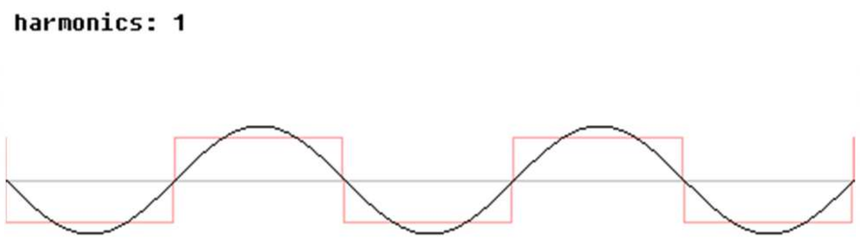
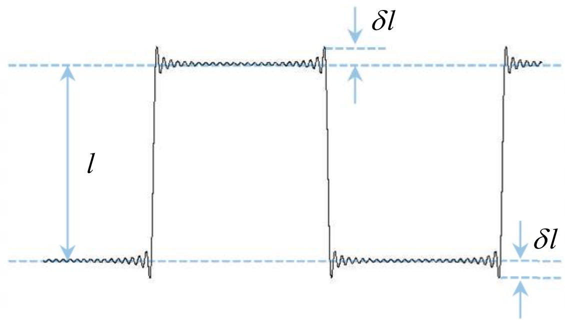
Chapter 2. Non-linear Stability and Hyperbolic PDE

Chap. 2-1. G-W Phenomenon, Godunov's Theorem and Monotonicity

- ***Gibbs-Wilbraham Phenomenon***
  - **Approximation of a profile including discontinuity by Fourier Series (or interpolating techniques using basis functions)**
    - Oscillations occurs across discontinuity with *O(1)* → It never dies out even if the number of basis function is increasing.
    - Henry Wilbraham (1848), J. Willard Gibbs (1899)



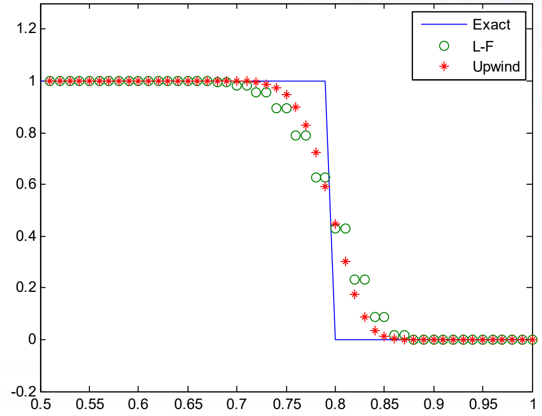
<Animation of the Gibbs phenomenon>



  - **Magnitude of overshoot/undershoot:** $(\delta l\,/\,l)\ \sim\ \pm 14\%$
  - **Locally converge (or L_1, L_2 convergence) but not uniformly (L_∞ convergence)**
    → warning to naïve capturing discontinuities by simply increasing the number of interpolating function or mesh point

- ***First-Order Scheme and Numerical Diffusion***
  - **For $u_t + au_x = 0$ with Upwind or L-F scheme**

$$\left.\begin{aligned} F_{j+1/2,up} &= \frac{a}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{|a|}{2}\Delta u_{j+1/2}^n \\ F_{j+1/2,L-F} &= \frac{a}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{\Delta x}{2\Delta t}\Delta u_{j+1/2}^n \end{aligned}\right\} \text{ with } u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2} - F_{j-1/2}\right)$$



  - **Modified equations**

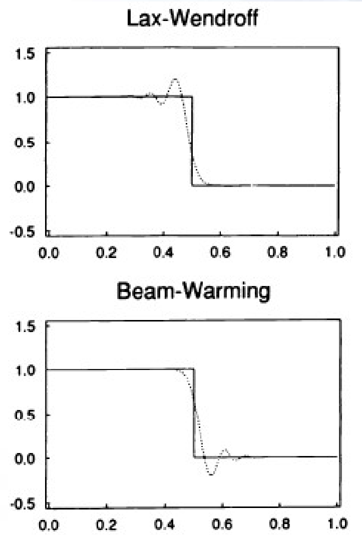
$$u_t + au_x = \begin{cases} \dfrac{a\Delta x}{2}\left(1-\sigma^2\right)u_{xx} + \ldots & \text{for upwind} \\ \dfrac{\Delta x^2}{2\Delta t}\left(1-\sigma^2\right)u_{xx} + \ldots & \text{for L-F} \end{cases} \rightarrow u_t + au_x = c_1 u_{xx} \ldots$$

  - **Leading error term is numerical dissipative → smooth transition across discontinuity without oscillations**
    - Excessive numerical dissipation of $O(\Delta x)$
      - Unacceptable loss of accuracy → Too many grid points
      - Viscous computation and resolution of boundary layer requires at least 2nd-order accuracy.

- ***Second-Order Scheme and Numerical Dispersion***
  - **With L-W or B-W scheme**

$$\bullet\ F_{j+1/2,L-W} = \frac{a}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{a^2\Delta t}{2\Delta x}\Delta u_{j+1/2}^n \quad \bullet\ F_{j+1/2,B-W} = au_j^n + \frac{a(1-\sigma)}{2}\Delta u_{j-1/2}^n \quad \text{with } a > 0$$

- ***Second-Order Scheme and Numerical Dispersion (cont'd)***



  - **Modified equations**

$$u_t + a u_x = \begin{cases} -\dfrac{a\Delta x^2}{6}\left(1-\sigma^2\right)u_{xxx}\cdots & \text{for L-W} \\ \dfrac{a\Delta x^2}{6}\left(2-3\sigma+\sigma^2\right)u_{xxx}\cdots & \text{for B-W} \end{cases}$$

$$u_t + a u_x = c_1 u_{xxx} \xrightarrow[u(x,t)\sim\hat{u}(\omega,t)e^{i\omega x}]{\text{discrete FT}} \hat{u}(\omega,t)_t + i\left(a\omega + c_1\omega^3\right)\hat{u}(\omega,t) = 0$$

$$\rightarrow \hat{u}(\omega,t) \sim e^{-i\left(a\omega+c_1\omega^3\right)t} \quad \text{vs.} \quad \hat{u}_{ex}(\omega,t) \sim e^{-ia\omega t}$$

  - **Numerical dispersion relation :** $a(\omega) = a\omega + c_1\omega^3$ vs. $a_{ex}(\omega) = a\omega$
    For each Fourier component with *ω*, group velocity $a_g(\omega) \equiv da(\omega)/d\omega$

$$\bullet\ a_g(\omega) = a + 3c_1\omega^2 \approx a \text{ for small wave number (long wave length)}$$

$$\neq a \text{ for large wave number (short wave length)}$$

For $a > 0$, $c_1 < 0$ for $L-W \rightarrow$ lagging error

$c_1 > 0$ for $B-W \rightarrow$ leading error

- ***Observation***
  - **Numerical oscillations across discontinuity occurs regardless of differencing type (central or upwind) once the order of accuracy is greater than one.**

- ***Godunov's Barrier Theorem on Monotonicity***
  - **General form of one-step numerical schemes for $u_t + au_x = 0$**

$$\sum_q \beta_q u_{j+q}^{n+1} = \sum_q \alpha_q u_{j+q}^n \text{ with } \beta_q \;\; st. \;\; u_j^{n+1} \text{ can be uniquely obtained (or } \mathrm{B}_{im}\mathbf{u}^{n+1} = A_{ex}\mathbf{u}^n \text{)}$$

$$\xrightarrow{\text{Linear mapping}} u_j^{n+1} = \sum_q c_{jq} u_{j+q}^n, \quad c_{jq} = c_{jq}\left(\Delta x, \Delta t, a\right)$$

  - Upwind : $u_j^{n+1} = \frac{1}{2}(\sigma + |\sigma|)u_{j-1}^n + (1 - |\sigma|)u_j^n + \frac{1}{2}(|\sigma| - \sigma)u_{j+1}^n$
  - L-W : $u_j^{n+1} = \frac{\sigma}{2}(\sigma+1)u_{j-1}^n + (1-\sigma^2)u_j^n + \frac{\sigma}{2}(\sigma-1)u_{j+1}^n$
- **Conditions for consistency and accuracy**
  - For $u_j^{n+1} = \sum_q c_q u_{j+q}^n$

$$u_j + u_t\Delta t + u_{tt}\frac{\Delta t^2}{2} + \ldots = u_j - au_x\Delta t + a^2 u_{xx}\frac{\Delta t^2}{2} + \ldots = \sum_q c_q\left[u_j + q\Delta x u_x + \frac{(q\Delta x)^2}{2}u_{xx} + \ldots\right]$$

  - Consistency : $\sum_q c_q = 1$ **Eq. (*1)**
  - First-order accuracy : $\Delta x \sum_q qc_q = -a\Delta t \rightarrow \sum_q qc_q = -\sigma$ **Eq. (*2)**
  - Second-order accuracy : $\Delta x^2 \sum_q q^2 c_q = a^2\Delta t^2 \rightarrow \sum_q q^2 c_q = \sigma^2$ **Eq. (*3)**

Chap. 2-1. G-W Phenomenon, Godunov's Theorem and Monotonicity

- (Godunov's barrier theorem) For the fully discretized *linear* scheme of $u_j^{n+1} = Ru_j^n \equiv \sum_q c_q u_{j+q}^n$,

  it can not be better than first-order accurate if the scheme is not oscillatory.

  - (Positivity condition) If R is stable in maximum-norm, c_q should be non-negative.

    (pf) Suppose it is not true, then there exists some negative c_q.

    Choose u_{j+q}^n st. $u_{j+q}^n = \begin{cases} 1 & \text{if } c_q > 0 \\ -1 & \text{if } c_q < 0 \end{cases}$

    Then, $\left\| u^n \right\|_\infty = 1$ and $u_j^{n+1} = Ru_j^n = \sum_q c_q u_{j+q}^n = \sum_q |c_q|$

    Since the scheme is consistent, $\sum_q c_q = 1$ and

    $$1 = \left| \sum_q c_q \right| \le \sum_q \left| c_q \right| \quad \rightarrow \quad \left\| u^n \right\|_\infty \le \sum_q \left| c_q \right| = u_j^{n+1} \le \left\| u^{n+1} \right\|_\infty$$

    Thus, c_q should be non-negative.

  - The scheme cannot be better than first-order accurate.

    (pf) Since $c_q > 0$, define positive α_q with $c_q = \alpha_q^2$, and

    $\alpha_q = \sqrt{c_q}, \quad \beta_q = q\sqrt{c_q}$

    For second-order accuracy, we have

    $\sum_q \alpha_q^2 = 1$ (consistency), $\sum_q \alpha_q \beta_q = -\sigma$ (1st-order), $\sum_q \beta_q^2 = \sigma^2$ (2nd-order)

By Cauchy-Schwartz inequality,

$$\left(\sum_q \alpha_q \beta_q\right)^2 \le \left(\sum_q \alpha_q^2\right)\left(\sum_q \beta_q^2\right)$$

But equality holds only if $\beta_q = k\alpha_q$ for some 'constant' k.

This is possible only if α_q has only one non-zero value.

From Eq. $(*1)$, $c_{\tilde{q}} = 1$ for the specific $\tilde{q}$, and, from Eq. $(*2)$, $\sigma = -\tilde{q} = -k$.

But this cannot be satisfied for arbitrary value of a, Δx, Δt (or equivalently, σ).

- **Constructive implication of Godunov's barrier theorem**
  - To obtain more than 2nd-order scheme without oscillations, the scheme should be 'non-linear' even for linear equation. → Aside from the linear stability for the linear difference schemes, such as Von Neumann stability, the need to develop non-linear stability theory becomes apparent.
    - Maximum-norm boundedness means that a computed result is non-oscillatory.
- **Oscillation check of first- or second-order linear schemes from the view point of positivity condition**

• $Upwind: u_j^{n+1} = \frac{1}{2}(\sigma + |\sigma|)u_{j-1}^n + (1-|\sigma|)u_j^n + \frac{1}{2}(|\sigma| - \sigma)u_{j+1}^n$

• $L\text{-}F: u_j^{n+1} = \frac{1+\sigma}{2}u_j^n + \frac{1-\sigma}{2}u_{j+1}^n$

• $L\text{-}W: u_j^{n+1} = \frac{\sigma(1+\sigma)}{2}u_{j-1}^n + (1-\sigma^2)u_j^n - \frac{\sigma(1-\sigma)}{2}u_{j+1}^n$