n+1} = \tilde{\rho}^n - \dfrac{2\nu_y}{\tilde{c}^2} f\tilde{p}^n,\ \ \tilde{p}^{n+1} = \left(1 - 2\nu_y f\right)\tilde{p}^n$

  Additional damping to deal with strong pressure field perturbation: $\mathbf{B}\Delta\mathbf{U}_{i+1/2} \rightarrow g\mathbf{B}\Delta\mathbf{U}_{i+1/2}$

$$\tilde{\rho}^{n+1} = \left(1 - 2\nu_y\left(1 - g\right)\right)\tilde{\rho}^n - \frac{2\nu_y}{c^2} f\tilde{p}^n,\ \ \tilde{p}^{n+1} = \left(1 - 2\nu_y f\right)\tilde{p}^n + 2\nu_y\left(\gamma - 1\right)\left[\frac{\tilde{u}^2 + \tilde{v}^2}{2\tilde{c}^2}\right](1 - g)f\tilde{p}^n$$

- **Numerical flux at a cell-interface**

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{b_1 \times \mathbf{F}_i - b_2 \times \mathbf{F}_{i+1}}{b_1 - b_2} + \frac{b_1 \times b_2}{b_1 - b_2}\Delta\mathbf{U}^* - \frac{b_1 \times b_2}{b_1 - b_2} \times \frac{1}{1 + |\tilde{M}|}\mathbf{B}\Delta\mathbf{U}$$

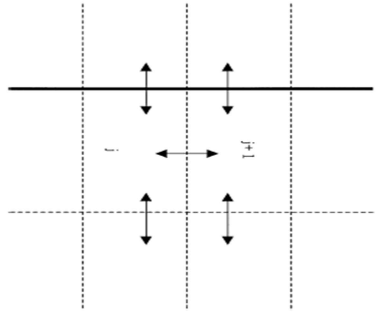
Chap. 3-2. Methods to design $\mathbf{F}^n_{i+1/2}$: II. Flux Difference Splitting

$$\text{with } \Delta\mathbf{U}^* = \begin{bmatrix} \Delta\rho \\ \Delta(\rho u) \\ \Delta(\rho v) \\ \Delta(\rho H) \end{bmatrix}, \quad \mathbf{B}\Delta\mathbf{U}_{i+1/2} = g\left(\Delta\rho - f\frac{\Delta p}{\tilde{c}^2}\right)\begin{bmatrix} 1 \\ \tilde{u} \\ \tilde{v} \\ \tilde{H} \end{bmatrix} + \tilde{\rho}\begin{bmatrix} 0 \\ \Delta u - n_x\Delta U \\ \Delta v - n_y\Delta U \\ \Delta H \end{bmatrix}$$

- Design of f and g

$$- f = \begin{cases} 1 & \tilde{u}^2 + \tilde{v}^2 = 0 \\ \left|\tilde{M}\right|^h & \text{elsewhere} \end{cases}, \quad h = 1 - \min\left(P_{i+1/2,j}, P_{i,j+1/2}, P_{i,j-1/2}, P_{i+1,j+1/2}, P_{i+1,j-1/2}\right)$$

$$\text{with } P_{i+1/2,j} = \min\left(\frac{p_{i,j}}{p_{i+1,j}}, \frac{p_{i+1,j}}{p_{i,j}}\right)$$



$$- g = \begin{cases} \left|\tilde{M}\right|^{1-P_{i+1/2,j}} & \tilde{M} \neq 0, \\ 1 & \tilde{M} = 0. \end{cases}$$

- The maximum and minimum wave speeds (b_1, b_2) are the same as (λ_L, λ_R).

- **Characteristics of RoeM**
  - Total enthalpy preservation for inviscid steady flow (by formulation based on *H*)
  - Cure of shock instability/carbuncle phenomenon (by *f, g*).
  - Exclusion of expansion shock, stability of expansion (by b_1, b_2)
  - Exact capturing of SW and CD → good for N-S Computations (by *f, g*)

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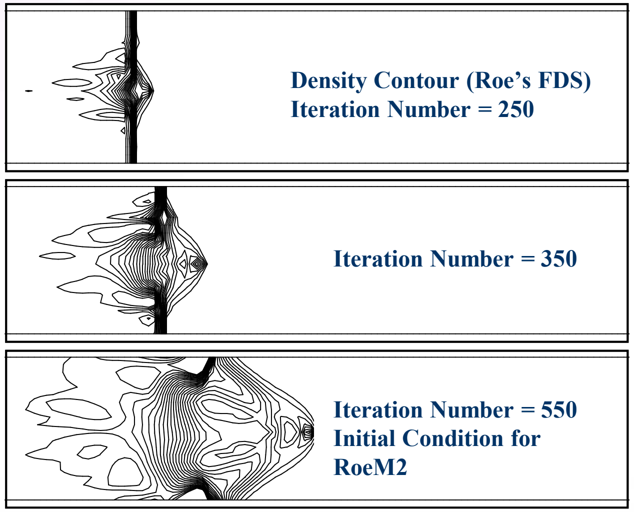
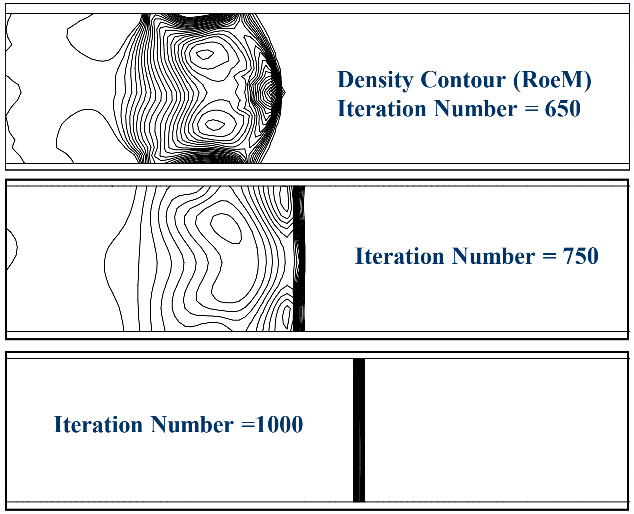
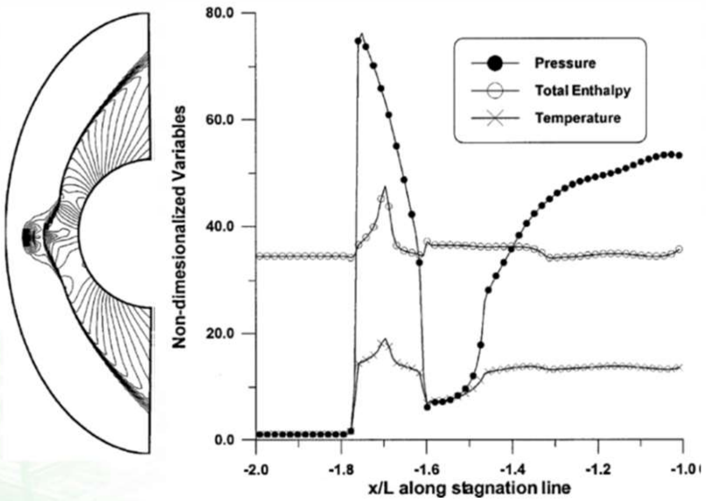


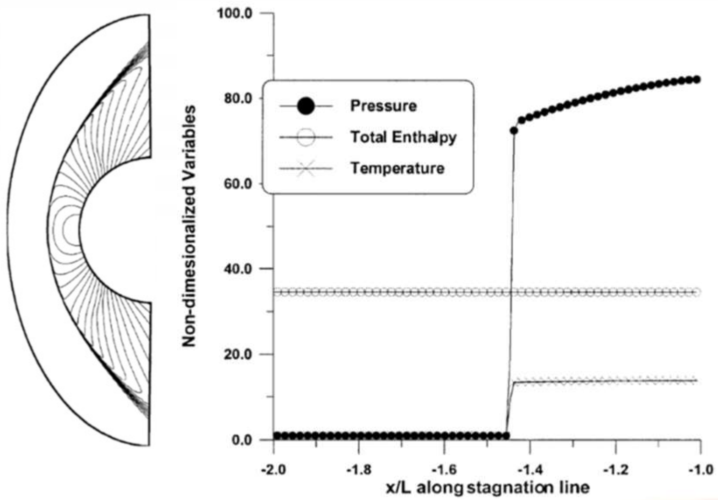
Roe's FDS



RoeM Scheme

Quirk Test





Supersonic flow with *M=8* around cylinder