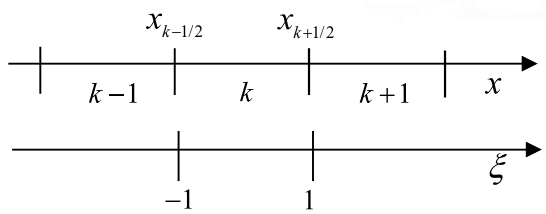
Chap. 5-3. Discretization with Modal Basis Function

- **Transformation from Physical Domain to Reference Domain**
  - Standard way to use numerical quadrature developed in reference domain
  - Transformation of $T_k = [x_{k-1/2}, x_{k+1/2}]$ in physical domain to standard element $\hat{T} = [-1,\ 1]$ in reference domain

• $\mathbf{T}: x \in T_k \mapsto \xi \in \hat{T}$ with the relation of $x(\xi) = x_k + \dfrac{h_k}{2}\xi$



or $\xi(x) = \dfrac{2}{h_k}(x - x_k),\ h_k = x_{k+1/2} - x_{k-1/2},\ J \equiv \dfrac{dx}{d\xi}$

• Normalized Legendre polynomials in $\hat{T} = [-1,1]$ with $\int_{-1}^{1} \hat{\phi}_i \hat{\phi}_j d\xi = 2\delta_{ij}$

$$\hat{\phi}_0(\xi) = 1,\ \hat{\phi}_1(\xi) = \xi \times \sqrt{3},\ \hat{\phi}_2(\xi) = \frac{1}{2}(3\xi^2 - 1) \times \sqrt{5},\ \hat{\phi}_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi) \times \sqrt{7}, \ldots$$

• $m_{ij}^k = \int_{T_k} \phi_i(x)\phi_j(x)dx = \int_{\hat{T}} (\phi_i\phi_j) \circ \mathbf{T}\,|J|\,d\xi \xrightarrow{|J| = \left|\frac{dx}{d\xi}\right|} \int_{-1}^{1} \phi_i(x(\xi))\phi_j(x(\xi)) \left|\dfrac{dx}{d\xi}\right| d\xi$

$$= \frac{h_k}{2}\int_{-1}^{1} \hat{\phi}_i(\xi)\hat{\phi}_j(\xi)\,d\xi = h_k\delta_{ij} \rightarrow \mathbf{M}_k = [m_{ij}^k] = h_k\mathbf{I}$$

- ***Choice of Numerical Flux, H***
  - **Basic elements**

• Conservation: $H_{kl}(q_-, q_+) = -H_{lk}(q_-, q_+)$

• Consistency or Lipschitz continuity

$H(q,q) = f(q)$ or $\left|H(q_-, q_+) - f(q)\right| \le K \max(|q_+ - q|, |q_- - q|)$ for some $K > 0$

Chap. 5-3. Discretization with Modal Basis Function

- Monotone flux (Harten, Hyman and Lax)

$$\frac{\partial H(q_-,q_+)}{\partial q_-} \geq 0 \text{ and } \frac{\partial H(q_-,q_+)}{\partial q_+} \leq 0$$

- **For scalar conservation law**
  - Upwind flux

$$\bullet\ H(q_-,q_+) = \frac{1}{2}(f(q_-)+f(q_+)) - |a_{1/2}|(q_+ - q_-) \text{ with } a_{1/2} = \begin{cases} \dfrac{f(q_+)-f(q_-)}{q_+ - q_-} & q_- \neq q_+ \\ \left.\dfrac{\partial f}{\partial q}\right)_{q=q_-} & q_- = q_+ \end{cases}$$

  - Local Lax-Friedrich Flux

$$\bullet\ H(q_-,q_+) = \frac{1}{2}(f(q_-)+f(q_+)) - |a_{1/2}|(q_+ - q_-) \text{ with } a_{1/2} = \max_{q\in[q_-,q_+]}\left|\frac{\partial f(q)}{\partial q}\right| = \max(|f'(q)|_{q_-}, |f'(q)|_{q+})$$

- **For Euler equations**
  - Approximate Riemann solvers: Roe, RoeM, HLLE, and so on
  - AUSM-type fluxes: AUSM+, AUSMPW+, AUSM+up, and so on
  - Local Lax-Friedrich (LLF)

- ***Accuracy Requirements of Domain and Boundary Integrals***
  - **1-D Semi-discrete form:** $\mathbf{M}_k \dfrac{\partial \mathbf{q}_k}{\partial t} = -\int_{\partial T_k} H_k^h(q_{k-}^h, q_{k+}^h)\,\mathbf{\Phi}\,dx + \int_{T_k} f_k^h(q_k^h)\dfrac{\partial \mathbf{\Phi}}{\partial x}dx$

diagonal matrix $h_k\mathbf{I}$ $\rightarrow$ time integration only

point values $\rightarrow$ no integration

need numerical integration

Chap. 5-3. Discretization with Modal Basis Function

- Only domain integration is necessary in 1-D formulation.
- For *Pn* polynomial approximation for $(n+1)^{\text{th}}$-order spatial accuracy,

  . accuracy of $f_k^h(q_k^h)$ should be one-degree higher than that of q_k^h .

  . $f_k^h(q_k^h)$ is $(n+1)^{\text{th}}$-order polynomial and $\partial\phi/\partial x$ is $(n\text{-}1)^{\text{th}}$-order polynomial.

  → numerical quadrature that is exact up to $(2n)^{\text{th}}$-order polynomial is necessary.
- In general case, domain integral needs numerical quadrature that is exact up to $(2n)^{\text{th}}$-order polynomial, and quadrature for boundary integral should be exact up to $(2n+1)^{\text{th}}$-order polynomial.

- ***Gauss Quadrature***
  - **Numerical integration of** $\int_a^b g(x)dx,\ \int_a^b g(x)w(x)dx$ **with polynomial interpolation**

• n - *th* order ploynomial interpolation to approximate $g(x)$ with *fixed* interpolation points $\{x_i\}_{0\le i\le n}$

$$\text{- } g(x) \cong \sum_{i=0}^{n} g(x_i)\phi_{n,i}(x) \ \text{ with } \phi_{n,i}(x) = \frac{\Pi_{j\neq i}(x-x_j)}{\Pi_{j\neq i}(x_i-x_j)},\ \phi_{n,i}(x_j)=\delta_{ij}$$

$$\rightarrow\ I=\int_a^b g(x)dx=\sum_{i=0}^{n} c_i g(x_i)+\mathrm{Er}(g)\ \text{ with } c_i=\int_a^b \phi_{n,i}(x)dx, \text{ which is exact up to } g(x)\in V^n$$

Similarly, $g(x)w(x) \cong \sum_{i=0}^{n} g(x_i)\phi_{n,i}(x)w(x)$ with $w(x)\ge 0$ on $[a,b]$. Thus, we have

$$I=\int_a^b g(x)w(x)dx=\sum_{i=0}^{n} c_i g(x_i)+\mathrm{Er}(g)\ \text{ with } c_i=\int_a^b \phi_{n,i}(x)w(x)dx, \text{ which is exact up to } g(x)\in V^n$$

- Newton-Cotes formulas can be used for equally spaced interpolation points with $h=x_{i+1}-x_i$

Chap. 5-3. Discretization with Modal Basis Function

. 2-point trapezoidal rule: $\int_{x_i}^{x_{i+1}} g(x)dx = \frac{h}{2}\left(g(x_i) + g(x_{i+1})\right) + O(h^2)$

. 3-point Simson's rule: $\int_{x_i}^{x_{i+2}} g(x)dx = \frac{h}{3}\left(g(x_i) + 4g(x_{i+1}) + g(x_{i+2})\right) + O(h^4)$

. Simson's $\frac{3}{8}$ rule: $\int_{x_i}^{x_{i+3}} g(x)dx = \frac{3h}{8}\left(g(x_i) + 3g(x_{i+1}) + 3g(x_{i+2}) + g(x_{i+3})\right) + O(h^4)$

- Consider the same ploynomial interpolation with *variable* interpolation points $\{x_i\}_{0\le i\le n}$

  - $g(x) \cong \sum_{i=0}^{n} g(x_i)\phi_{n,i}(x)$ and $\int_a^b g(x)dx = \sum_{i=0}^{n} c_i g(x_i) + \mathrm{Er}(g)$ gives $2n+2$ DOFs for $\{c_i, x_i\}_{0\le i\le n}$.

  $\rightarrow$ Is it possible to determine $\{x_i, c_i\}_{0\le i\le n}$ such that $\sum_{i=0}^{n} c_i g(x_i)$ is exact up to $g(x) \in V^{2n+1}$?

  - Choose $\{x_i\}_{0\le i\le n}$ as the zeros of orthogonal polynomial $\phi_{n+1}(x)$ with $\int_a^b \phi_i(x)\phi_j(x)w(x)dx = 0$ if $i \ne j$

  If $g(x) \in V^{2n+1}$, $g(x) = q(x)\phi_{n+1}(x) + r(x)$ with $q(x), r(x) \in V^n$. Since $\phi_{n+1}(x)$ is orthogonal,

  $I = \int_a^b g(x)w(x)dx = \int_a^b q(x)\phi_{n+1}(x)w(x)dx + \int_a^b r(x)w(x)dx = \int_a^b r(x)w(x)dx.$

  If we choose the zeros of $\phi_{n+1}(x)$,

  $I = \int_a^b g(x)w(x)dx = \int_a^b r(x)w(x)dx,$ and $I = \sum_{i=0}^{n} c_i g(x_i)w(x_i) = \sum_{i=0}^{n} c_i q(x_i)\phi_{n+1}(x_i)w(x_i) + \sum_{i=0}^{n} c_i r(x_i)w(x_i)$

  $= \sum_{i=0}^{n} c_i r(x_i)w(x_i)$, which holds exactly since $r(x) \in V^n$.

  For $\phi_{n+1}(x)$ and $\{x_i\}_{0\le i\le n}$, the Jacobi polynomial, $P_{n+1}^{(\alpha,\beta)}(x)$, and its roots which do not contain both end points, are used. This set of $\{x_i\}_{0\le i\le n}$ is known as the Gauss points.

Chap. 5-3. Discretization with Modal Basis Function

- **Gauss-Lobatto points**
  - For the purpose of stability without sacrificing accuracy too much, end points need to be included in many cases. Since Gauss points do not contain end points, we consider the zeros of $\phi_{n-1}(x)w(x)$ to integrate $f(x)$ on $[-1,1]$, with $w(x)=(1-x^2)$, $\int_{-1}^{1}\phi_i(x)\phi_j(x)w(x)dx=0$ if $i\neq j$

  - If $f(x)\in V^{2n-1}$, $f(x)=q(x)\phi_{n-1}(x)w(x)+r(x)$ with $q(x),r(x)\in V^{n-2}$. Since $\phi_{n-1}(x)$ is orthogonal,

    $I=\int_{-1}^{1}f(x)dx=\int_{-1}^{1}q(x)\phi_{n-1}(x)w(x)dx+\int_{-1}^{1}r(x)w(x)dx=\int_{-1}^{1}r(x)w(x)dx,$

    and by choosing the zeros of $\phi_{n-1}(x)w(x)$,

    $I=\int_{-1}^{1}f(x)dx=\int_{-1}^{1}r(x)w(x)dx$, and $I=\sum_{i=0}^{n}c_i f(x_i)=\sum_{i=0}^{n}c_i q(x_i)\phi_{n-1}(x_i)w(x_i)+\sum_{i=0}^{n}c_i r(x_i)=\sum_{i=0}^{n}c_i r(x_i)$

    which holds exactly since $r(x)\in V^{n-2}$.

  - $I=\int_{-1}^{1}f(\xi)d\xi\approx\sum_{i=1}^{n+1}w_i f(\xi_i)$ from Gauss-Legendre-Lobatto (GLL, or Gauss-Lobatto_GL) points

    . ξ_i is determined from the roots of the Jacobi polynomial, $(1-\xi^2)P_n^{(\alpha,\beta)\prime}(\xi)=0$

    $\left(\text{or } (1-\xi^2)P_{n-1}^{(\alpha+1,\beta+1)}(\xi)=0\right)$, and w_i can be determined as c_i.

    Thus, aside from two end points $(\xi=\pm1)$, $(n-1)$ interior GL points are just the Gauss points of $P_{n-1}^{(\alpha+1,\beta+1)}(\xi)$.

- **Numerical quadrature applied to the domain integral**
  - $\int_{T_k} f_k^h(q_k^h)\dfrac{\partial\Phi}{\partial x}dx=\int_{-1}^{1}f_k^h\left(\left(\sum_{l=1}^{n+1}q_k^{(l)}\hat{\phi}_l(\xi)\right)\right)\dfrac{\partial\hat{\Phi}}{\partial\xi}d\xi\cong\sum_{i=1}^{n+1}w_i\, f_k^h\left(\sum_{l=1}^{n+1}q_k^{(l)}\hat{\phi}_k(\xi_i)\right)\left.\dfrac{\partial\Phi}{\partial\xi}\right|_{\xi=\xi_i}$

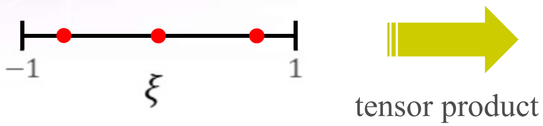
Chap. 5-3. Discretization with Modal Basis Function

- Numerical quadrature applied to the domain integral (cont'd)

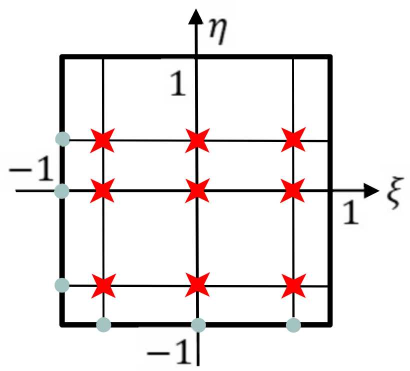
n Points [(2*n*-1)-order)]	Quadrature weights	Quadrature Points
2 (3)	w_1 = 1.000000000, w_2 = 1.000000000	$\xi_1 = -0.577350269$, $\xi_2 = 0.577350269$
3 (5)	w_1 = 0.555555556, w_2 = 0.888888889 w_3 = 0.555555556	$\xi_1 = -0.774596669$, $\xi_2 = 0.000000000$ $\xi_3 = 0.774596669$
4 (7)	w_1 = 0.347854845, w_2 = 0.652145155 w_3 = 0.652145155, w_4 = 0.347854845	$\xi_1 = -0.861136312$, $\xi_2 = -0.339981044$ $\xi_3 = 0.339981044$, $\xi_4 = 0.861136312$

- ***Quadrature rules for multi-dimensional elements***
  - **Quadrilateral/hexahedral elements**
    - Straightforward extension by dimensional splitting approach → tensor product from the 1-D Gauss quadrature
    - 1-D Gauss quadrature: $I = \int_{-1}^{1} f(\xi)d\xi = \sum_i w_i f(\xi_i)$
    - 2-D quadrature rule on quadrilateral elements: $w_{\xi,i}, w_{\eta,j}, \xi_i, \eta_j$ from a 1-D quad. rule
$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta = \int_{-1}^{1}\sum_i w_{\xi,i} f(\xi_i,\eta)d\eta = \sum_{i,j} w_{\xi,i}w_{\eta,j} f(\xi_i,\eta_j)$$
    - 3-D quadrature rule on hexahedral elements: $w_{\xi,i}, w_{\eta,j}, w_{\zeta,k}, \xi_i, \eta_j, \zeta_k$ from a 1-D quad. rule
$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta)d\xi d\eta d\zeta = \int_{-1}^{1}\int_{-1}^{1}\sum_i w_{\xi,i} f(\xi_i,\eta,\zeta)d\eta d\zeta = \sum_{i,j,k} w_{\xi,i}w_{\eta,j}w_{\zeta,k} f(\xi_i,\eta_j,\zeta_k)$$

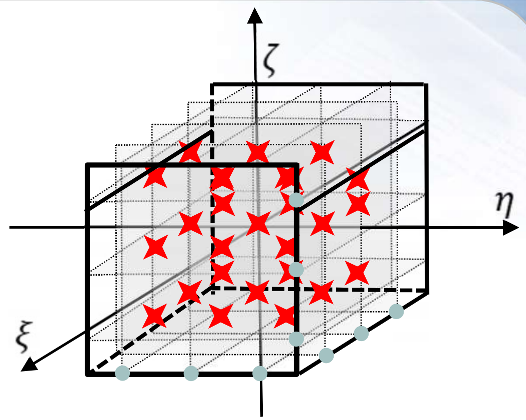
Chap. 5-3. Discretization with Modal Basis Function



< 1-D Gauss quadrature >



< Quadrature on quadrilateral >



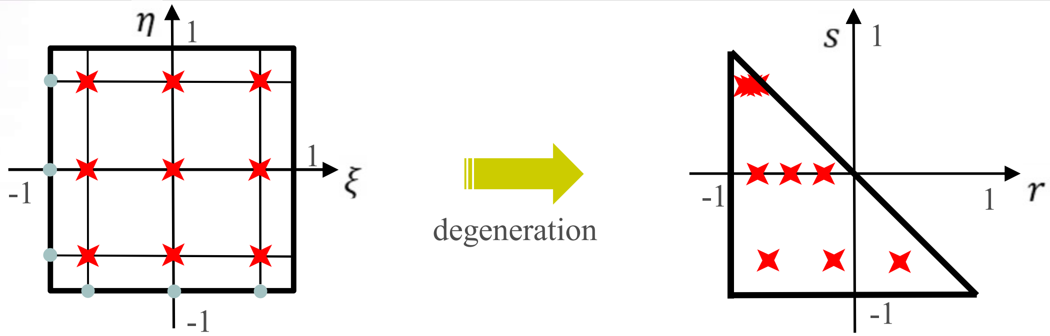
< Quadrature on hexahedral >

- **Triangular/tetrahedral/pyramid elements**
  - (Option 1) Degeneration from quadrilateral/hexahedral elements
    - Degeneration from quadrilateral to triangle by the mapping function, **T**, to transform an edge of quad. element on a point of tri. element
    - To get quadrature weights of the triangle, multiply the weight by Jacobian to quadrature rule for quad.
      - $\mathbf{T} : \mathbf{r} = (r,s) \in \hat{\hat{T}} \mapsto \mathbf{x} = (\xi,\eta) \in \hat{T}$

with the relation of $\begin{cases} r = \left[\xi(1-\eta)-(1+\eta)\right]/2 \\ s = \eta \end{cases}, \; |J| = \left|\dfrac{\partial(r,s)}{\partial(\xi,\eta)}\right| = \dfrac{1}{2}(1-\eta)$

$$I = \int_{\hat{\hat{T}}} f(r,s)d\mathbf{r} = \int_{\hat{T}} f(r,s)\circ T\,|J|\,d\mathbf{x} = \int_{\hat{T}} \hat{f}(\xi,\eta)\,|J|\,d\mathbf{x} = \sum_{i,j} w_{\xi,i} w_{\eta,j} \hat{f}(\xi_i,\eta_j)\,|J(\xi_i,\eta_j)|$$

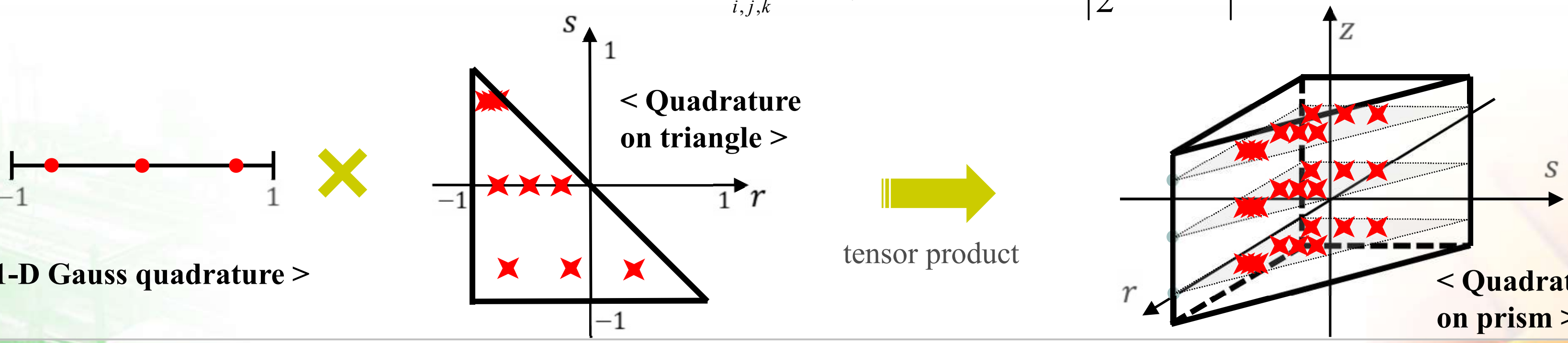
  - (Option 2) quadrature rules directly defined on the reference triangle/tetrahedron by imposing geometric symmetry and numerical stability



< Quadrature on quadrilateral >

< Quadrature degenerated to triangle >

- **Prism**
  - Tensor product from the quadrature rules of 1-D element and triangle
  - 1-D Gauss quadrature: $I = \int_{-1}^{1} f(\xi)d\xi = \sum_i w_i f(\xi_i)$
  - 2-D quadrature on triangle: $I = \int_{\hat{T}} f(r,s)d\mathbf{r} = \int_{\hat{T}} \hat{f}(\xi,\eta)\,|J|\,d\mathbf{x} = \sum_{i,j,k} w_{\xi_i} w_{\eta_j} \hat{f}(\xi_i,\eta_j)\left|\frac{1}{2}(1-\eta_j)\right|$
  - 3-D quadrature on prism:
    $I = \int_{\hat{T}} f(r,s,z)d\mathbf{r} = \int_{\hat{T}} \hat{f}(\xi,\eta,\zeta)\,|J|\,d\mathbf{x} = \sum_{i,j,k} w_{\xi_i} w_{\eta_j} w_{\zeta_k} \hat{f}(\xi_i,\eta_j,\zeta_k)\left|\frac{1}{2}(1-\eta_j)\right|$



< 1-D Gauss quadrature >

< Quadrature on triangle >

< Quadrature on prism >

- ***Stability for DG Method***
  - **For the semi-discrete form,**

$$\frac{d\mathbf{q}_k}{dt} = \mathbf{M}_k^{-1}\left(-\int_{\partial T_k} H_k^h(q_{k-}^h, q_{k+}^h)\mathbf{\Phi}\, dx + \int_{T_k} f_k^h(q_k^h)\frac{\partial \mathbf{\Phi}}{\partial x} dx \right) = -\mathbf{R}(\mathbf{q})$$

    - Explicit Euler time integration is unconditionally unstable.
  - **Runge-Kutta time integration by TVD stability**
    - 3-stage, 3rd-order accurate TVD-RK

$$\mathbf{q}_k^{(1)} = \mathbf{q}_k^n - \Delta t\, \mathbf{R}\left(\mathbf{q}_k^n\right), \quad \mathbf{q}_k^{(2)} = \frac{3}{4}\mathbf{q}_k^n + \frac{1}{4}\mathbf{q}_k^{(1)} - \frac{1}{4}\Delta t\, \mathbf{R}\left(\mathbf{q}^{(1)}\right),$$

$$\mathbf{q}_k^{n+1} = \frac{1}{3}\mathbf{q}_k^n + \frac{2}{3}\mathbf{q}_k^{(2)} - \frac{2}{3}\Delta t\, \mathbf{R}\left(\mathbf{q}^{(2)}\right).$$

    - 5-stage, 4rd-order accurate strong stability preserving Runge-Kutta method (SSP-RK(5,4))

$$\mathbf{q}_k^{(1)} = \mathbf{q}_k^n - 0.391752226571890\Delta t\, \mathbf{R}(\mathbf{q}_k^n),$$

$$\mathbf{q}_k^{(2)} = 0.444370493651235\mathbf{q}_k^n + 0.555629506348765\mathbf{q}_k^{(1)} - 0.368410593050371\Delta t\, \mathbf{R}(\mathbf{q}^{(1)}),$$

$$\mathbf{q}_k^{(3)} = 0.620101851488403\mathbf{q}_k^n + 0.379898148511597\mathbf{q}_k^{(2)} - 0.251891774271694\Delta t\, \mathbf{R}(\mathbf{q}^{(2)}),$$

$$\mathbf{q}_k^{(4)} = 0.178079954393132\mathbf{q}_k^n + 0.821920045606868\mathbf{q}_k^{(3)} - 0.544974750228521\Delta t\, \mathbf{R}(\mathbf{q}^{(3)}),$$

$$\mathbf{q}_k^{n+1} = 0.517231671970585\mathbf{q}_k^{(2)} + 0.096059710526147\mathbf{q}_k^{(3)} - 0.063692468666290\Delta t\, \mathbf{R}(\mathbf{q}^{(3)}) + 0.386708617503269\mathbf{q}_k^{(4)} - 0.226007483236906\Delta t\, \mathbf{R}(\mathbf{q}^{(4)}).$$

Chap. 5-3. Discretization with Nodal Basis Function

- **Runge-Kutta time integration to improve stability (cont'd)**
  - Severe restriction for the maximum allowable time-step size

$$\Delta t = \frac{CFL}{2n+1}\frac{\Delta x}{\left|\lambda_c\right|_{\max}} \sim O(1/n) \text{ for P}n \text{ approximation,}$$

where $\left|\lambda_c\right|_{\max}$ indicates the maximum eigenvalue of the flux Jacobian $\frac{\partial f}{\partial q}$.

- ***Construction of 1-D Nodal Basis Function***
  - **Lagrange polynomial based on solution points**

• $q_k^h(x,t) = \sum_{i=1}^{ndof(n)} q_{k,i}(t) l_{k,i}(x)$, where $l_{k,i}(x) = \prod_{j=1, j\neq i}^{ndof(n)} \frac{x - x_{k,j}}{x_{k,i} - x_{k,j}}$, $ndof(n) = \dim(V^n) = n+1$.

In reference domain $\xi \in [-1,1]$, $q_k^h(\xi,t) = \sum_{i=1}^{n+1} q_{k,i}(t) l_i(\xi)$, where $l_i(\xi) = \prod_{j=1, j\neq i}^{n+1} \frac{\xi - \xi_j}{\xi_i - \xi_j}$.

  - **Choice of solution points**
    - Lebesgue constant Λ to measure the interpolation error

• Consider the projection operator, $\Pi : V \to V^n (\subset V)$ such that $\Pi(q) = q^h$

For $q^h = \sum_{i=1}^{ndof(n)} q_i(t) l_i(\xi)$ and q^* with $\left\|q - q^*\right\|_\infty = \min\left\{\left\|q - \tilde{q}\right\|_\infty : \tilde{q} \in V^n\right\}$, we have

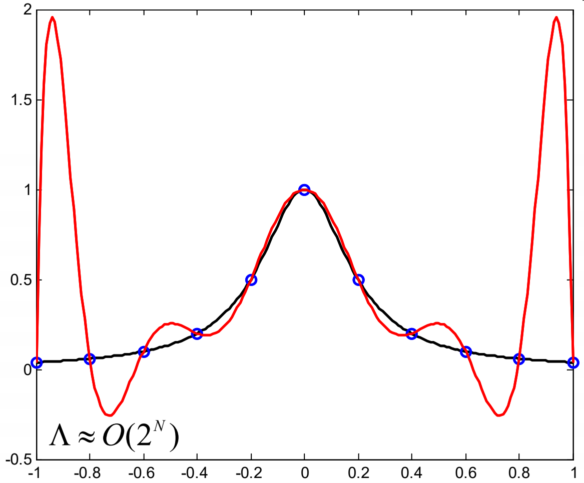
$$\left\|q - q^h\right\|_\infty \le \left\|q - q^*\right\|_\infty + \left\|q^* - q^h\right\|_\infty \le \left\|q - q^*\right\|_\infty + \left\|\Pi(q^*) - \Pi(q)\right\|_\infty \le \left(1 + \left\|\Pi\right\|_\infty\right)\left\|q - q^*\right\|_\infty \le \left(1 + \Lambda\right)\left\|q - q^*\right\|_\infty,$$

where $\Lambda = \left\|\Pi\right\|_\infty = \max_{\xi\in[-1,1]} \sum_{m=1}^{ndof(n)} \left|l_m(\xi)\right|$. Thus, choose solution points to minimize the Lebesque constant Λ.

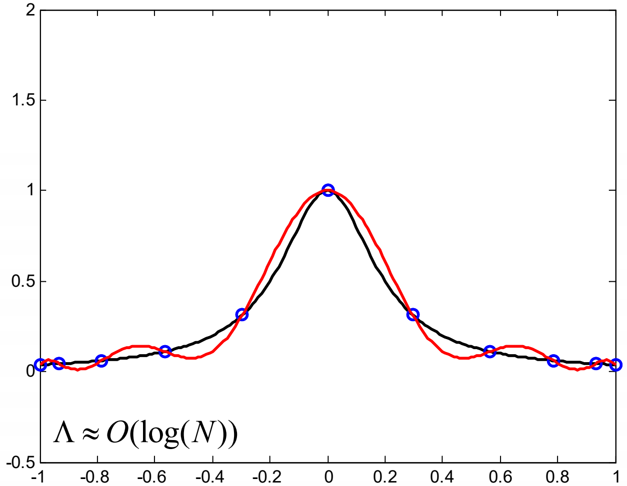
Chap. 5-3. Discretization with Nodal Basis Function

- **Choice of solution points (cont'd)**
  - Effects of solution points
    - Runge phenomenon

      Approximate the Runge function ($f(\xi) = 1/(1+25\xi^2)$) by Lagrange ploynomial basis $l_i(\xi)$ obtained by *uniform* solution points $\xi_k = 2k/n - 1,\ \ 0 \le k \le n \ \rightarrow$ unbounded oscillatory behavior at the edges of the interval as the number of solution points are increasing.



< Interpolation over equidistant points >



< Interpolation over Gauss-Lobatto points >

    - Runge phenomenon can be cured by choosing solution points that minimize the Lebesque constant.

      • From $\phi_i(\xi) = w_{i,1}l_1(\xi) + w_{i,2}l_2(\xi) + \ldots + w_{i,N}l_N(\xi),\ \ 1 \le i \le N(= ndof(n)),\ w_{i,j} = \phi_i(\xi_j)$ for solution points

      $\xi = \xi_j. \rightarrow$ With $V = [v_{ij}] \equiv [\phi_i(\xi_j)],\ \mathbf{\Phi} = V\mathbf{L}$. Thus, $l_i(\xi) \sim \dfrac{1}{\det(V)}$ and solution points maximizing the determinant of V (or minimizing the Lebesque constant) are the Gauss-Lobatto points.

Chap. 5-4. Relationship between Modal and Nodal Form

- ***Approximation with Modal and Nodal Basis Function***
  - Modal and nodal representation: $q_k^h(x,t) = \sum_{l=1}^{ndof(n)} q_k^{(l)}(t)\phi_l(x) = \sum_{i=1}^{ndof(n)} q_{k,i}(t) l_{k,i}(x)$
  - Solution in reference domain: $q_k^h(\xi,t) = \sum_{l=1}^{ndof(n)} q_k^{(l)}(t)\hat{\phi}_l(\xi) = \sum_{i=1}^{ndof(n)} q_{k,i}(t)\hat{l}_i(\xi)$
  - For each solution point $\xi = \xi_j$, $\sum_{l=1}^{ndof(n)} q_k^{(l)}(t)\hat{\phi}_l(\xi_j) = \sum_{i=1}^{ndof(n)} q_{k,i}(t)\hat{l}_i(\xi_j) = q_{k,j}(t)$

  Thus, we have '$\mathbf{q}_{\text{nodal}} = V^T \mathbf{q}_{\text{modal}}$'

  with $\mathbf{q}_{\text{nodal}} = \left[q_{k,j}(t) \right]_{ndof(n)}^T$, $\mathbf{q}_{\text{modal}} = \left[q_k^{(i)}(t) \right]_{ndof(n)}^T$, and $V = [\hat{\phi}_i(\xi_j)]_{ndof(n)\times ndof(n)}$.

  Also, from $q_k^h(\xi,t) = \mathbf{L}^T \mathbf{q}_{\text{nodal}} = \mathbf{L}^T V^T \mathbf{q}_{\text{modal}} = \mathbf{\Phi}^T \mathbf{q}_{\text{modal}}$, we have '$\mathbf{\Phi} = V\mathbf{L}$'

  with $\mathbf{L} = \left[\hat{l}_i(\xi) \right]_{ndof(n)}^T$, $\mathbf{\Phi} = \left[\hat{\phi}_j(\xi) \right]_{ndof(n)}^T$.

- $V = [v_{ij}] \equiv [\hat{\phi}_i(\xi_j)]$**, generalized *Vandermonde* matrix, plays a role of transforming the modal and nodal forms.**
  - Mass matrix in nodal form

$$m_{ij}^k = \int_{T_k} l_i(x) l_j(x) dx = \frac{h_k}{2}\int_{-1}^{1} \hat{l}_i(\xi)\hat{l}_j(\xi)d\xi = \frac{h_k}{2}\int_{-1}^{1}\left[\sum_{p=1}^{ndof(n)} V_{i,p}^{-1}\hat{\phi}_p(\xi)\right]\left[\sum_{q=1}^{ndof(n)} V_{j,q}^{-1}\hat{\phi}_q(\xi)\right]d\xi$$

$$= \frac{h_k}{2}\sum_{p=1}^{ndof(n)}\sum_{q=1}^{ndof(n)} V_{j,q}^{-1}V_{i,p}^{-1}\left(\int_{-1}^{1}\hat{\phi}_q(\xi)\hat{\phi}_p(\xi)d\xi\right) = h_k \sum_{p=1}^{ndof(n)} V_{i,p}^{-1} {V^{T}}_{p,j}^{-1} \rightarrow \mathbf{M}_k = h_k (V^T V)^{-1}$$