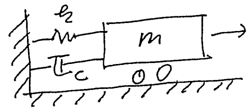
"Complex Frequency Response"

The topic of "<u>response of SDOF systems to harmonic excitation</u>" is extremely important, not only because many SDOF structures are subjected to harmonic excitation, but also because the results of this topic can also be extended to treat MDOF structures and structures subjected to more complex types of excitation.



$$m\ddot{u} + c\dot{u} + ku = \underbrace{p_0 \cos(\Omega t)}_{= p(t)} \quad \text{--- (16)}$$

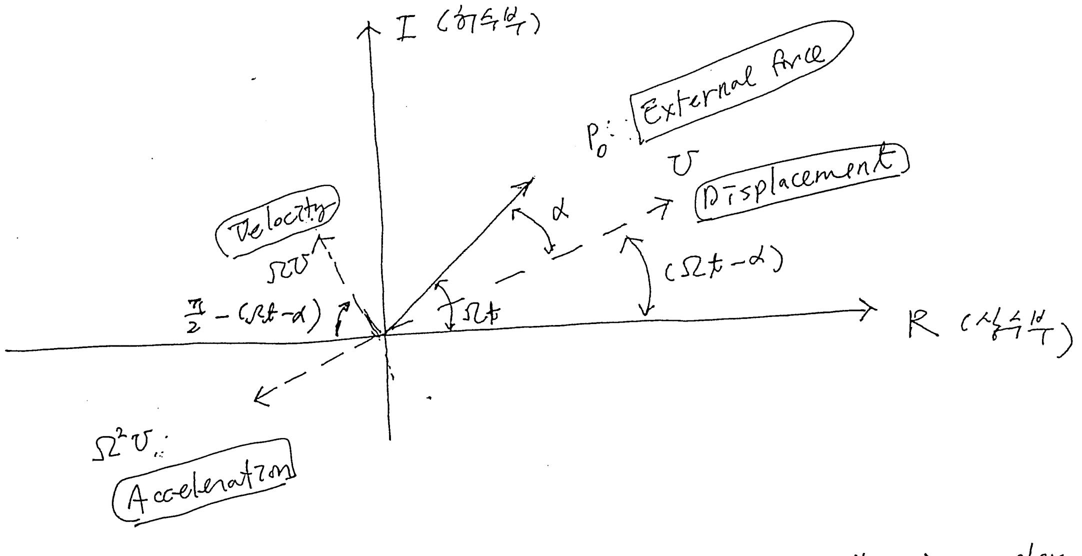
$$u_p = \underbrace{U}_{\substack{\text{steady state}\\\text{response amplitude}}} \cos(\Omega t - \underbrace{\alpha}_{\substack{\text{phase angle of the response}\\\text{to the excitation}}}) \quad \text{---- (17)}$$

← 왜 이런 solution form이 가능한가? ↑ steady state response

① damping term의 존재

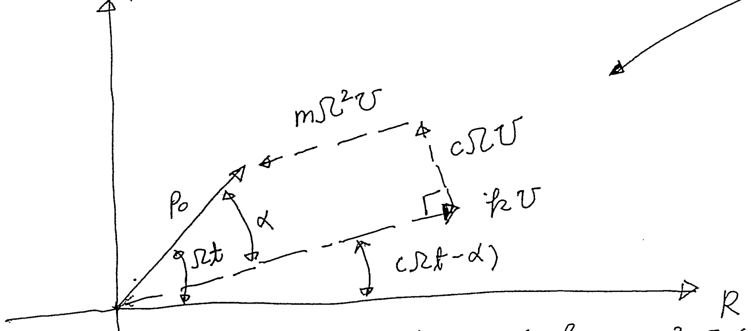
② $u_p = A \sin \Omega t + B \cos \Omega t$

$$\left.\begin{aligned} \dot{u}_p &= -\Omega U \sin(\Omega t - \alpha) \\ \ddot{u}_p &= -\Omega^2 U \cos(\Omega t - \alpha) \\ &= -\Omega^2 u_p \end{aligned}\right\} \quad \text{--- (18)}$$



: <u>Rotating vectors</u> representing p, u, $\dot{u}$, and $\ddot{u}$ in complex plane.

↳ 실수축에 대한 정사영에 주목 : (17)식을 (16)식에 대입했을 때 force polygon의 구성요소에 상응 (동적 평형 방정식을 만족하는)



: Force vector polygon (when $m\Omega^2 U < ku$) or <u>$\Omega < \omega_n$</u>

From "geometry",

$$\begin{cases} p_0^2 = (m\Omega^2 U - kU)^2 + (c\Omega U)^2 \\ \tan\alpha = \dfrac{c\Omega U}{kU - m\Omega^2 U} = \dfrac{c\Omega}{k - m\Omega^2} \end{cases}$$

"동일한 결과가 얻어짐"

이하 생략

The use of vectors in the complex plane greatly simplifies many frequency response calculations.

$$m \ddot{u}_R + c \dot{u}_R + k u_R = p_0 \cos \Omega t \quad \text{---- (24)}$$

The suscript R (for projection onto the real axis) is used to designate the steady-state motion due to $\cos \Omega t$ excitation.

$$u_R = U \cos(\Omega t - \alpha) \quad \text{---- (25)}$$

Likewise,

(sin Ωt 입력에 대한 경우)

$$\begin{cases} m \ddot{u}_I + c \dot{u}_I + k u_I = p_0 \sin \Omega t & \text{--- (26)} \\ u_I = U \sin(\Omega t - \alpha) & \text{-- (27)} \end{cases}$$

Now, if Eq. (26) is multiplied by $i = \sqrt{-1}$ and added to Eq. (24), and Euler's formula is used, there results

$$m \ddot{\bar{u}} + c \dot{\bar{u}} + k \bar{u} = \bar{p} = p_0 e^{i\Omega t} \quad \text{--- (28)}$$

$$\bar{u} = u_R + i u_I \quad \text{--- (29)}$$

where a bar denotes a vector in the complex plane. Eq. (28) is called the complex equation of motion, and the vector $\bar{u}$ is called the complex response.

It is understood that the actual steady-state response will be given by either the real part of $\bar{u}$ or its imaginary part, depending upon whether the excitation is of $\cos \Omega t$ or $\sin \Omega t$.

The steady-state solution of Eq. (28) may be assumed to have the form

$$\bar{u} = \bar{U} e^{i\Omega t} \quad \text{--- (30)}$$

← 좌변의 미계수의 'odd, even'에 무관한 solution form 임에 주목

where $\bar{U}$ is the complex amplitude.

↳ (최대 진폭과 위상정보가 모두 포함되어 있음)

The complex amplitude $\bar{U}$ may also written

$$\bar{U} = U e^{-i\alpha} \qquad \text{--- (31)}$$

where U and α are the same amplitude and phase angle introduced in Eq. (17).

Note: $\bar{u} = \bar{U} e^{i\Omega t} = \underset{\sim}{U e^{-i\alpha}} e^{i\Omega t} = U e^{i(\Omega t - \alpha)}$

$$= U \{ \cos(\Omega t - \alpha) + i \sin(\Omega t - \alpha) \}$$

(30) → (28) 에 대입하여 정리,

$$\bar{U} = \frac{p_o}{(k - m\Omega^2) + i c\Omega} \qquad \text{--- (32)}$$

Eq. static response

$$= \frac{p_o / k}{\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right) + i\, 2 \cancel{m \omega_n} \zeta \frac{\Omega}{\cancel{k}\, \omega_n}} = \frac{U_0}{(1 - r^2) + i(2\zeta r)}$$

Define

$$\bar{H}(\Omega) \equiv \frac{\bar{U}}{U_0} = \frac{1}{(1 - r^2) + i(2\zeta r)} \qquad \text{--- (33)}$$

the complex frequency response (function) ← (정상 상태 응답의 진폭과 위상 정보를 제공)

- A summary of a couple of results from the theory of complex numbers.

i) <u>Rectangular</u> and <u>polar</u> representation.

$$\bar{A} = A_R + i A_I \qquad \text{--- (34a)}$$

$$= \underline{A e^{i\alpha}} \qquad \text{--- (34b)}$$

$$A \equiv |\bar{A}| = \sqrt{A_R^2 + A_I^2} \qquad \text{--- (34c)}$$

$$\alpha = \tan^{-1}\left(\frac{A_I}{A_R}\right) \qquad \text{--- (34d)}$$

ii) Quotient of two complex numbers

$$\frac{\bar{B}}{\bar{A}} = \frac{B e^{i\beta}}{A e^{i\alpha}} = \left(\frac{B}{A}\right) e^{i(\beta-\alpha)} \qquad \text{--- (35)}$$

polar form

Back to the problem.

Then, the amplitude and the phase of the quotient in (33) is

$$\bar{H}(\Omega) = \frac{\bar{U}}{U_0} = \frac{e^{i(0)} = 1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}\ \underbrace{\left(\frac{(1-r^2)}{\sqrt{\quad ''\quad}} + i\,\frac{2\zeta r}{\sqrt{\quad ''\quad}}\right)}_{e^{i\alpha}}}$$

(where $\alpha = \tan^{-1}\left(\dfrac{2\zeta r}{1-r^2}\right)$

$$= \frac{1}{\sqrt{\quad ''\quad}}\, e^{-i\alpha}$$

동일한 결과!

$|\bar{H}(\Omega)|$

따라서, $\bar{U} = \underbrace{U_0 |\bar{H}(\Omega)|}_{U} e^{-i\alpha} = U e^{-i\alpha}$

$\bar{u} = \bar{U} e^{i\Omega t} = U e^{-i\alpha} e^{i\Omega t} = U e^{i(\Omega t-\alpha)}$

In summary, the four steps employed in using complex vectors to determine the steady-state responses are:

1. Write the differential equation in terms of complex excitation and complex response, Eq. (28)

2. Assume a solution with complex amplitude $\bar{U}$ as in Eq. (30).

3. Substitute the assumed response into the differential equation to obtain an expression for $\bar{H}(\Omega)$ $\left(= \frac{\bar{U}}{U_0}\right)$.

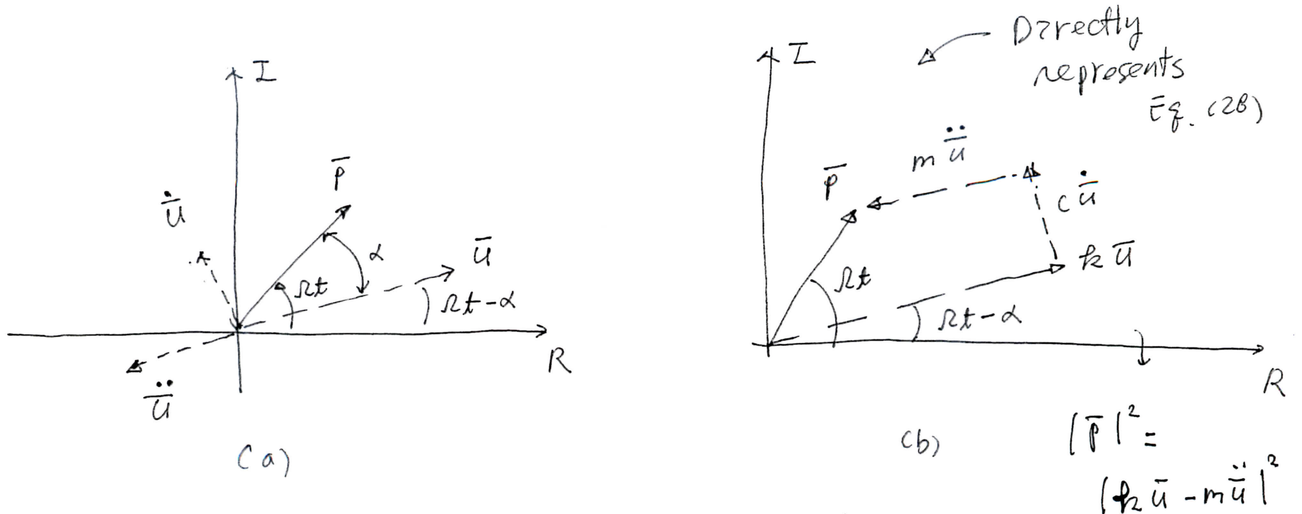
4. Use Eqs. (34) and (35) to get the amplitude and phase of the complex frequency response.

The force vector polygon presented previously can now be related directly to the complex differential equation, Eq. (28).

$$\bar{u} = \bar{U}\, e^{i\Omega t}$$

$$\dot{\bar{u}} = i\Omega \bar{U} e^{i\Omega t} = i\Omega \bar{u} = \Omega \cdot \bar{u}\, e^{i(\frac{\pi}{2})} = \Omega \bar{U} e^{i(\Omega t + \frac{\pi}{2})} \quad 90^\circ$$

$$\ddot{\bar{u}} = \ldots = -\Omega^2 \bar{u} = \Omega^2 \bar{u}\, \underbrace{e^{i(\pi)}}_{(-1)} = \Omega^2 \bar{U} e^{i(\Omega t + \pi)} \quad 180^\circ$$



$|\bar{P}|^2 = |k\bar{u} - m\ddot{\bar{u}}|^2 + |c\dot{\bar{u}}|^2$

: Complex vector notation for rotating vectors

The vector response plot (Nyquist plot or Argand plot)

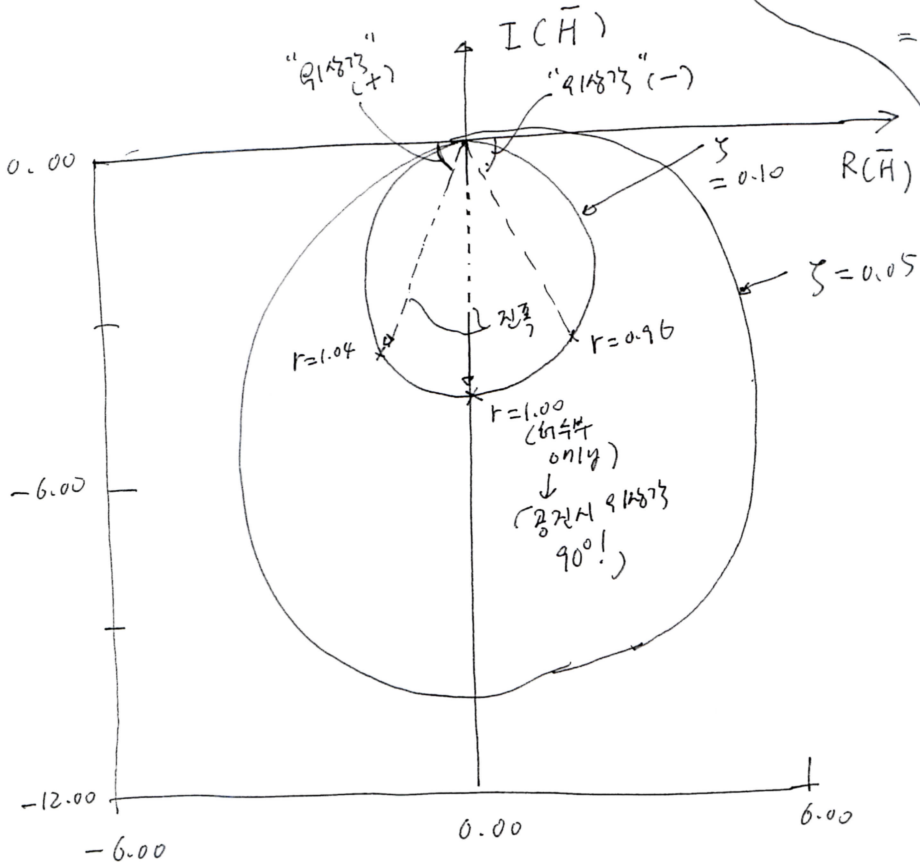
↳ 진폭과 위상 정보를 하나의 그림으로서
복소평면 상에 그린 그림, (33)식 이용

HW #.

$\zeta = 0.1,\ 0.05$ 에 대하여 vector response diagram을 작성할 것.

$$\bar{H}(\Omega) = \frac{1}{(1-r^2)+i(2\zeta r)} = \frac{(1-r^2)-i(2\zeta r)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

where $r = \frac{\Omega}{\omega_n}$



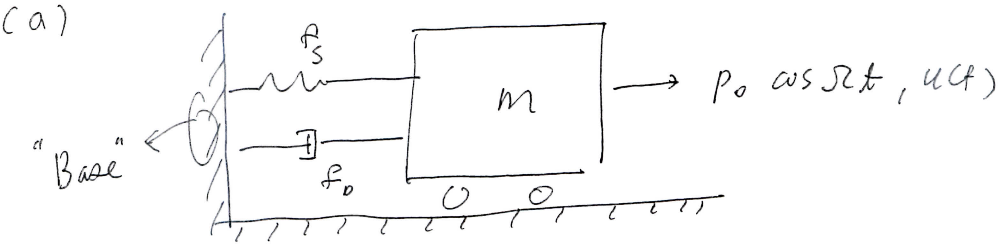
Application to vibration isolation problem

- (a) Force transmitted to the base (force transmissibility)
- (b) Moving base

It is <u>convenient</u>, although not essential, to use complex frequency response technique to solve these two problems.

" 이미 Chopra 책에서 구한 것 "

(a)



The force transmitted to the base <u>in vector form</u>,

$$\bar{f}_{tr} = \bar{f}_S + \bar{f}_D = k\bar{u} + c\dot{\bar{u}} \qquad \left(\begin{aligned}\bar{u} &= \bar{U} e^{i\Omega t}\\ \dot{\bar{u}} &= i\Omega \bar{U} e^{i\Omega t}\end{aligned}\right.$$

$$= (k + ic\Omega)\bar{U} e^{i\Omega t}$$

$$\bar{U} = U_0 \bar{H}(\Omega)$$

$$= \frac{(k + ic\Omega)U_0}{(1-r^2) + i(2\zeta r)} \times e^{i\Omega t} = \left[\frac{1 + i(2\zeta r)}{(1-r^2) + i(2\zeta r)}\right] k U_0 e^{i\Omega t}$$

polar form $Ae^{i\alpha}$

polar form $Be^{i\beta}$

결국 $\dfrac{A}{B} k U_0 e^{i(\Omega t - \alpha - \beta)}$

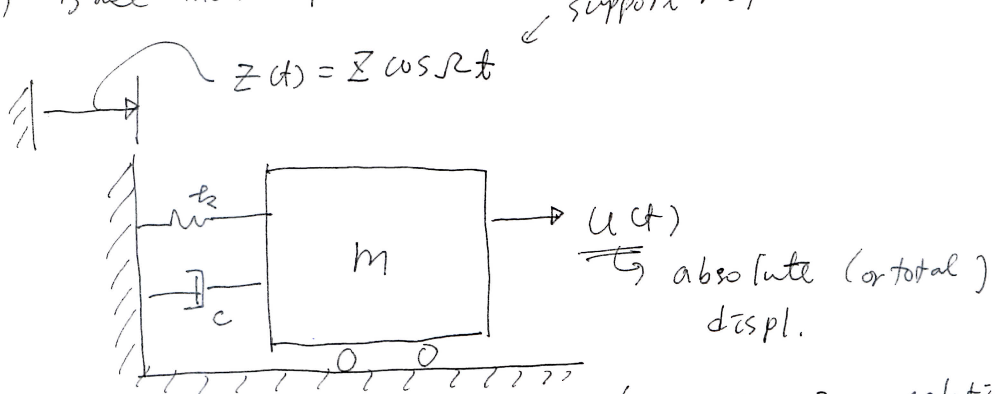
이 절대값을 취하면

$\left(|e^{i*}| = 1.0\right)$

$$\text{따라서} \quad \left|\bar{f}_{tr}\right| = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2) + (2\zeta r)^2}} \times k U_0$$

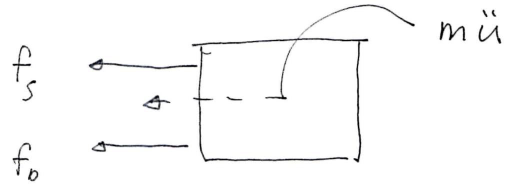
$$\therefore TR \equiv \frac{\bar{f}_{tr}}{kU_0} = \boxed{}$$

(b) Base motion



$(w = u - z$ = relative displ.$)$

$u = w + z$



$$f_s = k(u-z) = kw$$
$$f_D = c(\dot{u}-\dot{z}) = c\dot{w}$$

$\overset{+}{\leftarrow}\ \sum X_i = 0, \quad m\ddot{u} + c(\dot{u}-\dot{z}) + k(u-z) = 0$

$$m\ddot{u} + c\dot{u} + ku = c\dot{z} + kz \quad \text{--- (a)}$$

or $\quad m(\ddot{w}+\ddot{z}) + c\dot{w} + kw = 0$

$$m\ddot{w} + c\dot{w} + kw = -m\ddot{z} \quad \text{--- (b)}$$

위 두 식을 복소방정식으로 표시하면 $\quad \bar{z} = Z e^{i\Omega t}$

$$\begin{cases} m\ddot{\bar{u}} + c\dot{\bar{u}} + k\bar{u} = c\dot{\bar{z}} + k\bar{z} = (k + i\Omega c) Z e^{i\Omega t} & \text{--- (44)} \\ m\ddot{\bar{w}} + c\dot{\bar{w}} + k\bar{w} = -m\ddot{\bar{z}} = (+\Omega^2 m) Z e^{i\Omega t} & \text{--- (45)} \end{cases}$$

Assuming complex steady-state responses of the form

$$\bar{u} = \bar{U} e^{i\Omega t}, \quad \bar{w} = \bar{W} e^{i\Omega t} \quad \text{--- (46)}$$

We get the complex frequency response functions,

$$\frac{\bar{U}}{Z} = \frac{k + ic\Omega}{(k - m\Omega^2) + ic\Omega}, \quad \frac{\bar{W}}{Z} = \frac{m\Omega^2}{(k - m\Omega^2) + ic\Omega}$$

以下 생략

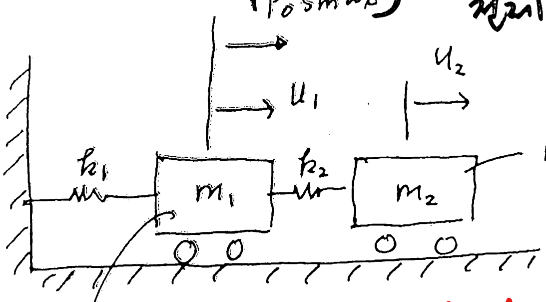
< ~~수강~~ 보충자료 > 12강 'Passive' ①

↓

Two-DOF systems without damping : Tuned Mass Damper

Harmonic motion을 제시 ($P_0 \sin\omega t$)

(TMD : Vibration absorbers to be in resonance with a particular mode of the structure. 20세기 이후 특허)



mass to control vibration of m_1

main system

$u = \sum \phi_m \times q_m$ $= [\,:\,] \times \sin\omega t$

$q_m(t) = \square \times \sin\omega t$

$\ddot{q}_m(t) + \omega_m^2 q_m(t) = \phi_m^T \begin{Bmatrix} P_0 \\ 0 \end{Bmatrix} \times \sin\omega t = \Gamma_m \cdot \sin\omega t$ 근거

Eq. of motion :

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{Bmatrix} P_0 \\ 0 \end{Bmatrix} \sin\omega t \quad \text{--- (1)}$$

Because the system is undamped, the steady-state solution can be assumed as

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} u_{10} \\ u_{20} \end{Bmatrix} \times \sin\omega t \quad \text{--- (2)}$$

비감쇠이므로

P.68, Eq (3.1.9) 상기

(모드중첩 mode superposition method 적용가능)

(Damping 역시 모형에 포함 가능)

Substituting this into Eq. (1),

$$\begin{bmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{20} \end{bmatrix} = \begin{bmatrix} P_0 \\ 0 \end{bmatrix} \quad \text{--- (3)}$$

or

$$[\,\underline{k} - \omega^2 \underline{m}\,] \begin{bmatrix} u_{10} \\ u_{20} \end{bmatrix} = \begin{Bmatrix} P_0 \\ 0 \end{Bmatrix} \quad \text{--- (4)}$$

Cramer's rule에 불과

$$\begin{Bmatrix} u_{10} \\ u_{20} \end{Bmatrix} = \frac{\operatorname{adj}[\underline{k} - \omega^2 \underline{m}]}{\det[\underline{k} - \omega^2 \underline{m}]} \times \begin{Bmatrix} P_0 \\ 0 \end{Bmatrix} \quad \text{--- (5)}$$

HW 확인요

or

$$u_{10} = \frac{P_0 (k_2 - m_2\omega^2)}{m_1 m_2 (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \; ; \; u_{20} = \frac{P_0 k_2}{(\quad " \quad)} \quad \text{--- (6)}$$

(ω_1 및 ω_2는 $\det[\underline{k} - \omega^2 \underline{m}] = 0$의 해) [2-DOF 계의 1차, 2차 frequency.]

고유치 해석 결과

「TMD의 강성과 질량이 main system의 1/2 인 때」 ↳ Not good! (too heavy)

Example : $m_1 = 2m$, $m_2 = m$, $k_1 = 2k$, $k_2 = k$ 이면

$\omega_1 = \sqrt{k/2m}$, $\omega_2 = \sqrt{2k/m}$ 이 되므로 이들 (6)식에 대입하면

$= \sqrt{4\left(\frac{k}{2m}\right)} = 2\omega_1$

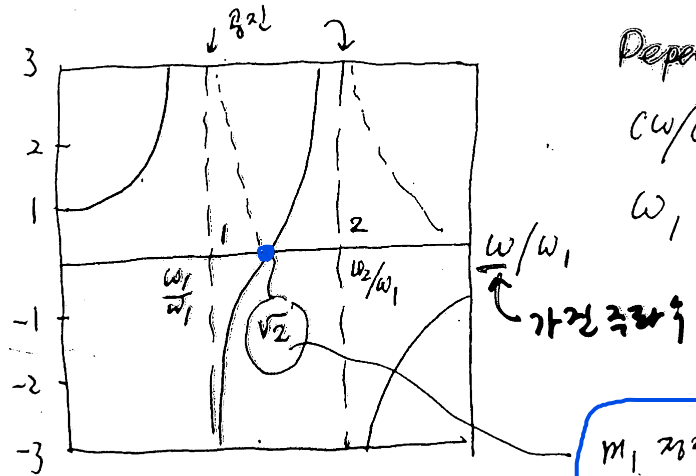
소위 mass ratio $\mu = 0.50$

$$\frac{u_{10}}{(u_{1st})_0} = \frac{1 - \frac{1}{2}(\omega/\omega_1)^2}{[1-(\omega/\omega_1)^2][1-(\omega/\omega_2)^2]} \quad \cdots (7)$$

$$\frac{u_{20}}{(u_{2st})_0} = \frac{1}{[1-(\omega/\omega_1)^2][1-(\omega/\omega_2)^2]}$$

where $(u_{1st})_0 = p_0/2k$; $(u_{2st})_0 = p_0/2k$

Main system response



가진 주파수

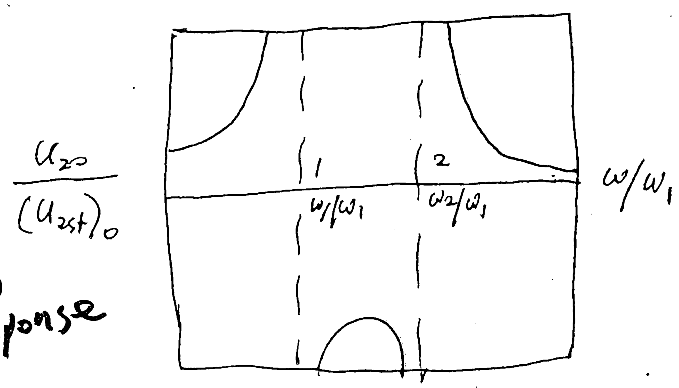
Depend on frequency ratios (ω/ω_1) and (ω/ω_2), not separately on ω, ω_1 and ω_2.

(note : $\omega_1 < \omega_2$)

m_1 정지 (TMD 설계 원리) → $\omega/\omega_1 = \sqrt{2}$ or $\omega_1 = \frac{\omega}{\sqrt{2}}$

↓

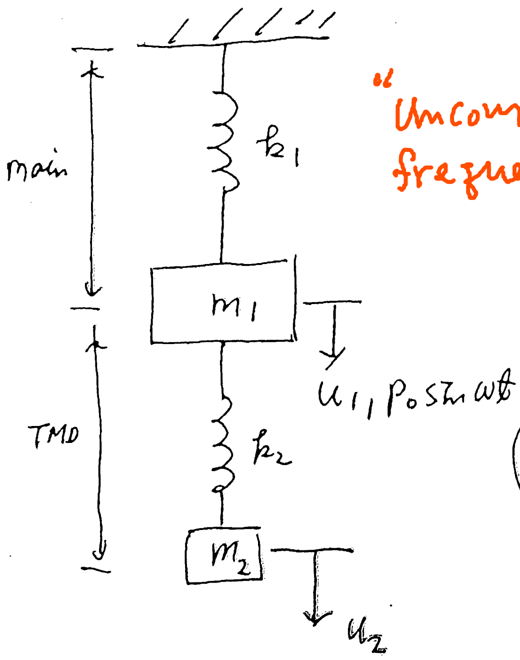
「the entire basis of TMD」



TMD response

(The basic principle of a vibration absorber)
without getting into the many important aspects of its practical design.

"symbol 변경"



Eq (6) 에서 $\omega_1^* = \sqrt{\frac{k_1}{m_1}}$, $\omega_2^* = \sqrt{\frac{k_2}{m_2}}$, $\mu = \frac{m_2}{m_1}$ 의 기호 도입

"Uncoupled" frequencies

mass ratio

$$u_{10} = \frac{p_0}{k_1} \times \frac{1-(\omega/\omega_2^*)^2}{[1+\mu(\omega_2^*/\omega_1^*)^2-(\omega/\omega_1^*)^2] \times [1-(\omega/\omega_2^*)^2] - \mu(\omega_2^*/\omega_1^*)^2} \quad \text{--- (a)}$$

$$u_{20} = \frac{p_0}{k_1} \times \quad \text{생략} \quad \text{--- (b)}$$

[$\omega = \omega_2^*$ 에서는 아예 완전히 정지]

(Specialization)

$\mu = 0.2$, $\omega_1^* = \omega_2^*$ 로 가정 (곧 TMD를 main system에 고유주파수에 tuning)

↳ 대부분 경우 사용

($\frac{1}{5}$) → Not good, too heavy!

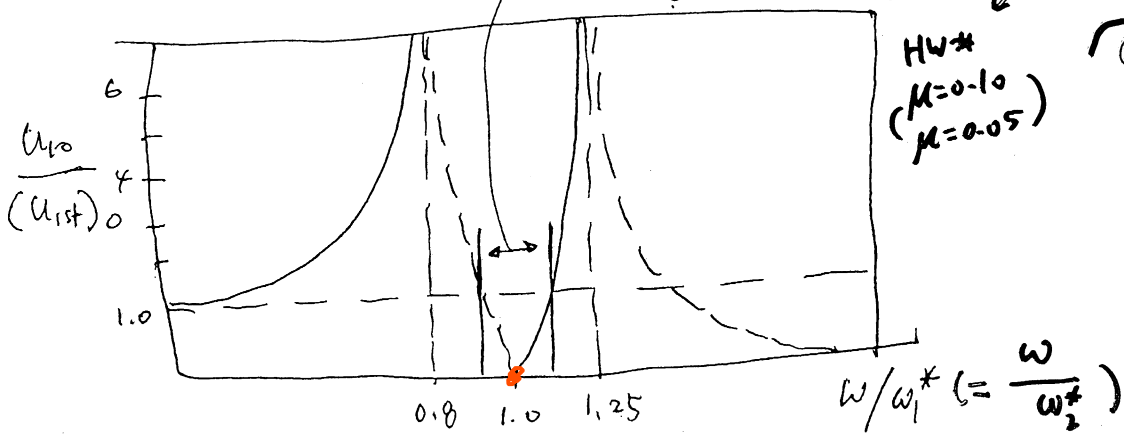
$$\frac{u_{10}}{(p_0/k_1)} = \frac{1-(\omega/\omega_1^*)^2}{[1+\mu-(\omega/\omega_1^*)^2][1-(\omega/\omega_1^*)^2]-0.2} \le 1.$$

$(u_{1st})_0$

μ: 0.2, $1+\mu$: 1.2

(Operating range): $(u_{10}/(u_{1st})_0 < 1.0)$

→ mass ratio가 작아지면 이 범위 더 좁아지게 됨



HW*
($\mu = 0.10$, $\mu = 0.05$)

(Operating range)
↑
그렇듯한 설정, 정하기 나름 (성능우주)

Note : ① At $\omega = \omega_1^*$, the response amplitude of the main mass alone is unbounded.

TMD 없이는 공진

"TMD가 가장 effective 한 경우"

Thus, if the exciting frequency (ω) is close to the natural frequency of the main system ω_1^* and operating restrictions make it impossible to vary either one, the TMD can be used to reduce the response of the main system to near zero.

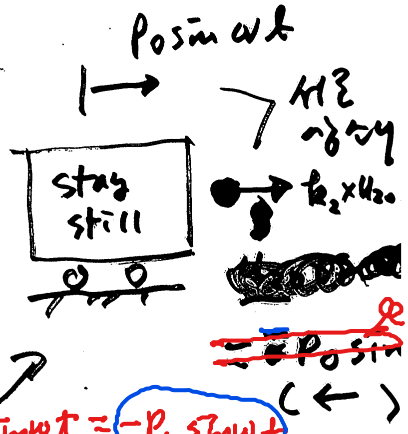
[운용조건상 가진주파수 및 main system 변수(고유주기)의 변동이 불가능한 경우 (즉 여건상 $\omega_1 \doteq \omega$ 조건이 불가피한 경우)]

② ~~What should be~~ the size of the absorber mass ?

To answer this question, use Eq. (b)

~~To answer this question~~, use Eq. (b) to determine ~~the motion~~ of the TMD at $\omega = \omega_2^*$;

$$u_{20} = -\frac{P_0}{k_2}$$



The force acting on the absorber mass is

$$k_2 u_{20} = \omega^2 m_2 u_{20} = -P_0$$

$k_2 u_2(t) = k_2 u_{20} \sin\omega t = -P_0 \sin\omega t$

This implies that the absorber exerts a force equal and opposite to the exciting force.

따라서 m_2 및 k_2 의 선택은 u_{20}의 허용치 한계에 좌우 된다.
다른 요인도 고려해야, mass가 클수록 실용적 불리 다수.
mass ratio가 너무 작으면 운동 주파수 범위도 협소해짐

→ 결국 TMD는 거의 일정 주파수로 운영되는 synchronous 기계류에 가장 큰 응용가능성이 있음

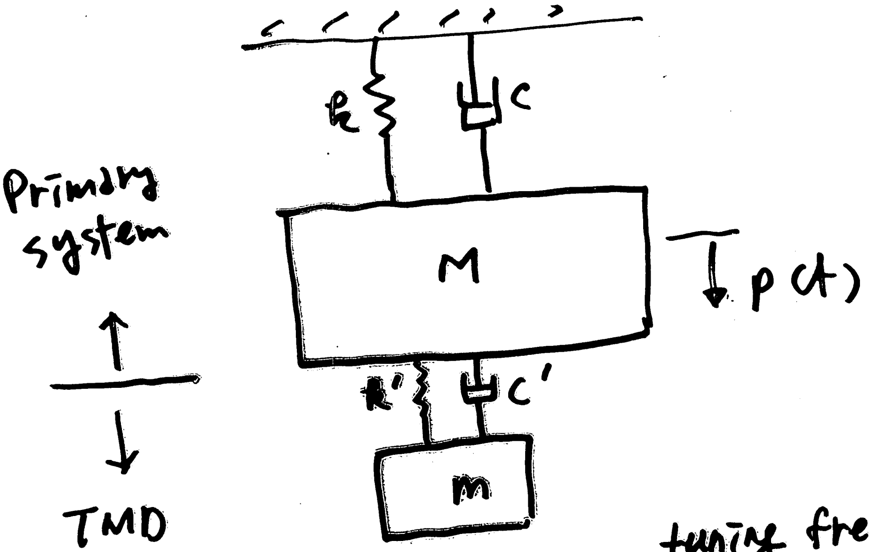
↳ 왜냐하면 TMD는 1개 주파수에 튜닝되어 협소 주파수에 대해서만 유효하므로

The use of Multiple TMD! ← use multi-

However, vibration absorbers are also used in situation where the excitation is not nearly harmonic.

ex): wind-induced vibration / floor vibration

"TMD를 활용한 실제 문제 해결의 향상":



Find optimal tuning frequency, mass (m) and damping parameters of a TMD.

Harmonic input

Ex) 조화하중 작용시 가속도응답을 최소화하기 위한 최적 tuning 조건 일례

i) $\dfrac{\omega_{TMD}}{\omega} = \dfrac{1}{\sqrt{1+\mu}}$

ii) $\xi_{TMD} = \sqrt{\dfrac{3\mu}{8(1+\mu/2)}}$

where $\mu = m/M$ = mass ratio

* 실제로 현장계측을 통하여 시행착오적으로 최적 tuning 조건을 찾음

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From Christopoulos and Filiatrault (2006)

Tuned Mass Dampers (TMDs)

동조질량 감쇠기.

1. Introduction

① TMDs : Vibration absorbes, relatively small mass-spring-dashpot systems calibrated to be in resonance with a particular mode of the structure on which they are installed. → 통상 기본(1차) 진동모드

(고층건물의 풍진동 저감, 바닥진동 저감, 최근 내진설계에도 응용) Less popular

② Frahm (1909)의 미국특허로 시작

(가령, 건물중량 50,000 ton에 mass ratio $\frac{1}{100}$ 이라면?)

③ 비교적 설치가 간편 (no need for an external power source) ATMD

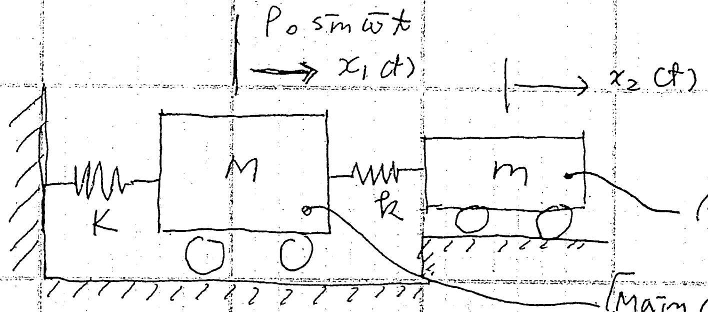
단점 : 여전히 큰 mass, 큰 공간 필요 등등

(모두 mode-superposition method로 일관된 해석이 가능) linear behavior를 전제로

passive vs active (수동 대 능동)

2. Theory of Undamped TMDs under Harmonic Loading
3. Theory of " " " Base Motion
4. Theory of Damped TMDs under Harmonic Loading
5. Application to Earthquake Engineering
6. Analysis of structures with TMDs
7. Seismic Response of Inelastic Buildings with TMDs
8. Design ~~considerations~~ procedure

2. Theory of "Undamped TMDs" under Harmonic Loading ↳ 가장 기본 — 수학적으로 가장 간단



(TMD system, Vibration absorber)

(Main (primary) system)

Eq. of motion

* (일반적으로 Mode superposition method 로 해석 가능)

$$\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K+k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} P_0 \\ 0 \end{bmatrix} \times \sin \bar{\omega} t \quad -(1)$$

「비감쇠이므로 정상상태응답은 간단한 형태가 됨」 조화진동 형태

$$\underline{X}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \times \sin \bar{\omega} t \quad \cdots (2)$$

(2) → (1), a_1 과 a_2 에 대해 풀면

$$\frac{a_1}{x_{st}} = \frac{\left(1 - \frac{\bar{\omega}^2}{\omega_a^2}\right)}{\left(1 - \frac{\bar{\omega}^2}{\omega_a^2}\right)\left(1 + \frac{k}{K} - \frac{\bar{\omega}^2}{\Omega_m^2}\right) - \frac{k}{K}} \quad \cdots (3)$$

$$\frac{a_2}{x_{st}} = \frac{1}{\left(1 - \frac{\bar{\omega}^2}{\omega_a^2}\right)\left(1 + \frac{k}{K} - \frac{\bar{\omega}^2}{\Omega_m^2}\right) - \frac{k}{K}} \quad \cdots (4)$$

where $x_{st} = \frac{P_0}{K}$

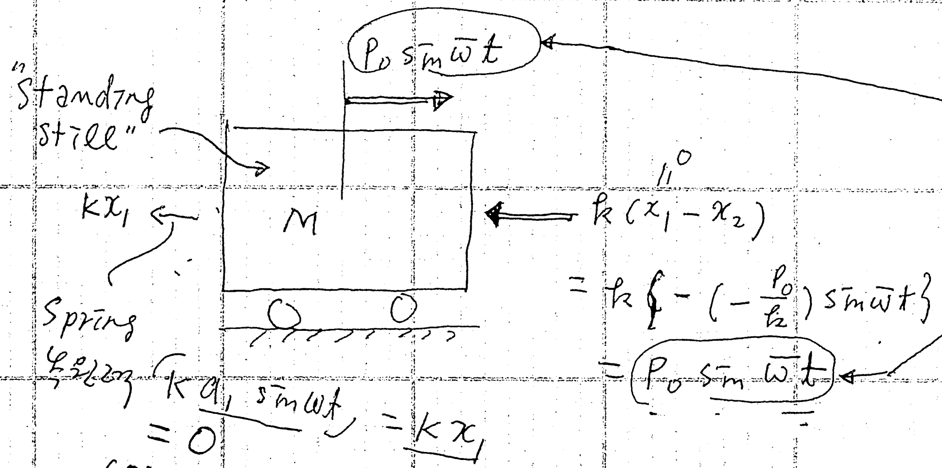
$\Omega_m^2 = \frac{K}{M}$

$\omega_a^2 = \frac{k}{m}$ "attached"

※ (TMD 주파수를 가진주파수에 동조시킴)

I. (i) (3)식에 의해, ($\omega_q = \sqrt{\frac{k}{m}} = \bar{\omega}$ 이면, $q_1 = 0$ 이 되어 주구조물 진동치 않음, 이 경우 (4)식에 의해

$$\frac{q_2}{x_{st}} = \frac{1}{0 - \frac{k}{K}}, \text{ or } q_2 = -\frac{K}{k}x_{st} = -\frac{P_0}{k} \quad ---(5)$$



II. (ii) $\omega_q = \Omega_m$ 일때 (TMD를 주구조체에 동조시킬 때)

즉, $\frac{k}{m} = \frac{K}{M}$

or, $\frac{k}{K} = \frac{m}{M} = \mu$ — — — (6) ※ (mass (or stiffness) ratio) (TMD 설계의 핵심변수)

From Eq. (3) and (4),

$$\frac{q_1}{x_{st}} = \frac{(1 - \frac{\bar{\omega}^2}{\omega_q^2})}{(1 - \frac{\bar{\omega}^2}{\omega_q^2})(1 + \mu - \frac{\bar{\omega}^2}{\omega_q^2}) - \mu} \quad ---(6)$$

$$\frac{q_2}{x_{st}} = \frac{1}{(\quad " \quad)} \quad ---(7)$$

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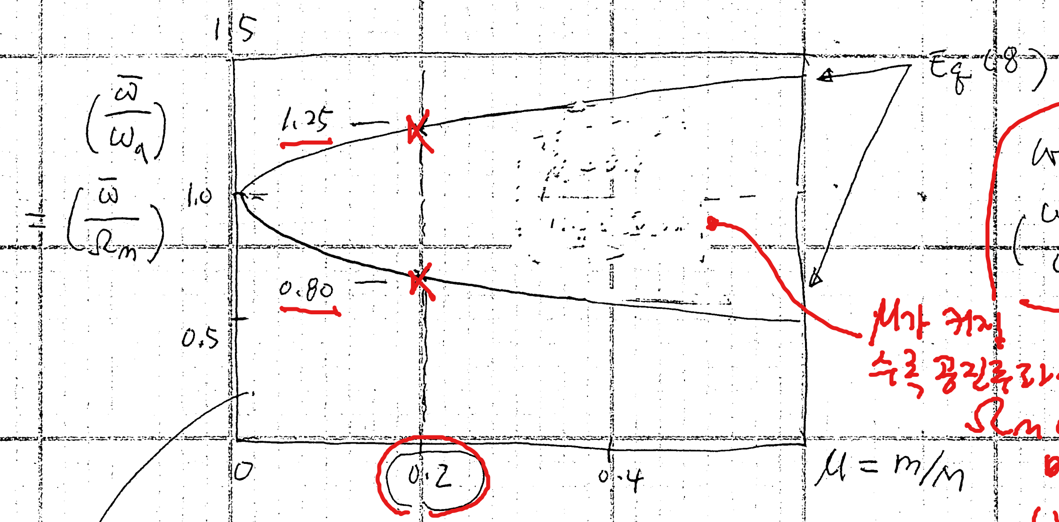
Note : (6), (7)식 분모 동일, $\left(\frac{\bar{\omega}}{\omega_a}\right)^2$에 대한 2차 방정식으로서 2개의 근이 존재,

$$\left(\frac{\bar{\omega}}{\omega_a}\right)^4 - \left(\frac{\bar{\omega}}{\omega_a}\right)^2 (2+\mu) + 1 = 0$$

$$\left(\frac{\bar{\omega}}{\omega_a}\right) = \sqrt{\left(1+\frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}}} \quad \left(= \frac{\bar{\omega}}{\Omega_m}\right) \quad ---(8)$$

(q_1, q_2를 ∞로 증폭시키는 주파수비로서 "주구조체와 TMD"가 결합된 시스템의 2개 고유주파수를 알 수 있게 해줌)

↳ (고유치해석 결과와 동일한 것임)



When $\mu = 0.2$, ($\omega_1 = 0.80\,\Omega_m$, $\omega_2 = 1.25\,\Omega_m$)

μ가 커질수록 공진주파수가 Ω_m에서 멀어진다 (비감쇠)

: $(\bar{\omega}/\omega_a)$ versus μ relationship.

("mass ratio $\mu = 0.2$로서 $\omega_a = \Omega_m$이 되도록 튜닝하면 가진주파수 $\bar{\omega}$가 Ω_m (또는 ω_a)의 0.80 및 1.25배가 되면 공진이 발생하고, 또한 이때에는, 이 2개의 공진주파수가 전체 시스템의 2개 고유주파수이다")

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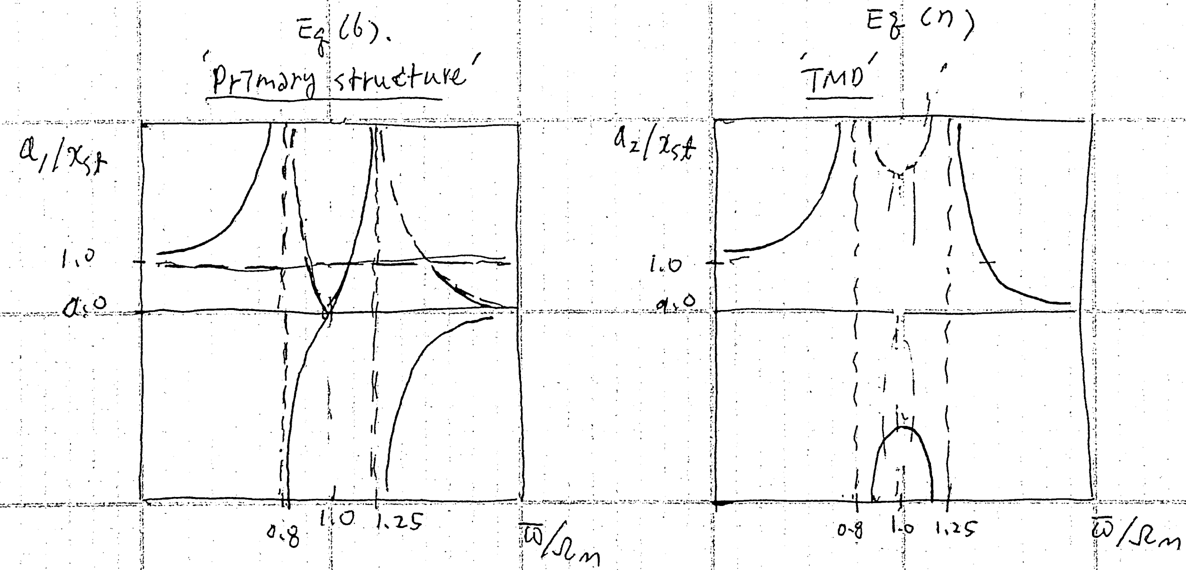


: Amplitude spectrum ($\omega_q = \Omega_m$, $\mu = 0.2$)

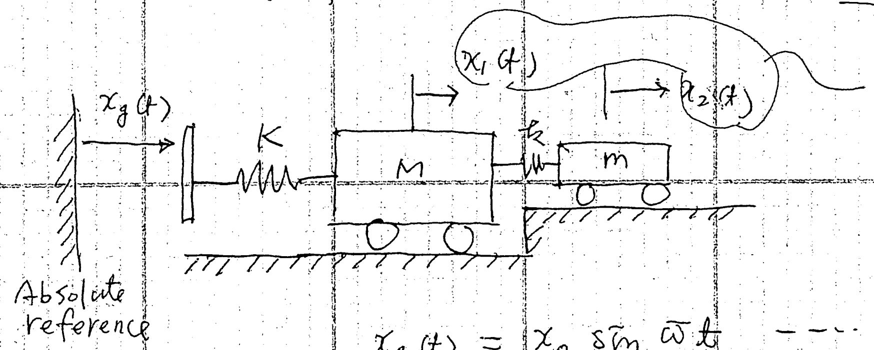
~~3 Theory of Damped TMDs under Harmonic Loading~~ P.4

3. Theory of Undamped TMDs under "Harmonic Base Motion"

→ 조화지반진동 → 노면조도 등에 의한 support motion

Just glance.

모두 상대변위



$$x_g(t) = x_0 \sin \bar{\omega} t \quad \cdots\cdots (1)$$

Equation of motion:

$$\begin{cases} M \dfrac{d^2}{dt^2}(x_g + x_1) + K x_1 + k(x_1 - x_2) = 0 \\ m \dfrac{d^2}{dt^2}(x_g + x_2) + k(x_2 - x_1) = 0 \end{cases}$$

$$\begin{cases} M \ddot{x}_1 + (K+k) x_1 - k x_2 = (M \bar{\omega}^2 x_0) \sin \bar{\omega} t \\ m \ddot{x}_2 - k x_1 + k x_2 = (m \bar{\omega}^2 x_0) \sin \bar{\omega} t \end{cases} \quad \cdots (2)$$

(2) 식에 $\left(\underline{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \times \sin \bar{\omega} t \right)$ 형태의 해를 대입하여 q_1, q_2를 구하면,

Solution form

(symbol은 이전과 동일)

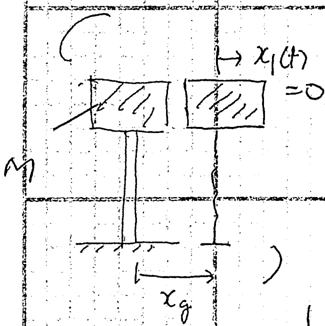
$$\frac{q_1}{x_0} = \frac{\bar{\omega}^2 \left[\frac{1}{\Omega_m^2} \left(1 - \frac{\bar{\omega}^2}{\omega_a^2} \right) + \frac{1}{\omega_a^2} \left(\frac{k}{K} \right) \right]}{\left(1 - \frac{\bar{\omega}^2}{\omega_a^2} \right) \left(1 + \frac{k}{K} - \frac{\bar{\omega}^2}{\Omega_m^2} \right) - \frac{k}{K}} \quad \text{--- (3)}$$

$$\frac{q_2}{x_0} = \boxed{\text{생략}}$$

i) The first tuning condition → $q_1 = 0$

↳ (main mass M의 지반에 대한 상대변위가 zero가 되도록, 즉 main mass M이 지반과 같이 rigid하게 거동하는 것으로서 "주구조체는 $\ddot{x}_g(t)$의 절대(total) 가속도를 경험하되 main-spring에 응력을 유발하지 않음"

↳ (stress-free)



↳ $q_1 = 0$, 즉 (3)식의 분자가 zero가 되도록 하려

$$\frac{1}{\cancel{\Omega_m^2}} \left(1 - \frac{\bar{\omega}^2}{\omega_a^2} \right) + \frac{1}{\cancel{\omega_a^2}} \left(\frac{k}{K} \right) = 0 \quad \text{--- (4)}$$

$k = m\omega_a^2$, $K = M\Omega_m^2$

$$1 - \frac{\bar{\omega}^2}{\omega_a^2} + \frac{m}{M} = 0, \quad \left(\frac{m}{M} = \mu\right)$$

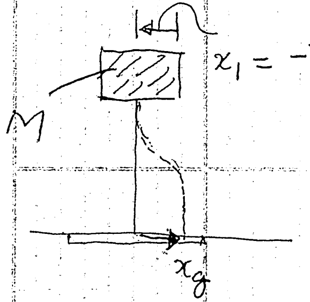
$$\left(\frac{\bar{\omega}^2}{\omega_a^2} \right) = \mu + 1$$

$$\therefore \omega_a = \frac{\bar{\omega}}{\sqrt{1+\mu}} \quad \text{--- (5)}$$

ii) The 2nd tuning condition → 주 질량이 정지할 조건

↳ Main mass 의 절대변위가 zero → $\left(x_1^{total} = x_1 + x_g = 0 \right)$

(곧 주질량이 정지할 조건)



따라서 $x_1(t) = -x_g(t)$ → $a_1 \sin \bar{\omega} t = -x_0 \sin \bar{\omega} t$

↳ (스프링에 $-k\,x_g(t)$ 의 응력은 유발이 됨.)

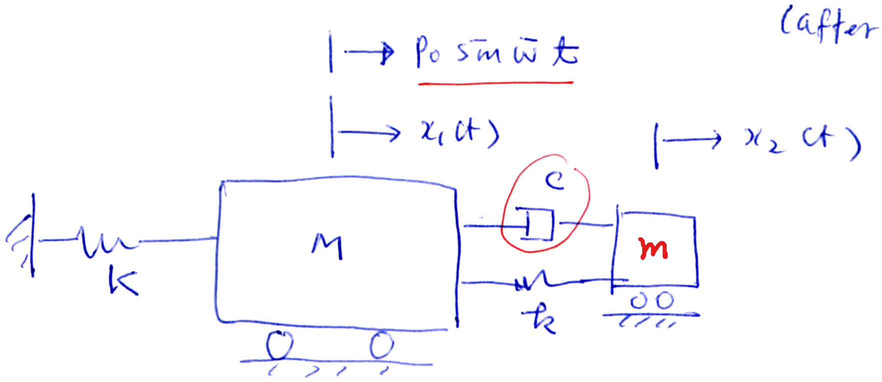
곧 $\left(a_1 = -x_0,\ \frac{a_1}{x_0} = -1 \right.$ 인 조건

(3) 식에 대입하여 정리

∴ $\omega_a = \bar{\omega}$ ---- (6)

4. Theory of "Damped" TMDs under Harmonic Loading

(after Den Hartog 1985) (1956)



Eq. of motion :

$$\begin{cases} M\ddot{x}_1 + (K+k)x_1 - kx_2 + c(\dot{x}_1 - \dot{x}_2) = P_0 \sin \bar{\omega} t \\ m\ddot{x}_2 + k(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) = 0 \end{cases} \quad \text{--- (1)}$$

(Mode superposition method 적용 가능)

정상상태응답은 $\bar{\omega}$에 대한 조화진동이므로 미지의 벡터성 복소수를 도입하여 표시하면 → (복소해석해 접근법)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{i\bar{\omega} t} \quad \text{--- (2)}$$

(왜 이런 solution form의 가정이 가능한가?) (진동과목 참고)

↑ (unknown complex number)

우리의 관심사는 main mass M의 진폭 (a_1)이므로

(2) → (1) 대입하여 a_1을 구하면,

→ 모두 대수 (+, −, ×, ÷) 연산

$$\frac{a_1}{x_{st}} \left(= \frac{|c_1|}{x_{st}} \right) = \left[\frac{\left(2\frac{c}{c_c}g\right)^2 + (g^2 - f^2)^2}{\left(2\frac{c}{c_c}g\right)^2 (g^2 - 1 + \mu g^2)^2 + [\mu f^2 g^2 - (g^2-1)(g^2-f^2)]^2} \right]^{1/2} \quad \text{--- (3)}$$

← (4개 변수의 함수 (μ, c/c_c, f, and g.)

where $\mu = \frac{m}{M}$ (= mass ratio)*

$\omega_a^2 = \frac{k}{m}$ (= natural freq. of TMD)

$\Omega_m^2 = K/M$ (= natural freq. of primary structure)

$f = \omega_a / \Omega_m$ (= natural frequency ratio)*

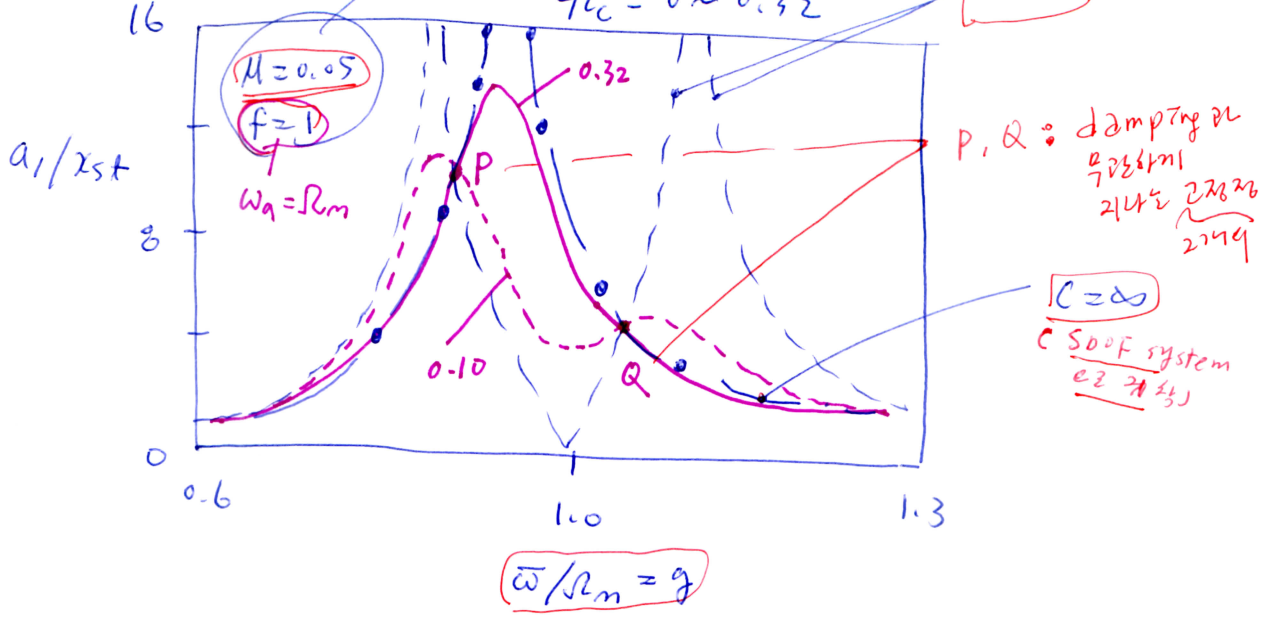
$g = \bar{\omega} / \Omega_m$ (= forcing " ")

$c_c = 2\omega_a m$ (= critical damping constant of TMD)

(독립변수) : f & $\frac{c}{c_c}$

• tuning 조건 (주구조체에 동조)

$c/c_c = 0 \sim 0.32$



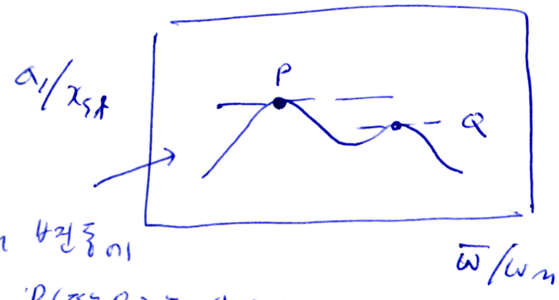
Note : i) $c=0$ → 비감쇠 TMD의 경우와 동일 (공진값 ∞)

ii) $c=\infty$ 이면, 2개 질량이 서로 합쳐져서 $(\frac{21}{20})M$ 의 질량을 갖는 SDOF계의 응답에 수렴 (공진값 ∞)

이 사이 어딘가에 'resonant peak'을 최소화시켜주는 c의 최적값이 존재할 것임

↳ The only purpose of a adding TMD

iii) 위 그래프의 4개 곡선은 damping 값과 상관없이 모두 P, Q 점을 지나고 있음. 만약 이들 위치가 정의될 수 있다면, 가장 바람직한 곡선은 P (또는) Q 점에서 접선 기울기가 수평이 되는 것이다.



(Natural frequency ratio) ↓ ω_a/Ω_m

f 값의 변동에 따라 P(또는 Q)는 상승하고 Q(또는 P)는 하강하는 거동)

① f를 적절히 설정하여 P, Q의 높이가 같아지도록 하면 이상적

② c/c_c 를 적절히 설정하여 응답곡선이 이들 점에서 접선기울기가 zero가 되도록 설정 (즉, P, Q 점이 peak 값이 되도록)

(i) The location of P and Q

Damping-Independent 하게 즉 (c/c_c) 항이 없어지는 조건을 찾아야.

(3) 식을 변경하여

$$\frac{a_1}{x_{st}} = \left[\frac{A\left(\frac{c}{c_c}\right)^2 + B}{C\left(\frac{c}{c_c}\right)^2 + D} \right]^{1/2} \quad \text{---- (4)}$$

($\frac{A}{C}$로 factor-out 가능해야!!)

where $A=(2g)^2$, $B=(g^2-f^2)^2$, $C=(2g^2)(g^2-1+\mu g^2)^2$
$D=(\mu f^2 g^2-(g^2-1)(g^2-f^2))^2$

만약 $\frac{A}{C}=\frac{B}{D}$ 라면, $\frac{B}{A}=\frac{D}{C}$ 의 관계가 성립, 따라서

$$\frac{a_1}{x_{st}} = \left[\frac{\left(\frac{A}{C}\right)\left(\frac{c}{c_c}\right)^2 + \frac{B}{C}}{\left(\frac{c}{c_c}\right)^2 + \frac{D}{C}} \right]^{1/2} = \left[\left(\frac{A}{C}\right) \left\{ \frac{\left(\frac{c}{c_c}\right)^2 + \frac{B}{A}}{\left(\frac{c}{c_c}\right)^2 + \frac{D}{C}} \right\} \right]^{1/2}$$

(4) 식의 우변

B/A

$\therefore \frac{B}{A} = \frac{D}{C}$ 가 되어야.

"대입"

Damping-Independent 해가 !!

$$\boxed{\frac{A}{C} = \frac{B}{D}} \quad \text{---- (5)}$$

(5) 식을 풀면,

$$g^4 - 2g^2\frac{1+f^2+\mu f^2}{2+\mu} + \frac{2f^2}{2+\mu} = 0 \quad \text{--- (6)}$$

이 2차방정식의 두 근이 고정점 P, Q를 줌

(6) 식의 두 근을 (3) 식에 대입하여 이 값을 등치시키면 (즉, P와 Q의 높이를 갖도록 하면)

$$f = \frac{1}{1+\mu} \quad \text{---- (7)}$$

a very simple relation.

(ii) Find the optimal damping $(c/c_{cr})_{opt}$

(ㄱ) 식을 (3)식에 대입하고 g에 대하여 미분한 값이 $(\bar{\omega}/\Omega_m)$

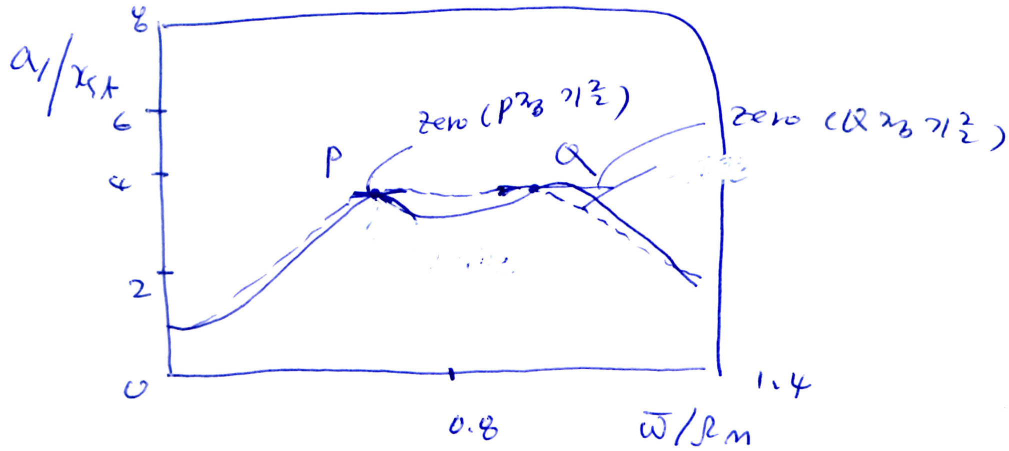
(ㄴ) 식에서 얻어진 두 곳에서 zero가 되도록 하여 $(c/c_{cr})_{opt}$를 구하면,

$$\left(\frac{c}{c_c}\right)^2_{opt-P} = \frac{\mu\left(3-\sqrt{\frac{\mu}{\mu+2}}\right)}{8(1+\mu)^3} \quad \text{--- (8)}$$

$$\left(\frac{c}{c_c}\right)^2_{opt-Q} = \frac{\mu\left(3+\sqrt{\frac{\mu}{\mu+2}}\right)}{8(1+\mu)^3} \quad \text{--- (9)}$$

별차 없음 → mean value

Ex) $\mu = 0.25$ (or $f = 0.80$) 일때



For practical applications, the two curves are almost identical. In practice, the mean value of Eq. (8) and (9) is used.

$$\left(\frac{c}{c_c}\right)_{opt} = \sqrt{\frac{3\mu}{8(1+\mu)^3}} \quad \text{---- (10)}$$

5. Application to Earthquake Engineering

상술한 최적기법은 (조화가진력에 대한 비감쇠 선형 SDOF계의 변위응답을 최소화하는 기준)에 근거한 것으로서 내진응용의 측면에서는 많은 개선의 논리가 있다.

i) 토건구조물은 거의 MDOF계임

ii) TMD를 설치해도 비선형 거동을 경험할 수 있음

iii) 지진하중은 random process loading임

iv) 변위 외의 다른 기준이 필요할 수도 (ex. 가속도 기준)

previous studies (Soong and Dargush 1997) 최적기준 요약

- 주구조물의 최소변위 (Den Hartog 1985, ...)
- ...
- Min. force in the main structure (Warburton 1982)

Table 1: Optimal tuning conditions for Damped TMDs attached to Undamped primary structure

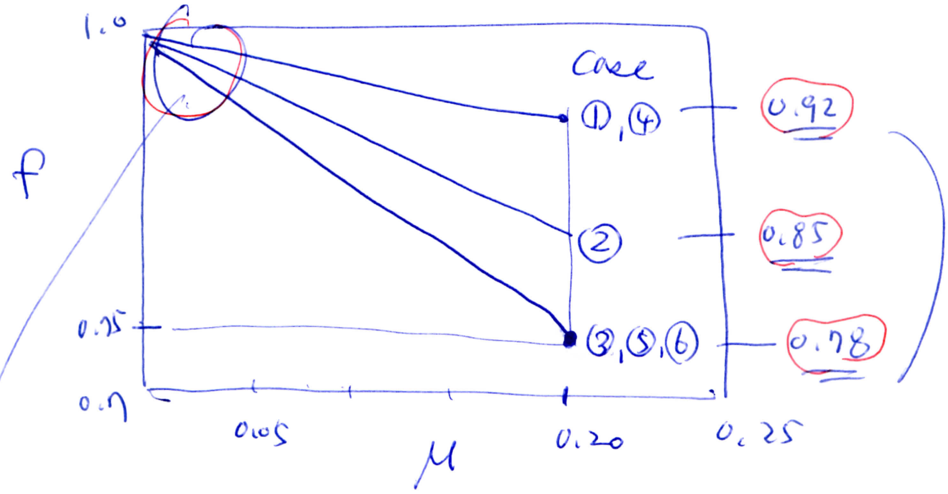
"동조주파수" "동조감쇠비"

Loading		Optimization criteria	Optimum tuning conditions $f\,(=\omega_a/\Omega_m)$	Optimum tuning conditions c/c_c
주구조 조화가진 (Den Hartog)	①	주구조 최소 상대 변위	$\left(\frac{1}{1+\mu}\right)$	$\sqrt{3\mu/8(1+\mu)^3}$
〃	②	주구조 최소 상대 가속도	$\left(\frac{1}{\sqrt{1+\mu}}\right)$	$\sqrt{3\mu/8(1+\mu/2)}$
조화 지반 가진	③	주구조 최소 상대 변위	$\frac{\sqrt{1-\mu/2}}{1+\mu}$	$\sqrt{\frac{3\mu}{8(1+\mu)(1-\mu/2)}}$
〃	④	주구조 최소 상대 가속도	$\frac{1}{1+\mu}$	$\sqrt{\frac{3\mu}{8(1+\mu)}}$
주구조 random 가진	⑤	주구조 상대변위 최소 RMS	$\frac{\sqrt{1-\mu/2}}{1+\mu}$	$\sqrt{\frac{\mu(1+3\mu/4)}{4(1+\mu)(1-\mu/2)}}$
random base acceleration	⑥	〃	$\frac{\sqrt{1-\mu/2}}{1+\mu}$	$\sqrt{\frac{\mu(1-\mu/4)}{4(1+\mu)(1-\mu/2)}}$

Notes : i) For very small TMDs ($\mu \fallingdotseq 0$) → "$f = 1.0$"

ii) Optimal damping constant 는 순전히 mass ratio에 의한 함수임.

iii) optimal frequency tuning 비교



(각종 mass ratio에서 큰 차이 없음)

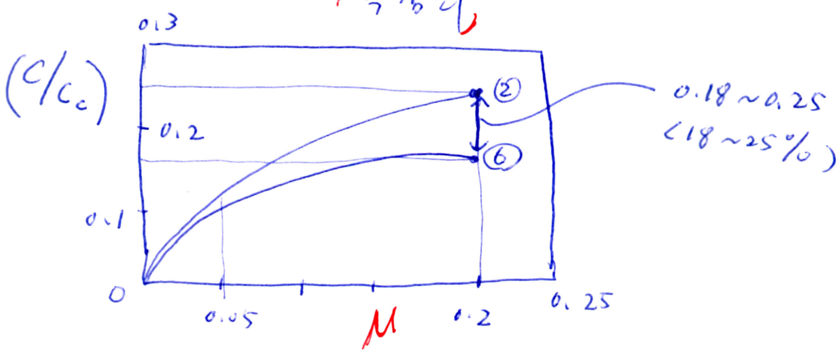
iv) Random base acceleration (case 6) 의 경우

$$(c/c_{cr})^2 = \frac{\mu(1-\mu/4)}{4(1+\mu)(1-\mu/2)} = \frac{\mu - \frac{1}{4}\mu^2}{4(1+\frac{1}{2}\mu - \frac{1}{2}\mu^2)}$$

$$= \frac{\mu - \frac{1}{4}\mu^2}{4 + 2\mu - 2\mu^2}$$

$\lim_{\mu \to 0} (c/c_{cr}) = 0$ (나머지 case도 동일)

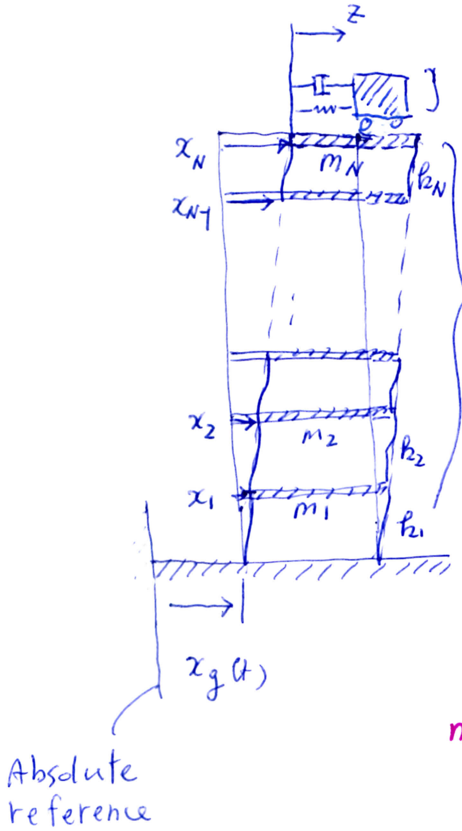
$\neq$ (무감쇠)



6. Analysis of structures with TMDs

○ 상기한 TMD tuning은 SDOF 계에만 유효
  ↳ 즉 단일 주파수에만 동조시킬 수 있음

○ 대부분의 building structures → MDOF 계 → (가장 지배적인 모드인 1차 진동모드에 동조시키는 것이 가장 효과적)

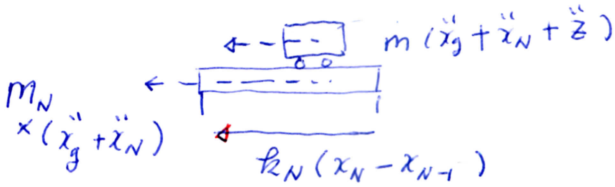


"TMDs on the roof" : m, c, k

★ z(t)는 지붕층 변위 $x_N(t)$에 대한 상대변위로 정의됨, $\underline{x}(t)$는 지반에 대한 상대변위로 정의됨.

Primary structure : $\underline{K}, \underline{M}$
(no damping)

"지붕층 운동방정식"



$$m_N \ddot{x}_N + k_N(x_N - x_{N-1}) = -m_N \times 1 \times \ddot{x}_g \underbrace{- m(\ddot{x}_g+\ddot{x}_N+\ddot{z})}_{(c\dot{z}+kz)}$$

: Modeling of MDOF structure with TMD on Roof

"(N+1)개의 2계 연립 상미분 문제"

$$\underline{M}\,\ddot{\underline{x}}(t) + \underline{K}\,\underline{x}(t) = -\underline{M}\,\underline{r}\,\ddot{x}_g(t) + \underline{p}(t) \quad \text{--- (1)}$$

← 주구조체 운동방정식

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{x}_N - m\ddot{x}_g \quad \text{--- (2)}$$

← TMD 운동방정식

$$\text{Where } \underline{p}(t) = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ c\dot{z}+kz \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -m(\ddot{z}+\ddot{x}_N+\ddot{x}_g) \end{bmatrix}}_{(2)\text{식 참고}}$$

"나머지 기호 같음"

Equivalent SDOF model by using the first mode shape $\underline{A}^{(1)}$,

or $\quad \underline{X}(t) = \underline{A}^{(1)} \times x_N(t) \quad \cdots (3)$

(3) → (1) 대입하고 $(\underline{A}^{(1)})^T$를 곱하면 (직교성 이용).
orthogonality

1차모드의 transpose

$$M_1 \ddot{x}_N(t) + K_1 x_N(t) = -\alpha_1 M_1 \ddot{x}_g + [?, ?, ??, 1.0] \times \underline{P}(t) \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ cz\ \text{위치} \end{bmatrix}$$

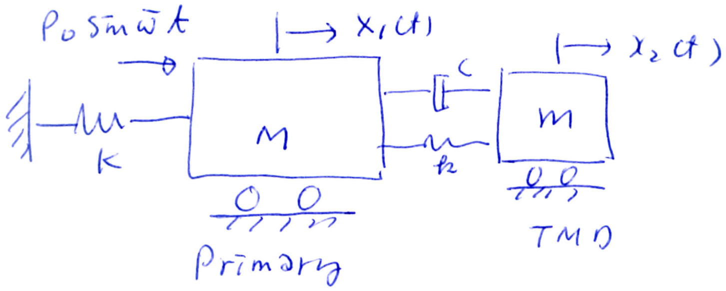
↳ 에만 중요

or
$$M_1 \ddot{x}_N(t) + K_1 x_N(t) = -\alpha_1 M_1 \ddot{x}_g + c\dot{z}(t) + k z(t) \quad \cdots (4)$$
$$m\ddot{z} + c\dot{z} + kz = -m\ddot{x}_N - m\ddot{x}_g \quad \cdots (2)$$

서로 couple 되어 있음 $\quad (*)$

where
$$\begin{cases} M_1 = (\underline{A}^1)^T \underline{M}\, \underline{A}^1 \\ K_1 = (\underline{A}^1)^T \underline{K}\, \underline{A}^1 \\ \alpha_1 = \dfrac{(\underline{A}^1)^T \underline{M}\, \underline{r}}{M_1} \end{cases}$$
(= modal part. factor in first mode)

Recall "조화하중을 받는 강체 TMD계의 운동방정식"



$$\begin{cases} M\ddot{x}_1(t) + Kx_1 = P_0 \sin\bar{\omega} t + c(\underbrace{\dot{x}_2 - \dot{x}_1}_{\dot{z}}) + k(\underbrace{x_2 - x_1}_{z}) \\ m\ddot{x}_2 + k(\underbrace{x_2 - x_1}_{z}) + c(\underbrace{\dot{x}_2 - \dot{x}_1}_{\dot{z}}) = 0 \end{cases} \quad (**)$$

("지붕층에 대한 상대값이 되어야.")

(만약 하중이 support motion $\ddot{x}_g(t)$로 작용한다면
위식은

주구조체
$$M(\cancel{\ddot{x}_g} + \ddot{x}_1) + Kx_1 = -M\ddot{x}_g + c(\underbrace{\dot{x}_2 - \dot{x}_1}_{\dot{z}}) + k(\underbrace{x_2 - x_1}_{z})$$

TMD
$$m(\cancel{\ddot{x}_g} + \cancel{\ddot{x}_1} + \ddot{z}) + c\dot{z} + kz = -m\ddot{x}_1 - m\ddot{x}_g \qquad (***)$$

$\ddot{z} = (\ddot{x}_2 - \ddot{x}_1)$

total accel

결국, (*), (**), (***) 식은 본질적으로 같은 운동방정식 형을 얻을 수 있다. 단 SDOF계의 질량이나 강성 파라메터를 modal 물성으로 대체하면 된다. 따라서 MDOF계의 1차진동모드에 대한 tuning은 선형 "2-DOF primary-TMD system"의 결과를 준용할 수 있다.

단, $\mu = \frac{m}{M_1}$; $\Omega_N = \omega_1$ ---- (5)

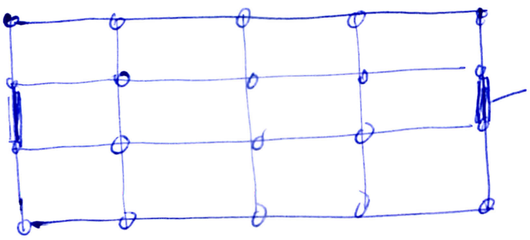
Note : i) This approach can also be used to mitigate the vibrations of any other mode

ii) A structure equipped with a TMD may, nevertheless, experience inelastic deformations during a strong earthquake.

↳ (강성 변화 (저하) → 주기변화 → may lose its effectiveness due to a de-tuning effect.)

7. An Example Response Analysis of Inelastic Buildings With TMDs (by Carr 2005)

Building model



($f = \frac{\omega_a}{\Omega_N}$ — $T_1 = 1.64$ (sec)

$T_a = \frac{2\pi}{\omega_a}$)

μ	T_a	c/c_c
0.05	1.72	0.13
0.10	1.80	0.17
0.20	1.97	0.21

Table의 case ④

Design information :

i) 10 story shear wall bldg

ii) $T_1 = 1.64$ (s) (gross section property-based)

iii) 남가주 수준의 지진위험도에 대하여 Canada 기준으로 설계

iv) 건물 1/2 part에 대하여 2차원 해석

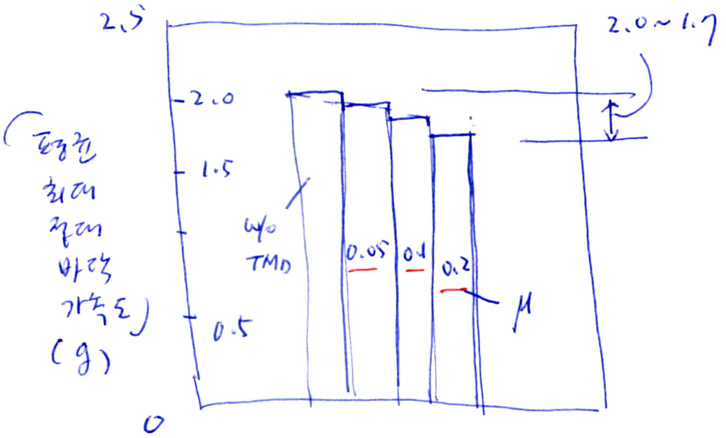
v) TMD 특성 : harmonic base acceleration → $f = \frac{1}{1+\mu}$; $c/c_c = \sqrt{\frac{3\mu}{8(1+\mu)}}$

vi) RUAUMOKO 이용 해석 ...

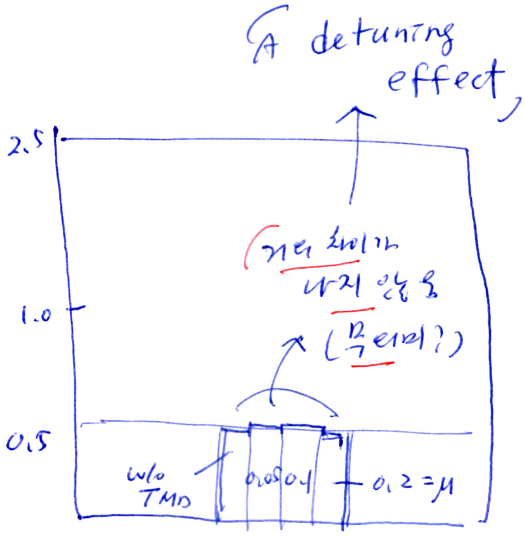
Input ground motions

↳ 인공지진동 입력 (a spectrum compatible Gaussian noise model, by Papageorgiou - Aki 1983)

Some of analysis results



: Elastic response
(무한한 강도로 가정)



: Inelastic response
(more realistic)

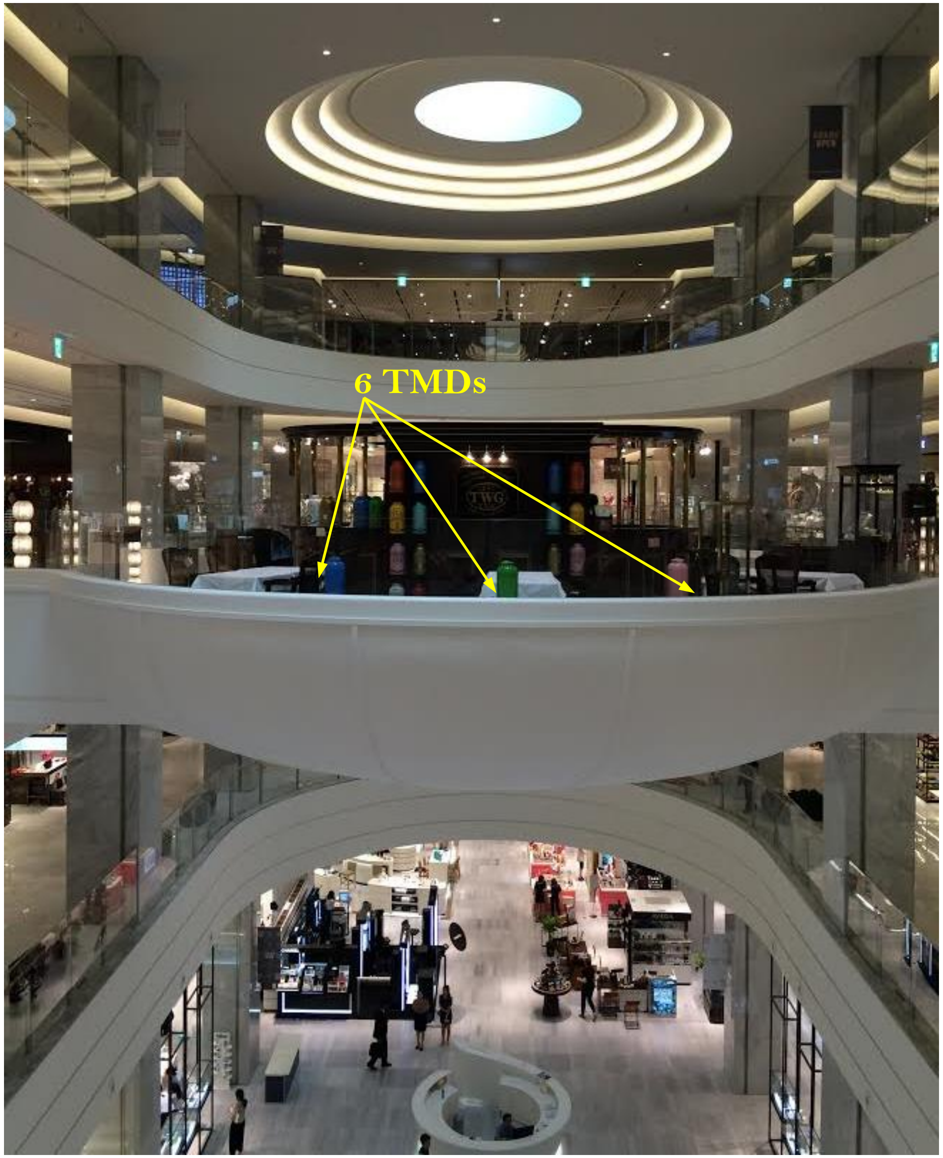
(개선 소지 상당!!?)

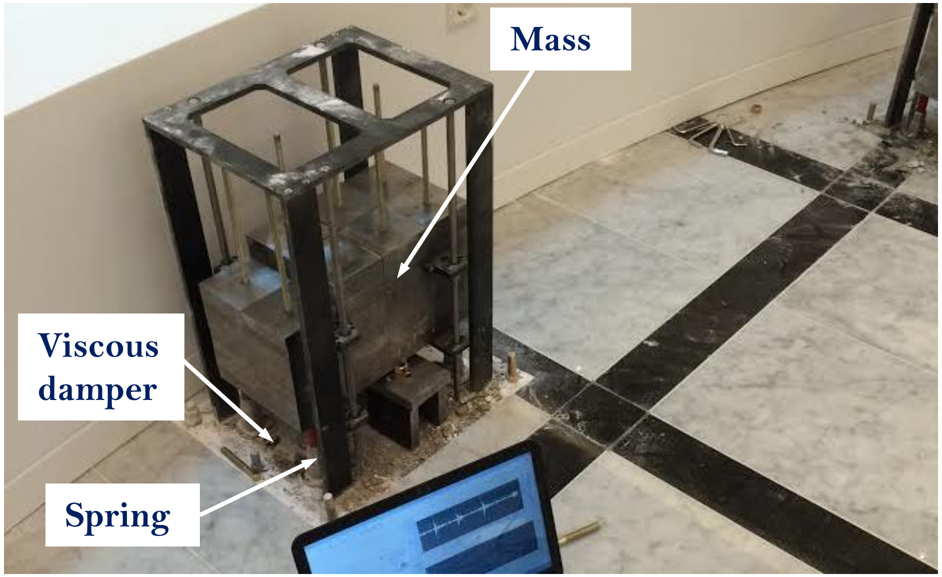
↓

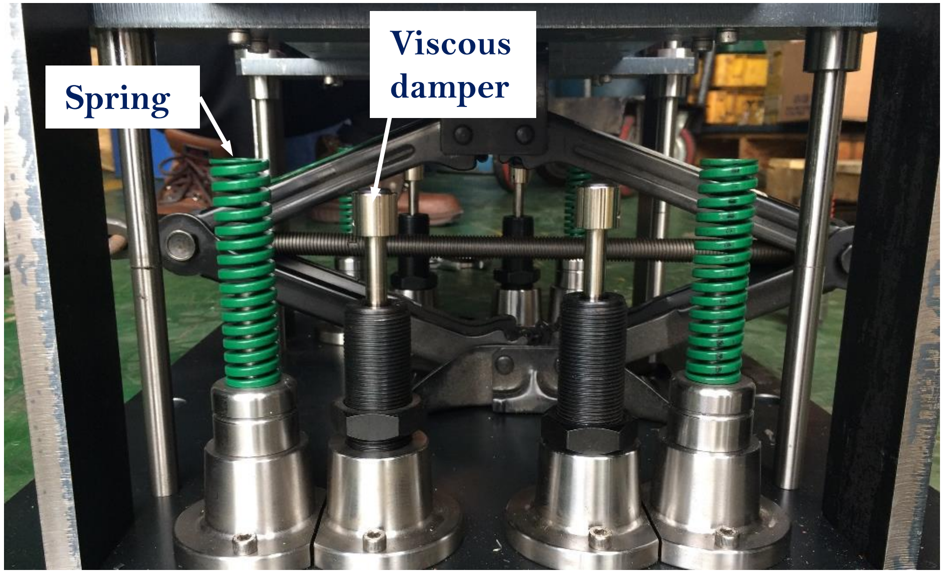
((상부구조물의 비선형해석을 통해 비선형거동을 미리
(pushover)
등가점성감쇠로 반영))

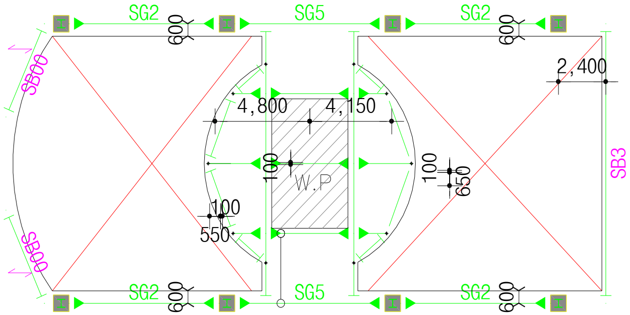
Part 1: Tuned Mass Dampers

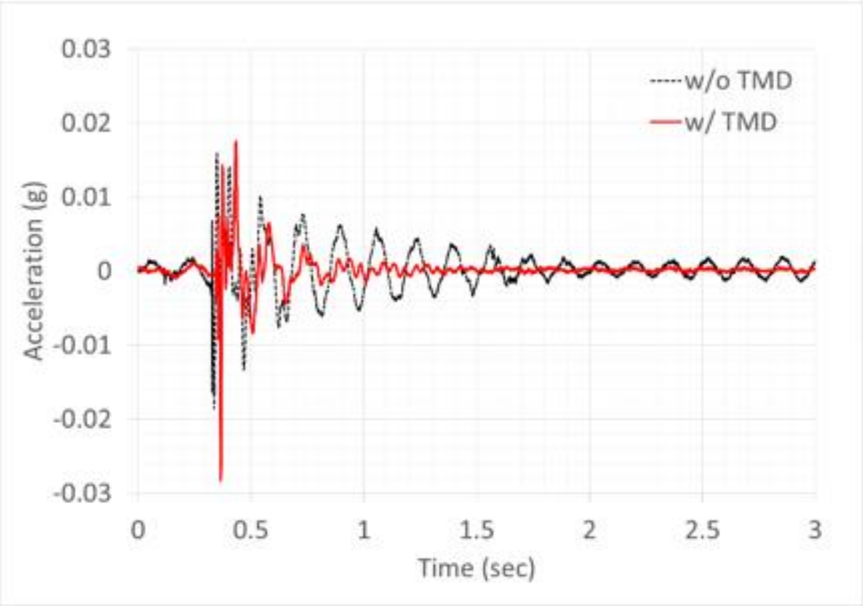
Application of TMDs to attenuate floor vibration ***by Steel Structures and Seismic Design Lab (2016)***








SG2
SG5
SG2
600
600
2,400
4,800
4,150
100
W.P
100
650
100
550
SB00
SB00
SB3
SG2
SG5
SG2
600
600


0.03
0.02
0.01
0
-0.01
-0.02
-0.03
Acceleration (g)
0
0.5
1
1.5
2
2.5
3
Time (sec)
w/o TMD
w/ TMD