Chap. 2-3. High Resolution Monotonic Schemes

- ***Critical Survey on High Resolution Monotonic Schemes***
  - **Basic analysis framework**
    - 1-D SCL of $\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \rightarrow u_j^{n+1}=u_j^n-\frac{\Delta t}{\Delta x}\left(F_{j+1/2}-F_{j-1/2}\right)$ or $\frac{du_j}{dt}=-\frac{1}{\Delta x}(F_{j+1/2}-F_{j-1/2})$
    - Monotonicity constraint as a way to design $F_{j+1/2}$

      $\min(u_{j-1},u_j,u_{j+1})\le u(x)\le \max(u_{j-1},u_j,u_{j+1})$ or $\min(u_j,u_{j+1})\le u_{j+1/2}\le \max(u_j,u_{j+1}),\ x_{j-1/2}\le x\le x_{j+1/2}$
    - One-dimensional l_1- or l_∞-stability is realized by imposing TVD stability (FCT, TVD, MUSCL, LED) or TVB stability (ENO and WENO).

    → Avoid numerical oscillations across discontinuous region while maintaining expected order of accuracy in smooth region
  - **Multi-dimensional extension**
    - 2-D SCL of $\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}=0 \rightarrow \frac{du_{i,j}}{dt}=-\frac{1}{\Delta x}(F_{i+1/2,j}-F_{i-1/2,j})-\frac{1}{\Delta y}(G_{i,j+1/2}-G_{i,j-1/2})$
    - Monotonicity constraint by dimensional splitting approach

      $(x\text{-}dir)\ \min(u_{i-1,j},u_{i,j},u_{i+1,j})\le u(x,y_j)\le \max(u_{i-1,j},u_{i,j},u_{i+1,j})$

      $\min(u_{i,j},u_{i+1,j})\le u_{i+1/2,j}\le \max(u_{i,j},u_{i+1,j}),\ x_{i-1/2,j}\le x\le x_{i+1/2,j}$

      $(y\text{-}dir)\ \min(u_{i,j-1},u_{i,j},u_{i,j+1})\le u(x_i,y)\le \max(u_{i,j-1},u_{i,j},u_{i,j+1})$

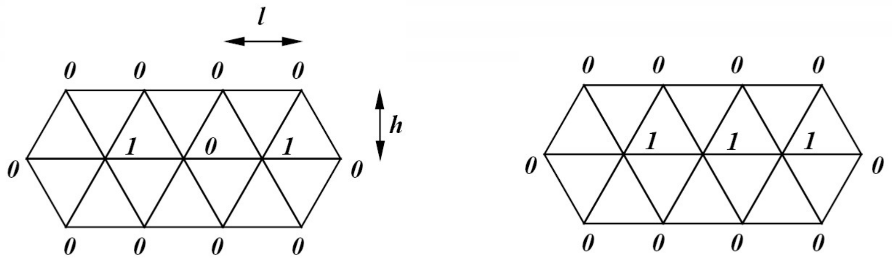
      $\min(u_{i,j},u_{i,j+1})\le u_{i,j+1/2}\le \max(u_{i,j},u_{i,j+1}),\ y_{i,j-1/2}\le y\le y_{i,j+1/2}$

Chap. 2-3. High Resolution Monotonic Schemes

- ***Critical Survey*** **(cont'd)**
  - **Any missing physics for multi-D flows?**
    - Multi-dimensional monotonicity and multi-dimensional numerical flux
  - **Intrinsic one-dimensionality of total variation**
    - $TV(u) = \int_{-\infty}^{\infty} \left| u'(x) \right| dx = \sum_{j=-\infty}^{\infty} \left| u_j - u_{j-1} \right| = 2\left(\sum \text{maxima} - \sum \text{minima} \right)$

      $\xrightarrow{TV \text{ in multiple dimensions}} TV(u) = \int_V \left\| \nabla u \right\|_p ds \;\; \text{with} \;\; \left\| \nabla u \right\|_1 = \left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right|, \;\; \left\| \nabla u \right\|_2 = \sqrt{\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2}, \ldots$
    - Two piecewise linear distributions (two-peaks and one-ridge) on equilateral triangular mesh with unit length (Jameson, 1995)

      $\rightarrow TV_{two-peaks} < TV_{one-ridge}$ confirming that straightforward extesnion of 1-D TVD is not proper to handle multi-dimensional oscillations.



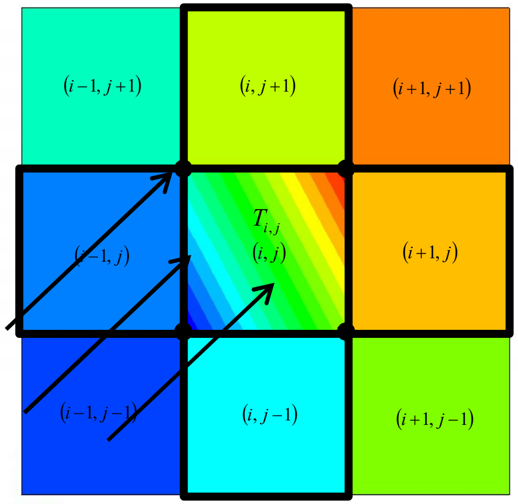
Two Peaks: $TV = 4 + 2\sqrt{3}$ (L_1), 6 (L_2), or $2 + 2\sqrt{3}$ (L_∞).

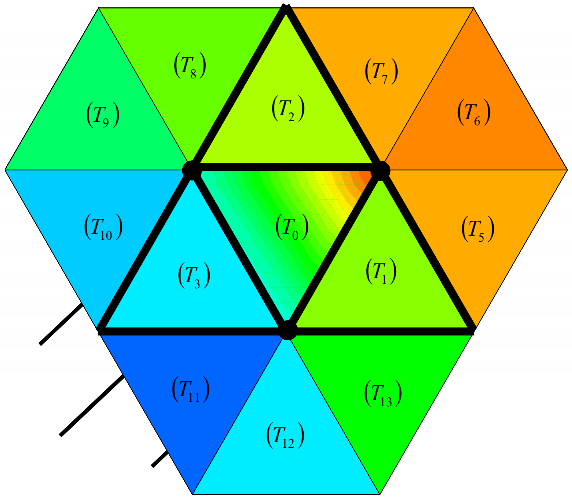
One Ridge: $TV = 6 + \sqrt{3}$ (L_1), 7 (L_2), or $5 + 3\sqrt{3}$ (L_∞).

- 2-D TVD scheme is at most first-order accurate. (Goodman and Leveque, 1985)

Chap. 2-3. High Resolution Monotonic Schemes

- *MLP Schemes*
  - **More refined numerical strategy is necessary for non-oscillatory schemes in multi-dimensional flows.**
    - Most oscillation-free schemes have been developed largely based on one dimensional flow physics$\left(u_t + f(u)_x = 0\right)$.
    - Dimensional splitting TVD schemes do not guarantee monotonicity at vertex point.
    - See the works by C. Kim *et al.* (2005, 2008, 2010), and others
  - **Massing flow physics in non grid-aligned flow distributions**





< Do the values at vertexes satisfy monotonicty? >

- Role of vertex values under linear distribution
- Local extrema always appear at vertex point under linear distribution. → treat cell-centered and cell-vertex point simultaneously

- MLP condition: $u_{neighbor}^{\min} \le u(x,y)\big|_{(x,y)\in T_{i,j} \text{ or } T_0} \le u_{neighbor}^{\max}$
  - Applying MLP condition to each vertex point of the target cell($T_{i,j}$ or T_o) to have

    $$\min\left(u_{i,j},u_{i+1,j},u_{i,j+1},u_{i+1,j+1}\right) \le u(x,y)\big|_{i+1/2,j+1/2} \le \max\left(u_{i,j},u_{i+1,j},u_{i,j+1},u_{i+1,j+1}\right)$$

    $$\min_{T_k\in S_{v_j}}(u_k) \le u(x,y)\big|_{v_j} \le \max_{T_k\in S_{v_j}}(u_k) \quad \text{with } S_{v_j} = \{T_k \,|\, v_j \in T_k \text{ for } v_j \text{ of } T_0\}$$

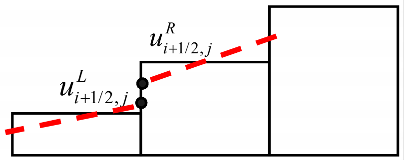
    $\rightarrow$ Impose the above inequality as additional constraint in designing MLP limiters

- ***MLP Limiters on Structured Meshes***
  - **TVD-MUSCL vs. MLP**

From MUSCL-type slope limiting, the left/right cell-interface values are given by

$$u_{i+1/2,j}^{L} = u_{i,j} + 0.5\phi_{i,j}(r_L)\Delta u_{i-1/2,j} \text{ and } u_{i+1/2,j}^{R} = u_{i+1,j} - 0.5\phi_{i+1,j}(r_R)\Delta u_{i+3/2,j}$$

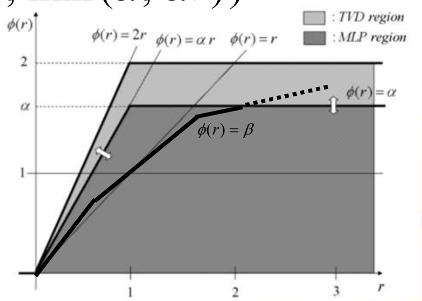
with $\phi(r) = r\phi(1/r)$ and $r_L = \Delta u_{i+1/2,j} / \Delta u_{i-1/2,j}, r_R = \Delta u_{i+1/2,j} / \Delta u_{i+3/2,j}$



- TVD limiting region based on 1-D analysis: $\Phi_{TVD}(r) = \max(0, \min(2, 2r))$
- MLP limiting region to treat multi-dimensionality: $\Phi_{MLP}(r) = \max(0, \min(\alpha, \alpha r))$
  - Basic form: $\phi_{MLP}(r) = \max(0, \min(\alpha, \alpha r, \beta(r)))$

    α : multi-dimensional restriction coefficient to determine the upper bound of MLP limiting region

    β : charactersitic interpolation coefficient to determine local variation (slope or higher than 2nd-order)



[TVD vs. MLP limiting region]

Chap. 2-3. High Resolution Monotonic Schemes

- **Determination of multi-dimensional restriction coefficient, α**
  - (S1) Each cell-vertex value of the target cell $T_{i,j}$ is estimated by

$$u_{i+k_1/2,j+k_2/2} = u_{i,j} + \Delta u^x_{i+k_1/2,j} + \Delta u^y_{i,j+k_2/2} = u_{i,j} + 0.5k_1\phi(r_x)\Delta u_{i-1/2,j} + 0.5k_2\phi(r_y)\Delta u_{i,j-1/2}$$

$$= u_{i,j} + \left(1 + r_{xy}\right)\Delta u^x_{i+k_1/2,j}$$

with $r_x = \Delta u_{i+1/2,j} / \Delta u_{i-1/2,j}$, $r_y = \Delta u_{i,j+1/2} / \Delta u_{i,j-1/2}$, $r_{xy} = \Delta u^y_{i,j+k_2/2} \Big/ \Delta u^x_{i+k_1/2,j}$ and $k_1, k_2 = \pm 1$

- Noth that, with linear subcell approximation, we only need to check the upper bound of the maximum vertex value and the lower bound of the minimum vertex value. Thus, r_{xy} is assumed to be positive.

  - (S2) Obtain the neighboring minimum and maximum values by checking all cell-averaged values sharing the same vertex point $(i + k_1/2,\ j + k_2/2)$

$$u^{\min}_{i+k_1/2,j+k_2/2} = \min\left(u_{i,j}, u_{i+k_1,j}, u_{i,j+k_2}, u_{i+k_1,j+k_2}\right),\ u^{\max}_{i+k_1/2,j+k_2/2} = \max\left(u_{i,j}, u_{i+k_1,j}, u_{i,j+k_2}, u_{i+k_1,j+k_2}\right)$$

  - (S3) Enforce the MLP limiting condition with $\phi\left(r\right) = \phi_{MLP}\left(r\right) = \min\left(\alpha, \alpha r\right)$ into $u_{i+k_1/2,j+k_2/2}$

$$u^{\min}_{i+k_1/2,j+k_2/2} \le u_{i+k_1/2,j+k_2/2} \le u^{\max}_{i+k_1/2,j+k_2/2} \rightarrow u^{\min}_{i+k_1/2,j+k_2/2} \le u_{i,j} + \left(1 + r_{xy}\right)\Delta u^x_{i+k_1/2,j} \le u^{\max}_{i+k_1/2,j+k_2/2}$$

$$\frac{u^{\min}_{i+k_1/2,j+k_2/2} - u_{i,j}}{1 + r_{xy}} \le \Delta u^x_{i+k_1/2,j} \le \frac{u^{\max}_{i+k_1/2,j+k_2/2} - u_{i,j}}{1 + r_{xy}} \quad \text{or}$$

$$\frac{u^{\min}_{i+k_1/2,j+k_2/2} - u_{i,j}}{1 + r_{xy}} \le 0.5k_1\phi_{MLP}(r_x)\Delta u_{i-1/2,j} \le \frac{u^{\max}_{i+k_1/2,j+k_2/2} - u_{i,j}}{1 + r_{xy}}$$

Determination of multi-dimensional restriction coefficient, α (cont'd)

- (S4) Obtain the range of α for local maximum case

$$\text{Since } \Delta u^{x}_{i+k_1/2,j} > 0 \text{ (local maximum)}, \; 0 \le 0.5 k_1 \phi_{MLP}(r_x) \Delta u_{i-1/2,j} \le \frac{u^{\max}_{i+k_1/2,j+k_2/2} - u_{i,j}}{1+r_{xy}}$$

$$\text{i) } 0 < r_x < 1, \;\; 0 \le 0.5 k_1 \alpha \Delta u_{i+1/2,j} \le \frac{u^{\max}_{i+k_1/2,j+k_2/2} - u_{i,j}}{1+r_{xy}} \;\rightarrow\; 0 \le \alpha \le \frac{2(u^{\max}_{i+k_1/2,j+k_2/2} - u_{i,j})}{k_1(1+r_{xy})\Delta u_{i+1/2,j}}$$

$$\text{ii) } r_x > 1, \;\; 0 \le \frac{k_1 \alpha}{2 r_x} \Delta u_{i+1/2,j} \le \frac{u^{\max}_{i+k_1/2,j+k_2/2} - u_{i,j}}{1+r_{xy}} \;\rightarrow\; 0 \le \alpha \le \frac{2 r_x (u^{\max}_{i+k_1/2,j+k_2/2} - u_{i,j})}{k_1(1+r_{xy})\Delta u_{i+1/2,j}}$$

$$\text{Thus, we have } 0 \le \alpha_{\text{lmax}} \le \frac{2\max(1, r_x)(u^{\max}_{i+k_1/2,j+k_2/2} - u_{i,j})}{k_1(1+r_{xy})\Delta u_{i+1/2,j}}$$

- (S5) Obtain the range of α for local minimum case

$$\text{Since } \Delta u^{x}_{i+k_1/2,j} < 0 \text{ (local minimum)}, \; \frac{u^{\min}_{i+k_1/2,j+k_2/2} - u_{i,j}}{1+r_{xy}} \le 0.5 k_1 \phi_{MLP}(r_x) \Delta u_{i-1/2,j} \le 0$$

$$\text{i) } 0 < r_x < 1, \;\; \frac{u^{\min}_{i+k_1/2,j+k_2/2} - u_{i,j}}{1+r_{xy}} \le 0.5 k_1 \alpha \Delta u_{i+1/2,j} \le 0 \;\rightarrow\; 0 \le \alpha \le \frac{2(u^{\min}_{i+k_1/2,j+k_2/2} - u_{i,j})}{k_1(1+r_{xy})\Delta u_{i+1/2,j}}$$

$$\text{ii) } r_x > 1, \;\; \frac{u^{\min}_{i+k_1/2,j+k_2/2} - u_{i,j}}{1+r_{xy}} \le \frac{k_1 \alpha}{2 r_x} \Delta u_{i+1/2,j} \le 0 \;\rightarrow\; 0 \le \alpha \le \frac{2 r_x (u^{\min}_{i+k_1/2,j+k_2/2} - u_{i,j})}{k_1(1+r_{xy})\Delta u_{i+1/2,j}}$$

$$\text{Thus, we have } 0 \le \alpha_{\text{lmin}} \le \frac{2\max(1, r_x)(u^{\min}_{i+k_1/2,j+k_2/2} - u_{i,j})}{k_1(1+r_{xy})\Delta u_{i+1/2,j}}$$

Chap. 2-3. High Resolution Monotonic Schemes

- **Determination of multi-dimensional restriction coefficient, α (cont'd)**
  - (S6) Obtain the range of α from (S4) and (S5) with $\alpha \geq 1$

$$1 \leq \alpha = \min(\alpha_{l\max}, \alpha_{l\min}) \leq \left| \frac{2\max(1, r_x)}{(1+r_{xy})\Delta u_{i+1/2,j}} \right| \min\left(\left| u^{\max}_{i+k_1/2, j+k_2/2} - u_{i,j} \right|, \left| u^{\min}_{i+k_1/2, j+k_2/2} - u_{i,j} \right| \right),$$

and choose the upper bound as $\alpha = \left| \frac{2\max(1, r_x)}{(1+r_{xy})\Delta u_{i+1/2,j}} \right| \min\left(\left| u^{\max}_{i+k_1/2, j+k_2/2} - u_{i,j} \right|, \left| u^{\min}_{i+k_1/2, j+k_2/2} - u_{i,j} \right| \right)$

  - Notice that if $r_{xy} = 0$ (1-D situation), it recovers the 1-D TVD region.

- **Determination of interpolation coefficient, β**
  - β using conventional TVD-MUSCL or LED slopes
    - MLP with minmod limiter is the same as TVD minmod limiter with α=1=β.
    - MLP with van Leer: $\beta_L = \frac{2 r_L}{1 + r_L}$, $\beta_R = \frac{2 r_R}{1 + r_R}$
    - MLP with superbee: $\beta_{L/R} = \max(1, r_{L/R})$
    - Any other TVD-MUSCL or LED slopes with $\beta(r)\big|_{r=1}$ =1 can be used.
  - β using polynomial interpolation
    - Consider higher-order polynomial of $u(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \quad x_{j-1/2} \leq x \leq x_{j+1/2}$
      - MLP3 (slope limiting using 2rd-order polynomial)

For $u(x) = a_0 + a_1 x + a_2 x^2$ with $u_{i+l} = \int_{x_{i+l-1/2}}^{x_{i+l+1/2}} u(x)dx \ (l = 0, \pm 1)$,

- **Determination of interpolation coefficient, β (cont'd)**
  - β using polynomial interpolation

    - MLP3 (slope limiting using 2rd-order polynomial)

$$u^L_{i+1/2} = u(x_{i+1/2}) = \frac{-u_{i-1} + 5u_i + 2u_{i+1}}{6} = u_i + 0.5\frac{\Delta u_{i-1/2} + 2\Delta u_{i+1/2}}{3} = u_i + 0.5\beta(r_L)\Delta u_{i-1/2}$$

Thus, we have $\beta_{L/R} = \beta(r_{L/R}) = \dfrac{1+2r_{L/R}}{3}$.

    - MLP5 (slope limiting using 4th-order polynomial)

Similarly, with $u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$ and $u_{i+l} = \int_{x_{i+l-1/2}}^{x_{i+l+1/2}} u(x)dx \ (l = 0, \pm 1, \pm 2)$,

$$u^L_{i+1/2} = u(x_{i+1/2}) = u_i + 0.5\frac{-2\Delta u_{i-3/2} + 11\Delta u_{i-1/2} + 24\Delta u_{i+1/2} - 3\Delta u_{i+3/2}}{30} = u_i + 0.5\beta(r_L)\Delta u_{i-1/2}$$

Thus, we have
$$\begin{cases} \beta_L = \beta(r_L) = \dfrac{-2/r_{L(i-1)} + 11 + 24r_L - 3r_L r_{L(i+1)}}{30}, \\ \beta_R = \dfrac{-2/r_{R(i+1)} + 11 + 24r_R - 3r_{R(i-1)} r_R}{30}. \end{cases}$$

- **MLP slope limiting in multiple dimensions**
  - x-directional cell-interface values and fluxes

$$\left.\begin{aligned} u^L_{i+1/2,j} &= u_{i,j} + 0.5\phi_{MLP(i,j)}(r_L)\Delta u_{i-1/2,j} = u_{i,j} + 0.5\max\left(0, \min\left(\alpha_L, \alpha_L r_L, \beta_L\right)\right)\Delta u_{i-1/2,j} \\ u^R_{i+1/2,j} &= u_{i+1,j} - 0.5\phi_{MLP(i+1,j)}(r_R)\Delta u_{i+3/2,j} = u_{i+1,j} - 0.5\max\left(0, \min\left(\alpha_R, \alpha_R r_R, \beta_R\right)\right)\Delta u_{i+3/2,j} \end{aligned}\right\} \rightarrow F_{i+1/2,j}(u^L_{i+1/2,j}, u^R_{i+1/2,j})$$

Chap. 2-3. High Resolution Monotonic Schemes

- ***MLP Limiters on Unstructured Meshes***
  - **Multi-dimensional unstructured meshes excludes direct estimation of α, β.**
    - Multi-dimensional limited linear reconstruction for the target cell T_O

      $$u_O(\mathbf{r}) = u_O + \nabla u_O \cdot (\mathbf{r} - \mathbf{r}_O) \;\rightarrow\; L\left[u_O(\mathbf{r})\right] = u_O + \phi_{\text{MLP},T_O} \nabla u_O \cdot (\mathbf{r} - \mathbf{r}_O)$$

      • Gradient ∇u_O is estimated by least square method using the cell-averaged values of the cell T_O and its neighborhoods T_A, T_B, T_C.

      • The same MLP condition is enforced to determine the limiter function ϕ_{MLP,T_O} since local extrema appear at vertex points.
    - Apply the MLP condition to each vertex $v_i \in T_O$

      $$u_{v_i}^{\min} \le L\left[u_O(\mathbf{r}_{v_i})\right] = u_O + \phi_{\text{MLP},v_i} \nabla u_O \cdot (\mathbf{r}_{v_i} - \mathbf{r}_O) \le u_{v_i}^{\max},$$

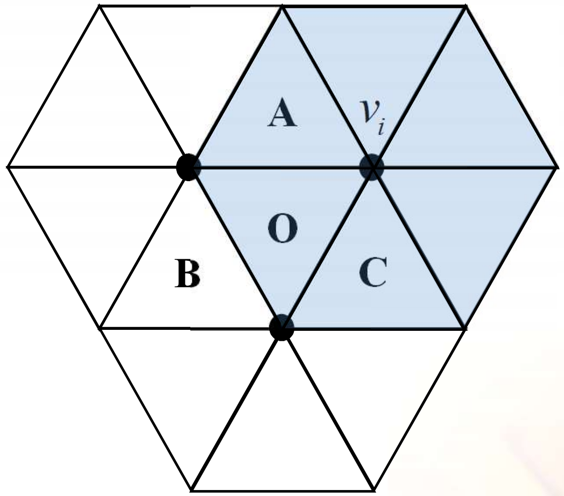
      where $u_{v_i}^{\min} = \min\limits_{T_k \in S_{v_i}}(u_k)$, $u_{v_i}^{\max} = \max\limits_{T_k \in S_{v_i}}(u_k)$,

      and $S_{v_i} = \{T_k \mid v_i \in T_k \text{ for } v_i \in T_o\}$.

      $$\rightarrow u_{v_i}^{\min} - u_O \le \phi_{\text{MLP},v_i} \nabla u_O \cdot (\mathbf{r}_{v_i} - \mathbf{r}_O) \le u_{v_i}^{\max} - u_O$$

      Since $\text{Sgn}(u_{v_i}^{\min} - u_O) \ne \text{Sgn}(u_{v_i}^{\max} - u_O)$,

      $$0 \le \phi_{\text{MLP},v_i} \le \max\left(\frac{u_{v_i}^{\min} - u_O}{\nabla u_O \cdot (\mathbf{r}_{v_i} - \mathbf{r}_O)}, \frac{u_{v_i}^{\max} - u_O}{\nabla u_O \cdot (\mathbf{r}_{v_i} - \mathbf{r}_O)}\right).$$



< Target triangular cell T_O and its neighborhoods T_A, T_B, T_C >

- **Determination of MLP-u slope limiters on unstructured meshes**
  - (S1) Estimate a local gradient ∇u_O using a least-square approach and reconstruct the linear distribution within the target cell T_O.

$$u_O(\mathbf{r}) = u_O + \nabla u_O \cdot (\mathbf{r} - \mathbf{r}_O)\big|_{A,B,C} \rightarrow [\mathbf{L}_1 \quad \mathbf{L}_2][\nabla u_O] = [\Delta u] \text{ with}$$

$$\mathbf{L}_1 = \begin{bmatrix} \Delta x_{OA} \\ \Delta x_{OB} \\ \Delta x_{OC} \end{bmatrix},\ \mathbf{L}_2 = \begin{bmatrix} \Delta y_{OA} \\ \Delta y_{OB} \\ \Delta y_{OC} \end{bmatrix},\ [\nabla u_O] = \begin{bmatrix} (\nabla u_O)_x \\ (\nabla u_O)_y \end{bmatrix} \text{ and } [\Delta u] = \begin{bmatrix} u_A - u_O \\ u_B - u_O \\ u_C - u_O \end{bmatrix}$$

Thus, we have
$$[\nabla u_O] = \frac{1}{l_{11} l_{22} - l_{12}^2} \begin{bmatrix} l_{22}(\mathbf{L}_1 \cdot [\Delta u]) & -l_{12}(\mathbf{L}_2 \cdot [\Delta u]) \\ l_{11}(\mathbf{L}_2 \cdot [\Delta u]) & -l_{12}(\mathbf{L}_1 \cdot [\Delta u]) \end{bmatrix} \text{ with } l_{ij} = \mathbf{L}_i \cdot \mathbf{L}_j.$$

  - (S2) For each vertex $v_i \in T_O$, search neighboring maximum and minimum cell-averaged values, and determine the permissible range of ϕ_{MLP,v_i} by applying the MLP condition.

$$u_{v_i}^{\min} = \min_{T_k \in S_{v_i}}(u_k),\ u_{v_i}^{\max} = \max_{T_k \in S_{v_i}}(u_k) \text{ with } S_{v_i} = \{T_k \mid v_i \in T_k \text{ for } v_i \in T_O\} \text{ gives}$$

$$0 \le \phi_{\text{MLP},v_i} \le \min(r_{v_i}, 1) \text{ with } r_{v_i} = \begin{cases} \dfrac{u_{v_i}^{\max} - u_O}{\nabla u_O \cdot (\mathbf{r}_{v_i} - \mathbf{r}_O)} & \text{if } \nabla u_O \cdot (\mathbf{r}_{v_i} - \mathbf{r}_O) > 0 \\ \dfrac{u_{v_i}^{\min} - u_O}{\nabla u_O \cdot (\mathbf{r}_{v_i} - \mathbf{r}_O)} & \text{if } \nabla u_O \cdot (\mathbf{r}_{v_i} - \mathbf{r}_O) < 0 \end{cases}$$

Similar to minmod limiting, r_{v_i} is the ratio of the maximum (or minimum) allowable variation to the estimated variation at the vertex $v_i \in T_O$.

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- **Determination of MLP-u slope limiters on unstructured meshes (cont'd)**
  - (S3) For each vertex $v_i \in T_O$, determine ϕ_{MLP,v_i}

i) MLP-u1 limiter $\rightarrow$ Take the upper bound of the limiting region

$$\phi_{\text{MLP-u1},v_i} = \Phi(r_{v_i}), \quad \text{where} \quad \Phi(r) = \begin{cases} \min(1, r) & r > 0, \\ 0 & r = 0. \end{cases}$$

ii) MLP-u2 limiter $\rightarrow$ Make MLP-u1 limiter function $\Phi_{\text{MLP-u1}}$ differentiable for steady-state computations

. Approximate $\min(1, r)$ by a smooth function $f_v(r) = (r^2 + 2r)/(r^2 + r + 2)$ _*Venkat*, and apply $f_v(r)$ to $\Phi_{\text{MLP-u1}}(r)$ with ε to avoid clipping

$$\Phi_{\text{MLP-u2}}\left(\frac{\Delta_+}{\Delta_-}\right) = \begin{cases} \dfrac{1}{\Delta_-}\left(\dfrac{\left(\Delta_+^2 + \varepsilon^2\right)\Delta_- + 2\Delta_-^2\Delta_+}{\Delta_+^2 + 2\Delta_-^2 + \Delta_-\Delta_+ + \varepsilon^2}\right) & r \neq 0, \\ 1 & r = 0. \end{cases},$$

where $r = \Delta_+ / \Delta_-$, $\Delta_+ = u_{v_j}^{\min} - u_O$ or $u_{v_j}^{\max} - u_O$, and $\Delta_- = \nabla u_O \cdot (\mathbf{r}_{v_i} - \mathbf{r}_O)$.

. ε is determined by relfecting local flow physics

- In nearly uniform regions, ε is large enough to prevent unnecessary operation of the limiter.
- In fluctuating regions, ε is smaller than local flow variation to activate the limiter.

$$\varepsilon_{MLP-u2}^2 = \frac{K_1}{1+r}(\Delta u_{v_i})^2 \quad \text{with} \quad r = \frac{\Delta u_{v_i}}{K_2(\Delta x)^{1.5}}, \quad \Delta u_{v_i} = u_{v_i}^{\max} - u_{v_i}^{\min}, \text{ and } K_1 = K_2 = 5.0$$

Chap. 2-3. High Resolution Monotonic Schemes

- **Determination of MLP-u slope limiters on unstructured meshes (cont'd)**
  - (S4) Determine the limited slope within the cell T_O to satisfy the local maximum principle.

$$L\left[u_O(\mathbf{r})\right] = u_O + \phi_{\mathrm{MLP},T_O} \nabla u_O \cdot (\mathbf{r} - \mathbf{r}_O) \quad \text{with } \phi_{\mathrm{MLP},T_O} = \min_{\forall v_i \in T_O} \phi_{\mathrm{MLP},v_i}$$

- **Stability of MLP schemes**
  - l_∞ stability by the local maximum principle

For multi-dimensional nonlinear scalar conservation law of $\partial u / \partial t + \partial f\left(u\right) / \partial x + \partial g\left(u\right) / \partial y = 0$, a fully discrete scheme using the MLP limiters for *linear reconstruction* over T_O satisfies the local maximum principle under a suitable CFL condition.

$$\textit{If } u_{neighbor}^{\min,n} \le u_{T_O}^{n} \le u_{neighbor}^{\max,n}, \ \textit{then } u_{neighbor}^{\min,n} \le u_{T_O}^{n+1} \le u_{neighbor}^{\max,n}$$

- $u_{neighbor}^{\min/max,n}$ is the minimum/maximum cell-averaged value over the MLP stencil, S_{T_j}, given by

$$S_{T_j} = \left\{ T_k \middle| v_i \in T_k \text{ for any } v_i \in T_j \right\}.$$

- ϕ_{MLP} is less diffusive than conventional limiters on regular triangular/tetrahedral meshes.

Performances of MLP

- Ex1: 2-D linear wave equation on triangular meshes

$$u_t + \mathbf{a}\cdot\nabla u = 0, \quad \mathbf{a} = (1,\ 2)$$

with

i) smooth IC:

$$u_0(x,y) = \sin(2\pi x)\sin(2\pi y)$$

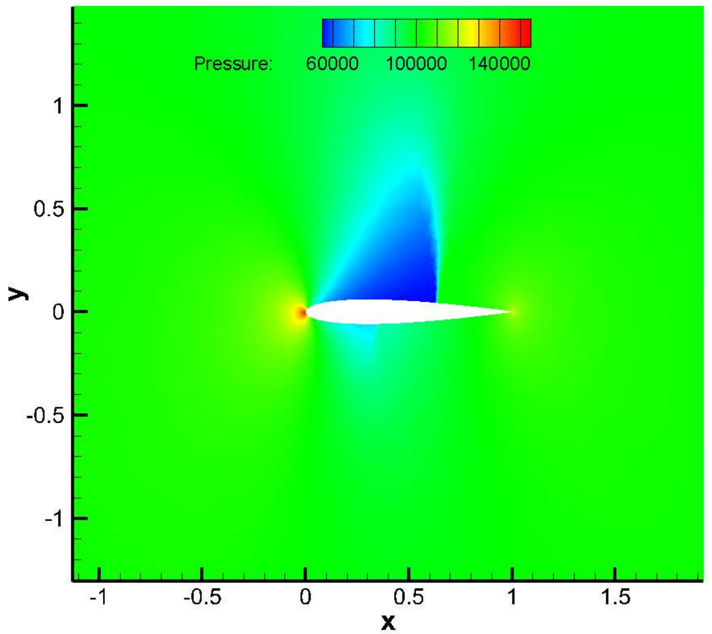
ii) discontinuous IC:

$$u_0(x,y) = \begin{cases} 1 & \text{if } -0.5 \le x, y \le 0.5 \\ 0 & \text{otherwise} \end{cases}$$

At $t = 1.0$

Scheme	Grid	L_∞	Order	L_1	Order
Barth's Limiter	20x20x2	4.0857E-01	-	1.8795E-01	-
	40x40x2	2.3275E-01	0.81	1.1570E-01	0.70
	80x80x2	1.3716E-01	0.76	6.6048E-02	0.81
	160x160x2	7.7957E-2	0.81	3.6023E-02	0.87
MLP-u1	20x20x2	1.7544E-01		3.3721E-02	
	40x40x2	6.6036E-02	1.41	6.5248E-03	2.37
	80x80x2	2.3412E-02	1.50	1.2439E-03	2.39
	160x160x2	8.0831E-03	1.53	2.5402E-04	2.29
MLP-u2	20x20x2	2.5789E-01		7.0173E-02	
	40x40x2	1.0727E-01	1.27	1.6780E-02	2.06
	80x80x2	4.3619E-02	1.30	4.1919E-03	2.00
	160x160x2	1.7198E-02	1.34	4.1919E-03	2.11
Unlimited	20x20x2	3.2729E-02		1.7253E-02	
	40x40x2	6.5138E-03	2.33	3.4741E-03	2.31
	80x80x2	1.4691E-03	2.15	8.0518E-04	2.11
	160x160x2	3.5323E-04	2.06	1.9720E-04	2.03

- **Performances of MLP**
  - Ex2: 2-D steady Euler euqations
    - Inviscid transonic flow over NACA0012 airfoil
    - Freestream Mach number = 0.8, angle of attack = 1.25°
    - RoeM numerical flux with LU-SGS implicit time integration
    - 24,041 triangular elements



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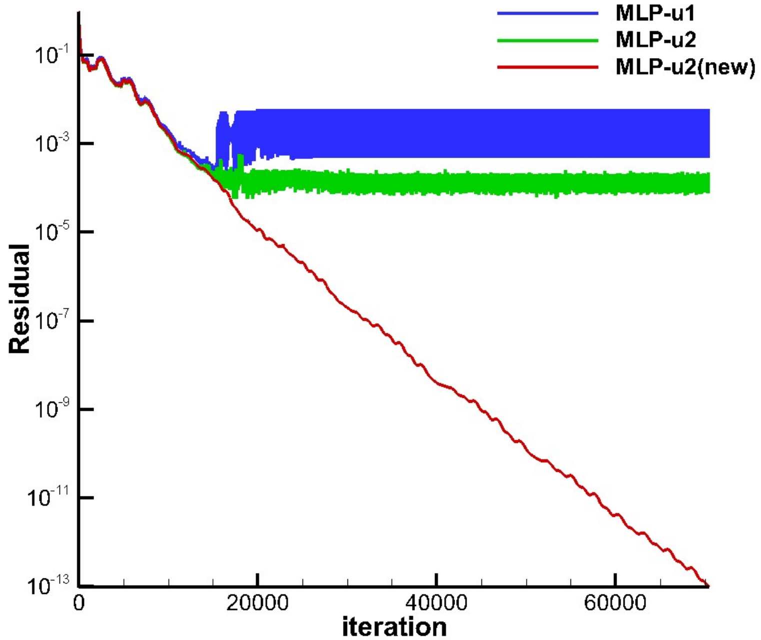

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