Chapter 4. Discretization of the 2-D Euler Eqns.

Chap. 4-1. Conservative Form of 2-D Euler Equations

- ***Conservative Form of Inviscid Governing Eqns***
  - **Mass conservation**

$$\frac{\partial}{\partial t}\int_{V_i}\rho dV+\int_{\partial V_i}\rho\mathbf{V}\cdot\mathbf{n}dS=0 \;\rightarrow\; \frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\mathbf{V}\right)=0$$

  - **Momentum conservation**

$$\frac{\partial}{\partial t}\int_{V_i}\rho\mathbf{V}dV+\int_{\partial V_i}\rho\mathbf{V}\left(\mathbf{V}\cdot\mathbf{n}\right)dS+\int_{\partial V_i}p\mathbf{n}dS=\mathbf{0} \;\rightarrow\; \frac{\partial}{\partial t}\left(\rho\mathbf{V}\right)+\nabla\cdot\left(\rho\mathbf{V}\mathbf{V}+p\mathbf{I}\right)=\mathbf{0}$$

  - **Energy conservation**

$$\frac{\partial}{\partial t}\int_{V_i}\rho EdV+\int_{\partial V_i}\rho E\mathbf{V}\cdot ndS+\int_{\partial V_i}p\mathbf{V}\cdot ndS=0 \;\rightarrow\; \frac{\partial}{\partial t}\left(\rho E\right)+\nabla\cdot\left(\rho\mathbf{V}H\right)=0$$

  - **Conservative differential form in *(x, y)* coordinate**

$$\frac{\partial\mathbf{U}}{\partial t}+\frac{\partial\mathbf{F}(\mathbf{U})}{\partial x}+\frac{\partial\mathbf{G}(\mathbf{U})}{\partial y}=\mathbf{0},\ \text{with }\mathbf{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho E\end{bmatrix},\ \mathbf{F}\left(\mathbf{U}\right)=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uH\end{bmatrix},\ \text{and }\mathbf{G}\left(\mathbf{U}\right)=\begin{bmatrix}\rho v\\ \rho vu\\ \rho v^2+p\\ \rho vH\end{bmatrix} \quad \text{(Eq.1)}$$

where $E=e+\frac{u^2+v^2}{2},\quad H=h+\frac{u^2+v^2}{2},\quad p=\left(\gamma-1\right)\rho\left[E-\frac{u^2+v^2}{2}\right]$

Chap. 4-1. Conservative Form of 2-D Euler Equations

- System is closed by eqn. of state for calorically perfect gas.

  $e = c_v T,\ \ h = e + \dfrac{p}{\rho} = c_p T,\ \ \gamma = \dfrac{c_p}{c_v} = 1.4,\ \ R = c_p - c_v,$ and $c = \sqrt{\gamma p / \rho} = \sqrt{\gamma R T}$

- **Hyperbolicity and non-dimensionalized governing eqns.**
  - $A(= \dfrac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}})$ and $B(= \dfrac{\partial \mathbf{G}(\mathbf{U})}{\partial \mathbf{U}})$ have real eigenvalues $(u-c, u, u, u+c)$, $(v-c, v, v, v+c)$ with linearly independent eigenvectors. $\rightarrow$ hyperbolic with respect to (x,t) and (y,t)

• From $\det(A - \lambda I) = 0$ with $A \equiv \dfrac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -u^2 + (\gamma-1)|\mathbf{V}|^2/2 & (3-\gamma)u & (1-\gamma)v & \gamma - 1 \\ -uv & v & u & 0 \\ u[(\gamma-1)|\mathbf{V}|^2/2 - H] & H - (\gamma-1)u^2 & (1-\gamma)uv & \gamma u \end{bmatrix},$

And from $A\mathbf{r}_i = \lambda_i \mathbf{r}_i$ and $\mathbf{l}_i^T A = \lambda_i \mathbf{l}_i^T$ with $\mathbf{l}_i^T \cdot \mathbf{r}_j = \delta_{ij}$, $\lambda_{1,2,3,4} = u-c, u, u, u+c$, we have

$$R = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4] = \begin{bmatrix} 1 & 1 & 0 & 1 \\ u-c & u & 0 & 1+c \\ v & v & 1 & v \\ H - uc & |\mathbf{V}|^2/2 & v & H + uc \end{bmatrix},$$

$$R^{-1} = [\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4]^T = \frac{\gamma-1}{2c^2} \begin{bmatrix} H + c(u-c)/(\gamma-1) & -u - c/(\gamma-1) & -v & 1 \\ 4c^2/(\gamma-1) - 2H & 2u & 2v & -2 \\ -2c^2 v/(\gamma-1) & 0 & 2c^2 v/(\gamma-1) & 0 \\ H - c(u+c)/(\gamma-1) & -u + c/(\gamma-1) & -v & 1 \end{bmatrix} \rightarrow R^{-1} A R = \Lambda = [\lambda_i]$$

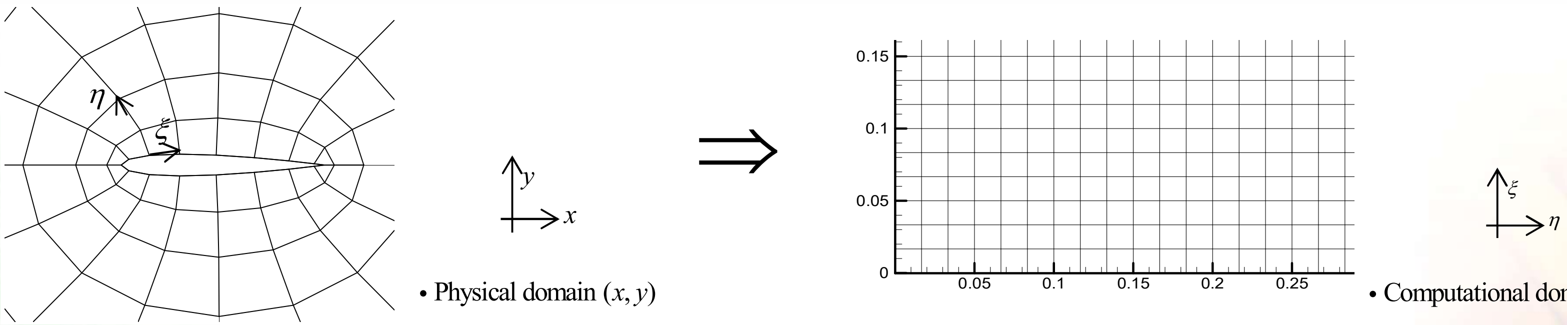
Chap. 4-1. Conservative Form of 2-D Euler Equations

- **Hyperbolicity and non-dimensionalized governing eqns. (cont'd)**
  - Non-dimensionalized governing eqns. for dynamic similarity

$$\text{From } x_i^* = \frac{x_i}{L},\ t^* = \frac{t}{L/V_\infty},\ u_i^* = \frac{u_i}{V_\infty},\ \rho^* = \frac{\rho}{\rho_\infty},\ p^* = \frac{p}{\rho_\infty V_\infty^2},\ \text{and } e^* = \frac{e}{V_\infty^2},$$

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{0} \rightarrow \frac{\partial \mathbf{U}^*}{\partial t^*} + \frac{\partial \mathbf{F}^*}{\partial x^*} + \frac{\partial \mathbf{G}^*}{\partial y^*} = \frac{\partial \mathbf{U}^*}{\partial t^*} + A\frac{\partial \mathbf{U}^*}{\partial x^*} + B\frac{\partial \mathbf{U}^*}{\partial y^*} = \mathbf{0},\quad A = \frac{\partial \mathbf{F}^*}{\partial \mathbf{U}^*} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}},\quad B = \frac{\partial \mathbf{G}^*}{\partial \mathbf{U}^*} = \frac{\partial \mathbf{G}}{\partial \mathbf{U}}$$

- ***Finite Difference Approximation***
  - **Transformation of governing eqns. from physical domain *(x, y)* to computational domain *(ξ, η)***
    - Differential form of governing eqns. is always kept.
    - A set of well-ordered grid points *(i, j)* over target geometry and domain for computation (or body-fitted curvelinear coordinates around target geometry) is always assumed.
      - From body-fitted stretched grid points in *(x, y)* to uniform rectangular grid points in *(ξ, η)*



• Physical domain (x, y)

• Computational domain (ξ, η)

Chap. 4-2. Finite Difference Approximation

- ***Finite Difference Approximation*** **(Cont'd)**
  - **Transformed governing eqns. are approximated by finite-difference formula in a point-wise manner.**
    - Without introducing computational cell and cell-averaged value
  - **For 2-D case with *(x, y, t)* → *(ξ, η, χ)***

• From $\chi = t,\ \xi = \xi(x, y, t)$ and $\eta = \eta(x, y, t)$,

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \chi} + \xi_t \frac{\partial}{\partial \xi} + \eta_t \frac{\partial}{\partial \eta},\quad \frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta},\quad \frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta} \qquad \text{(Eq.2)}$$

with a set of metric coefficients $(\xi_t, \eta_t, \xi_x, \eta_x, \xi_y, \eta_y$ or $x_t, y_t, x_\xi, y_\xi, x_\eta, y_\eta)$.

Incremental change in terms of metric coefficients is given by

$$\begin{bmatrix} d\chi \\ d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \xi_t & \xi_x & \xi_y \\ \eta_t & \eta_x & \eta_y \end{bmatrix} \begin{bmatrix} dt \\ dx \\ dy \end{bmatrix} \text{ and } \begin{bmatrix} dt \\ dx \\ dy \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x_t & x_\xi & x_\eta \\ y_t & y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} d\chi \\ d\xi \\ d\eta \end{bmatrix}.$$

• For transformation with $(x, y) \to (\xi, \eta)$,

define Jacobian of transformation, $J = \dfrac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x \eta_y - \xi_y \eta_x = \dfrac{\partial(x,y)}{\partial(\xi,\eta)}^{-1} = (x_\xi y_\eta - x_\eta y_\xi)^{-1}$

- $\int_\Omega f(x,y)\,dxdy = \int_\Omega f(x(\xi,\eta), y(\xi,\eta)) J^{-1} d\xi d\eta$, and $\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} = I$,

- $\xi_x = J y_\eta,\ \xi_y = -J x_\eta,\ \eta_x = -J y_\xi,\ \eta_y = J x_\xi$

- J simply indicates area ratio of computational cell to physical cell $\left(\Delta\xi\Delta\eta\big|_{(\xi_0,\eta_0)} = J_{(x_0,y_0)}\,\Delta x \Delta y\big|_{(x_0,y_0)}\right)$.

Chap. 4-2. Finite Difference Approximation

- Plugging $\frac{1}{J}(Eq.(2))$ into $Eq.(1)$ and taking differentiation by part gives

$$\frac{1}{J}\left[\frac{\partial \mathbf{U}}{\partial t}+\xi_t\frac{\partial \mathbf{U}}{\partial \xi}+\eta_t\frac{\partial \mathbf{U}}{\partial \eta}\right]+\frac{1}{J}\left[\xi_x\frac{\partial \mathbf{F}}{\partial \xi}+\eta_x\frac{\partial \mathbf{F}}{\partial \eta}\right]+\frac{1}{J}\left[\xi_y\frac{\partial \mathbf{G}}{\partial \xi}+\eta_y\frac{\partial \mathbf{G}}{\partial \eta}\right]=\mathbf{0}$$

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{U}}{J}\right)+\frac{\partial}{\partial \xi}\left(\xi_t\frac{\mathbf{U}}{J}\right)+\frac{\partial}{\partial \eta}\left(\eta_t\frac{\mathbf{U}}{J}\right)-\mathbf{U}\underline{\left[\frac{\partial}{\partial t}\left(\frac{1}{J}\right)+\frac{\partial}{\partial \xi}\left(\frac{\xi_t}{J}\right)+\frac{\partial}{\partial \eta}\left(\frac{\eta_t}{J}\right)\right]}$$

$$+\frac{\partial}{\partial \xi}\left(\xi_x\frac{\mathbf{F}}{J}\right)+\frac{\partial}{\partial \eta}\left(\eta_x\frac{\mathbf{F}}{J}\right)-\mathbf{F}\underline{\left[\frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right)+\frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right)\right]}$$

$$+\frac{\partial}{\partial \xi}\left(\xi_y\frac{\mathbf{G}}{J}\right)+\frac{\partial}{\partial \eta}\left(\eta_y\frac{\mathbf{G}}{J}\right)-\mathbf{G}\underline{\left[\frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right)+\frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right)\right]}=\mathbf{0}$$

Since each of three underlined terms vanishes identically,

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{U}}{J}\right)+\frac{\partial}{\partial \xi}\left[\frac{1}{J}\left(\xi_t\mathbf{U}+\xi_x\mathbf{F}+\xi_y\mathbf{G}\right)\right]+\frac{\partial}{\partial \eta}\left[\frac{1}{J}\left(\eta_t\mathbf{U}+\eta_x\mathbf{F}+\eta_y\mathbf{G}\right)\right]=\mathbf{0}$$

or $\frac{\partial \bar{\mathbf{U}}}{\partial t}+\frac{\partial \bar{\mathbf{F}}(\bar{\mathbf{U}})}{\partial \xi}+\frac{\partial \bar{\mathbf{G}}(\bar{\mathbf{U}})}{\partial \eta}=\mathbf{0}$ with $\bar{\mathbf{U}}=\frac{\mathbf{U}}{J}$, $\bar{\mathbf{F}}=\frac{\xi_t\mathbf{U}+\xi_x\mathbf{F}+\xi_y\mathbf{G}}{J}$, and $\bar{\mathbf{G}}=\frac{\eta_t\mathbf{U}+\eta_x\mathbf{F}+\eta_y\mathbf{G}}{J}$ (Eq.3)

$\left(cf:\ \frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}+\frac{\partial \mathbf{G}(\mathbf{U})}{\partial y}=\mathbf{0}\right)$

$\rightarrow$ conservation form does not change after non-singular coordinate transformation.

Chap. 4-2. Finite Difference Approximation

- **With no mesh motion with** $\xi = \xi(x,y),\ \eta = \eta(x,y)$

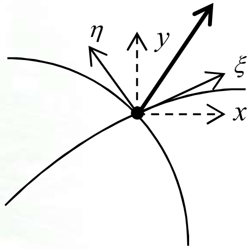
$$\frac{\partial \bar{\mathbf{U}}}{\partial t} + \frac{\partial \bar{\mathbf{F}}}{\partial \xi} + \frac{\partial \bar{\mathbf{G}}}{\partial \eta} = \mathbf{0},$$

where $\bar{\mathbf{U}} = \frac{1}{J}\mathbf{U} = \frac{1}{J}\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}$, $\bar{\mathbf{F}} = \frac{1}{J}\left(\xi_x \mathbf{F} + \xi_y \mathbf{G}\right) = \frac{1}{J}\begin{bmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ \rho U H \end{bmatrix}$,

and $\bar{\mathbf{G}} = \frac{1}{J}\left(\eta_x \mathbf{F} + \eta_y \mathbf{G}\right) = \frac{1}{J}\begin{bmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_y p \\ \rho V H \end{bmatrix}$ with $U = \xi_x u + \xi_y v,\ V = \eta_x u + \eta_y v.$

  - Contra-variant velocity (U, V): velocity component along the (ξ, η) direction or normal/ parallel to a cell-interface

$\mathbf{V} = (u,v)_{(x,y)} = u\hat{\mathbf{x}} + v\hat{\mathbf{y}}$ or $\mathbf{V} = (U,V)_{(\xi,\eta)} = U\hat{\boldsymbol{\xi}} + V\hat{\boldsymbol{\eta}}$



$$U = \mathbf{V}\cdot\hat{\boldsymbol{\xi}} = \mathbf{V}_{(x,y)}\cdot\left(\xi_x,\xi_y\right)_{(x,y)} = (u,v)\cdot\left(\xi_x,\xi_y\right) = u\xi_x + v\xi_y$$

$$V = \mathbf{V}\cdot\hat{\boldsymbol{\eta}} = \mathbf{V}_{(x,y)}\cdot\left(\eta_x,\eta_y\right)_{(x,y)} = u\eta_x + v\eta_y$$

Chap. 4-2. Finite Difference Approximation

- **With mesh motion and/or mesh deformation with** $\xi = \xi(x, y, t),\ \eta = \eta(x, y, t)$
  - Three underlined conservative terms are only associated with geometric motion of grid points (J and metric coefficients)

    → (Geometric Conservation Laws, GCL) corresponding discretized forms should be treated in a consistent manner to avoid numerical errors induced by grid motion

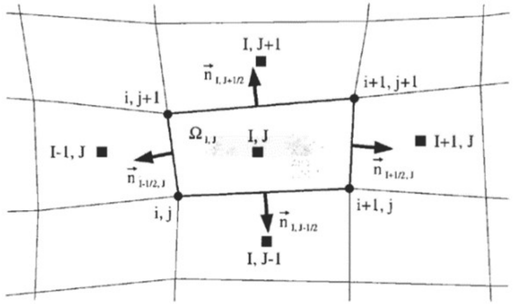
    $$\frac{\partial}{\partial t}\left(\frac{1}{J}\right) + \frac{\partial}{\partial \xi}\left(\frac{\xi_t}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\eta_t}{J}\right) = 0,\quad \frac{\partial}{\partial \xi}\left(\frac{\xi_x}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\eta_x}{J}\right) = 0,\quad \frac{\partial}{\partial \xi}\left(\frac{\xi_y}{J}\right) + \frac{\partial}{\partial \eta}\left(\frac{\eta_y}{J}\right) = 0$$

    - Discrete form of Eq. (3) should be satisfied under uniform freestream condition. → numerical disturbances should not be introduced by any mesh motion under uniform flows.

- **Discretization of transformed governing eqns.**

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{U}}{J}\right) + \frac{\partial}{\partial \xi}\left(\mathbf{F} y_\eta - \mathbf{G} x_\eta\right) + \frac{\partial}{\partial \eta}\left(-\mathbf{F} y_\xi + \mathbf{G} x_\xi\right) = \mathbf{0} \qquad \text{(Eq.4)}$$

- Metric coefficients are usually computed with $(x_\xi, x_\eta, y_\xi, y_\eta)$
- By considering a cell (i, j)/cell-interface ($\mathrm{i} \pm 1/2$ or $\mathrm{j} \pm 1/2$) in the computational domain, Eq. (4) is discretized to make a consistent comparison with finite volume discretization.



< Grid points and computational cells in physical domain >

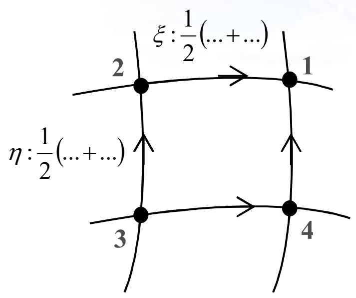
(Eq.4) $\rightarrow \frac{\partial}{\partial t}\left(\frac{\mathbf{U}}{J}\right) + \frac{\partial}{\partial \xi}\left(\mathbf{F} y_\eta - \mathbf{G} x_\eta\right) + \frac{\partial}{\partial \eta}\left(-\mathbf{F} y_\xi + \mathbf{G} x_\xi\right) = \mathbf{0}$ for the cell (i, j)

Chap. 4-2. Finite Difference Approximation

$$\frac{d}{dt}\left(\frac{\mathbf{U}_{i,j}}{J_{i,j}}\right)+\frac{\left(\mathbf{F}y_{\eta}-\mathbf{G}x_{\eta}\right)_{i\pm1/2,j}}{\Delta\xi}+\frac{\left(-\mathbf{F}y_{\xi}+\mathbf{G}x_{\xi}\right)_{i,j\pm1/2}}{\Delta\eta}=\mathbf{0} \quad \text{or} \quad \frac{\Delta\xi\Delta\eta}{J_{i,j}}\frac{d\mathbf{U}_{i,j}}{dt}+\Delta\eta\left(\mathbf{F}y_{\eta}-\mathbf{G}x_{\eta}\right)_{i\pm1/2,j}+\Delta\xi\left(-\mathbf{F}y_{\xi}+\mathbf{G}x_{\xi}\right)_{i,j\pm1/2}=\mathbf{0}$$

- $$\Delta\xi\Delta\eta\frac{1}{J_{i,j}}=\Delta\xi\Delta\eta(x_{\xi}y_{\eta}-x_{\eta}y_{\xi})_{i,j}=\Delta\xi\Delta\eta\left[\frac{1}{2}\left(\frac{(x_{i+1,j+1}-x_{i,j+1})}{\Delta\xi}+\frac{(x_{i+1,j}-x_{i,j})}{\Delta\xi}\right)\right]$$

$$\times\left[\frac{1}{2}\left(\frac{(y_{i,j+1}-y_{i,j})}{\Delta\eta}+\frac{(y_{i+1,j+1}-y_{i+1,j})}{\Delta\eta}\right)\right]-\Delta\xi\Delta\eta(x_{\eta}y_{\xi})_{i,j}=\left|(\Delta x,\Delta y)_{\rightarrow}\times(\Delta x,\Delta y)_{\uparrow}\right|=\Delta S_{i,j}$$



$\rightarrow$ area of the cell (i, j) in the physical domain

- $$\Delta\eta\left(\mathbf{F}y_{\eta}-\mathbf{G}x_{\eta}\right)_{i\pm1/2,j}=\left(\mathbf{F}\Delta y-\mathbf{G}\Delta x\right)_{i\pm1/2,j}$$

$$=\left[\mathbf{F}_{i+1/2,j}\left(y_{i+1,j+1}-y_{i+1,j}\right)-\mathbf{G}_{i+1/2,j}\left(x_{i+1,j+1}-x_{i+1,j}\right)\right]-\left[\mathbf{F}_{i-1/2,j}\left(y_{i,j+1}-y_{i,j}\right)-\mathbf{G}_{i-1/2,j}\left(x_{i,j+1}-x_{i,j}\right)\right]$$

with an esatimation of the cell-interface fluxes, $\mathbf{F}_{i\pm1/2,j}$, $\mathbf{G}_{i\pm1/2,j}$. And, similary for $\Delta\xi\left(\mathbf{F}y_{\xi}-\mathbf{G}x_{\xi}\right)_{i,j\pm1/2}$

yielding the sum of fluxes normal to the cell boundary

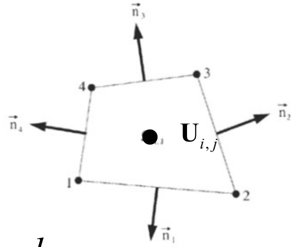
$\rightarrow$ This is identical to finite volume discretization in the physical domain.

Chap. 4-3. Finite Volume Discretization

- *Finite Volume Discretization*
  - **Directly apply the integral form of the conservation laws to each computational cell in the physical domain without coordinate transformation to computational domain**
    - Arbitrary shape of computational cell (rectangular, triangular, polygon,…) → suitable for complex geometry
  - **Integration of conservation law over computational cell $\Omega_{i,j}$**
    - Cell-averaged physical quantities defined at a cell-center (cell-centered FVM)

$$\iint\left(\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}\right)dS=\mathbf{0}\ \text{ over } \Omega_{i,j} \rightarrow \frac{\partial}{\partial t}\left(\int_{\Omega_{ij}}\mathbf{U}dS\right)+\iint\nabla\cdot(\mathbf{F},\mathbf{G})dS=\frac{\partial}{\partial t}\left(\int_{\Omega_{ij}}\mathbf{U}dS\right)+\oint_{\partial\Omega_{ij}}(\mathbf{F},\mathbf{G})\cdot\mathbf{n}dl=\mathbf{0}$$

$$\frac{\partial}{\partial t}\left(\int_{\Omega_{ij}}\mathbf{U}dS\right)\rightarrow\frac{d}{dt}\left(\mathbf{U}_{i,j}\Delta S_{i,j}\right)=\Delta S_{i,j}\frac{d\mathbf{U}_{i,j}}{dt}\ \text{ and }\ \oint_{\partial\Omega_{ij}}(\mathbf{F},\mathbf{G})\cdot\mathbf{n}dl\cong\sum_{1\sim4}\left(\mathbf{F}\Delta y-\mathbf{G}\Delta x\right)_{1\sim4}$$



$$\left(\mathbf{F}\Delta y-\mathbf{G}\Delta x\right)_{2\sim3}=\left(\mathbf{F}\Delta y-\mathbf{G}\Delta x\right)_{i+1/2,j}=\begin{bmatrix}\rho(u\Delta y-v\Delta x)\\ \rho u(u\Delta y-v\Delta x)+p\Delta y\\ \rho v(u\Delta y-v\Delta x)-p\Delta x\\ \rho H(u\Delta y-v\Delta x)\end{bmatrix}_{i+1/2,j}=\begin{bmatrix}\rho U\\ \rho uU+pn_x\\ \rho vU+pn_y\\ \rho UH\end{bmatrix}_{i+1/2,j}\cdot l_{23}$$

with $U=V\cdot\mathbf{n}, V=(u,v),\ \mathbf{n}=(\Delta y,-\Delta x)/l_{1\sim4},\ and\ l_{1\sim4}=\sqrt{\Delta x^2+\Delta y^2}$

→ Semi-discrete form: $\dfrac{d\mathbf{U}_{i,j}}{dt}=-\dfrac{1}{\Delta S_{i,j}}\sum_{1\sim4}\left(\mathbf{F}\Delta y-\mathbf{G}\Delta x\right)_{1\sim4}=-\mathbf{R}_{i,j}$ (residual vector of $\Omega_{i,j}$)

Chap. 4-3. Finite Volume Discretization

- Euler explicit with rectangular cell ($\Delta x, \Delta y = const.$) gives

$$\frac{\mathbf{U}_{i,j}^{n+1}-\mathbf{U}_{i,j}^{n}}{\Delta t}+\frac{1}{\Delta x}(\mathbf{F}_{i+1/2,j}-\mathbf{F}_{i-1/2,j})+\frac{1}{\Delta y}(\mathbf{G}_{i,j+1/2}-\mathbf{G}_{i,j-1/2})=\mathbf{0}.$$

- Comparison between FDM and FVM

$$\frac{\Delta\xi\Delta\eta}{J_{\mathrm{i,j}}}\frac{d\mathbf{U}_{\mathrm{i,j}}}{dt}+\Delta\eta\left(\mathbf{F}y_\eta-\mathbf{G}x_\eta\right)_{\mathrm{i\pm1/2,j}}+\Delta\xi\left(-\mathbf{F}y_\xi+\mathbf{G}x_\xi\right)_{\mathrm{i,j\pm1/2}}=\mathbf{0}\ \text{ vs. }\ \Delta S_{i,j}\frac{d\mathbf{U}_{i,j}}{dt}+\sum_{1\sim4}\left(\mathbf{F}\Delta y-\mathbf{G}\Delta x\right)_{1\sim4}=\mathbf{0}$$

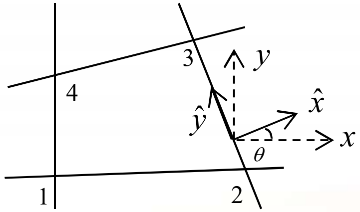
$$\frac{\Delta\xi\Delta\eta}{J_{\mathrm{i,j}}}\frac{d\mathbf{U}_{\mathrm{i,j}}}{dt}\leftrightarrow\Delta S_{i,j}\frac{d\mathbf{U}_{i,j}}{dt}\ \text{ and }\ \Delta\eta\left(\mathbf{F}y_\eta-\mathbf{G}x_\eta\right)_{\mathrm{i\pm1/2,j}}\leftrightarrow\left(\mathbf{F}\Delta y-\mathbf{G}\Delta x\right)_{23,14}$$

Flux evaluation via local coordinate

- $\dfrac{d\mathbf{U}_{i,j}}{dt}=-\dfrac{1}{\Delta S_{i,j}}\sum_{1\sim4}\left(\mathbf{F}\Delta y-\mathbf{G}\Delta x\right)_{1\sim4}$ in terms of global coordinate (x, y)
  - Introduce a local coordinate $(\hat{x},\hat{y})$ along each edge

For the edge 23 with $l_{23}=\sqrt{\Delta x^2+\Delta y^2}$, $\mathbf{n}=(\cos\theta,\sin\theta)=\left(\Delta y/l_{23},-\Delta x/l_{23}\right)$

Thus, $\dfrac{d\mathbf{U}_{i,j}}{dt}=-\dfrac{1}{\Delta S_{i,j}}\sum_{1\sim4}\left(\mathbf{F}\cos\theta+\mathbf{G}\sin\theta\right)_{1\sim4}l_{1\sim4}$



- Coordinate transform $T:(x,y)\rightarrow(\hat{x},\hat{y})$
  - $\mathbf{U}(x,y)\rightarrow\hat{\mathbf{U}}(\hat{x},\hat{y})$: $\hat{\mathbf{U}}=T\mathbf{U}=\begin{bmatrix}1&0&0&0\\0&\cos\theta&\sin\theta&0\\0&-\sin\theta&\cos\theta&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\rho\\\rho u\\\rho v\\\rho E\end{bmatrix}=\begin{bmatrix}\rho\\\rho(u\cos\theta+v\sin\theta)\\\rho(-u\sin\theta+v\cos\theta)\\\rho E\end{bmatrix}=\begin{bmatrix}\rho\\\rho U\\\rho V\\\rho E\end{bmatrix}$

Chap. 4-3. Finite Volume Discretization

- $(\mathbf{F}\cos\theta + \mathbf{G}\sin\theta) \rightarrow \hat{\mathbf{F}}: \ \hat{\mathbf{F}} = T(\mathbf{F}\cos\theta + \mathbf{G}\sin\theta)$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{bmatrix} \cos\theta + \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho uH \end{bmatrix} \sin\theta \right) = \begin{bmatrix} \rho U \\ \rho U^2 + p \\ \rho UV \\ \rho UH \end{bmatrix} = \mathbf{F}(\hat{\mathbf{U}})$$

- By summing all fluxes normal to the cell-boundary, we have

$$\frac{d\mathbf{U}_{i,j}}{dt} = -\frac{1}{\Delta S_{i,j}} \sum_{1\sim4} (\mathbf{F}\cos\theta + \mathbf{G}\sin\theta)_{1\sim4}\, l_{1\sim4} \text{ in } (x, y) \rightarrow \frac{d\hat{\mathbf{U}}_{i,j}}{dt} = -\frac{1}{\Delta S_{i,j}} \sum_{1\sim4} \mathbf{F}(\hat{\mathbf{U}})_{1\sim4}\, l_{1\sim4} \text{ in } (\hat{x}, \hat{y})$$

$$\rightarrow \frac{d\mathbf{U}_{i,j}}{dt} = -\frac{1}{\Delta S_{i,j}} \sum_{1\sim4} T^{-1}\mathbf{F}(\hat{\mathbf{U}})_{1\sim4}\, l_{1\sim4} \text{ with } T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$