Aeroelasticity

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Dynamic Aeroelasticity

Dynamic aeroelasticity

- Two principal phenomena

- Dynamic instability (flutter)
- Responses to dynamic load, or modified by aeroelastic effects

- Flutter ··· self-excited vibration of a structure arising from the interaction of aerodynamic elastic and internal loads

  "response" ··· forced vibration

  "Energy source" ··· flight vehicle speed

- Typical aircraft problems

- Flutter of wing
- Flutter of control surface
- Flutter of panel

Dynamic aeroelasticity

- Stability concept

If solution of dynamic system may be written or

$$y(x,t) = \sum_{k=1}^{N} \bar{y}_k(x) \cdot e^{(\sigma_k + i\omega_k)t}$$

a) $\sigma_k < 0, \omega_k \neq 0 \Rightarrow$ Convergent solution : “stable”

b) $\sigma_k = 0, \omega_k \neq 0 \Rightarrow$ Simple harmonic oscillation : ”stability boundary”

c) $\sigma_k > 0, \omega_k \neq 0 \Rightarrow$ Divergence oscillation : “unstable”

d) $\sigma_k < 0, \omega_k = 0 \Rightarrow$ Continuous convergence : “stable”

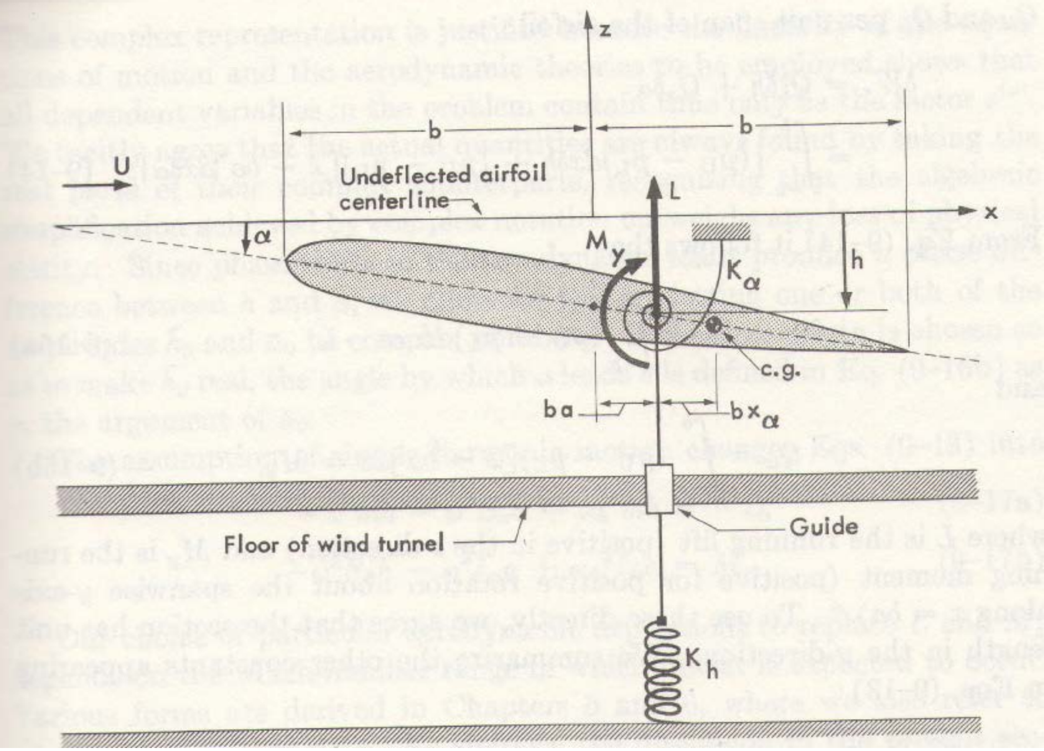
e) $\sigma_k = 0, \omega_k = 0 \Rightarrow$ Time independent solution : ”stability boundary”

f) $\sigma_k > 0, \omega_k = 0 \Rightarrow$ Continuous divergence : “unstable”

Dynamic aeroelasticity

- Flutter of a wing

Typical section with 2 D.O.F



K_α, K_h : torsional, bending stiffness

Dynamic aeroelasticity

- First step in flutter analysis

- Formulate eqns of motion
- The vertical displacement at any point along the mean aerodynamic chord from the equilibrium $z=0$ will be taken as $z_a(x,t)$

$$z_a(x,t) = -h - (x - x_{ea})\alpha$$

- The eqns of motion can be derived using Lagrange's eqn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$L = T - U$$

Dynamic aeroelasticity

- The total kinetic energy(T)

$$T = \frac{1}{2}\int_{-b}^{b} \rho \left(\frac{\partial z_a}{\partial t}\right)^2 dx$$

$$= \frac{1}{2}\int_{-b}^{b} \rho \left[\dot{h} + (x - x_{ea})\dot{\alpha}\right]^2 dx$$

$$= \frac{1}{2}\dot{h}^2 \underbrace{\int_{-b}^{b} \rho dx}_{\substack{m \\ \text{(airfoil mass)}}} + \dot{h}\dot{\alpha} \underbrace{\int_{-b}^{b} \rho (x - x_{ea}) dx}_{\substack{S_\alpha \\ \text{(static unbalance)}}} + \frac{1}{2}\dot{\alpha}^2 \underbrace{\int_{-b}^{b} (x - x_{ea})^2 dx}_{\substack{I_\alpha \\ \text{(mass moment of inertia} \\ \text{about c.g.)}}}$$

*Note) if $x_{ea} = x_{cg},$ then $S_\alpha = 0$ by the definition of c.g.

Therefore,

$$T = \frac{1}{2} m\dot{h}^2 + \frac{1}{2} I\dot{\alpha}^2 + S_\alpha \dot{h}\dot{\alpha}$$

Dynamic aeroelasticity

- The total potential energy (strain energy)

$$U = \frac{1}{2}k_h h^2 + \frac{1}{2}k_\alpha \alpha^2$$

- Using Lagrange's eqns with $L = T - U$

$$q_1 = h_1, q_2 = \alpha$$

$$\Rightarrow \begin{cases} m\ddot{h} + S_\alpha \ddot{\alpha} + k_h h = Q_h \\ S_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + k_\alpha \alpha = Q_\alpha \end{cases}$$

Where Q_h, Q_α are generalized forces associated with two d.o.f's h, α respectively.

Dynamic aeroelasticity

$$Q_h = -L = -L(\alpha, h, \dot{\alpha}, \dot{h}, \ddot{\alpha}, \ddot{h}, \cdots)$$

$$Q_\alpha = M_{ea} = M_{ea}(\alpha, h, \dot{\alpha}, \dot{h}, \ddot{\alpha}, \ddot{h}, \cdots)$$

Governing eqn.

$$\Rightarrow \begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} -L \\ M_{ea} \end{Bmatrix}$$

- For approximation, let's use quasi-steady aerodynamics

$$L = qSC_{L_\alpha}(\alpha + \frac{\dot{h}}{U_\infty})$$

$$M_{ac} = qS_c C_{m_{\dot{\alpha}}} \dot{\alpha}$$

$$M_{ea} = (x_{ea} - x_{ac}) \bullet L + M_{ac} = eqSC_{L_\alpha}(\alpha + \frac{\dot{h}}{U_\infty}) + qS_c C_{m_{\dot{\alpha}}} \dot{\alpha}$$

Dynamic aeroelasticity

*Note) Three basic classifications of unsteadiness (linearized potential flow)

i) Quasi-steady aero: only circulatory terms due to the bound vorticity. Used for characteristic freq. below $2Hz$ (e.g., conventional dynamic stability analysis)

ii) Quasi-unsteady aero: includes circulatory terms from both bound and wake vorticities. Satisfactory results for $2Hz < \omega_\alpha, \omega_h < 10Hz$. Theodorsen is one that falls into here. (without apparent mass terms)

iii) Unsteady aero: "quasi-unsteady"+"apparent mass terms" (non-circulatory terms, inertial reactions: $\dot{\alpha}$, $\ddot{h}$)

For $\omega > 10Hz$, for conventional aircraft at subsonic speed.

Dynamic aeroelasticity

Then, aeroelastic systems of equations becomes

$$\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} \dfrac{qSC_{L_\alpha}}{U_\infty} & 0 \\ -\dfrac{qSeC_{L_\alpha}}{U_\infty} & -qS_c C_{m_{\dot{\alpha}}} \end{bmatrix} \begin{Bmatrix} \dot{h} \\ \dot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & qSC_{L_\alpha} \\ 0 & K_\alpha - qSeC_{L_\alpha} \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- For stability, we can obtain characteristic eqn. of the system and analyze the roots.

  neglect damping matrix for first,

$$\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & qSC_{L_\alpha} \\ 0 & K_\alpha - qSeC_{L_\alpha} \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Much insight can be obtained by looking at the undamped system (Dowell, pp. 83)

Dynamic aeroelasticity

Set $\alpha = \bar{\alpha} e^{pt}, h = \bar{h} e^{pt}$

$$\Rightarrow \begin{bmatrix} (mp^2 + K_h) & (S_\alpha p^2 + qSC_{L\alpha}) \\ S_\alpha p^2 & (I_\alpha p^2 + K_\alpha - qSeC_{L_\alpha}) \end{bmatrix} \begin{Bmatrix} \bar{h} \\ \bar{\alpha} \end{Bmatrix} e^{pt} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For non-trivial solution,

Characteristic eqn., $\det(\Delta) = 0$

$$\underbrace{(mI_\alpha - S_\alpha)}_{A} p^4 + \underbrace{[K_h I_\alpha + (K_\alpha - qSeC_{L_\alpha})m - qSC_{L\alpha} S_\alpha]}_{B} p^2 + \underbrace{K_h (K_\alpha - qSeC_{L_\alpha})}_{C} = 0$$

$$\therefore p^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Dynamic aeroelasticity

The signs of *A*, *B*, *C* determine the nature of the solution.

$A > 0, C > 0$ (if $q < q_D$)

B Either (+) or (-)

$$B = mK_\alpha + K_h I_\alpha - [me + S_\alpha] qSC_{L_\alpha}$$

- If $[me + S_\alpha] < 0, B > 0$ for all q

- Otherwise $B < 0$ when

$$\frac{K_\alpha}{e} + \frac{K_h I_\alpha}{me} - \left[1 + \frac{S_\alpha}{me}\right] qSeC_{L_\alpha} < 0$$

Dynamic aeroelasticity

- Two possibilities for *B* (*B*>0 and *B*<0)

i) *B*>0:

① $B^2 - 4AC > 0, p^2$ are real, negative, so *p* is pure imaginary → neutrally stable

② $B^2 - 4AC < 0, p^2$ is complex, at least one value should have a positive real part → unstable

③ $B^2 - 4AC = 0$ → stability boundary

- Calculation of q_F

$Dq_F^2 + Eq_F + F = 0$ ←(from $B^2 - 4AC = 0,$ stability boundary)

$$q_F = \frac{-E \pm \sqrt{E^2 - 4DF}}{2D}$$

Dynamic aeroelasticity

where,

$$D \equiv \left\{ \left[me + S_\alpha \right] S C_{L_\alpha} \right\}^2$$

$$E \equiv \left\{ -2\left[me + S_\alpha \right]\left[mK_\alpha + K_h I_\alpha \right] + 4\left[mI_\alpha - S_\alpha^2 \right] eK_h \right\} S C_{L_\alpha}$$

$$F \equiv \left[mK_\alpha + K_h I_\alpha \right]^2 - 4\left[mI_\alpha - S_\alpha^2 \right] K_h K_\alpha$$

① At least, one of the q_F must be real and positive in order for flutter to occur.

② If both are, the smaller is the more critical.

③ If neither are, flutter does not occur.

④ If $S_\alpha \leq 0$ (c.g. is ahead of e.a), no flutter occurs(mass balanced)

Dynamic aeroelasticity

ii) $B<0$: B will become (-) only for large q

$B^2-4AC=0$ will occur before $B=0$ since $A>0, C>0$

$\therefore$ To determine q_F, only $B>0$ need to be calculated.

Examine p as q increases

Low $q \rightarrow p=\pm i\omega_1, \pm i\omega_2 (B^2-4AC>0)$

Higher $q \rightarrow p=\pm i\omega_1, \pm i\omega_2 (B^2-4AC=0) \rightarrow$ stability boundary

More higher $q \rightarrow p=-\sigma_1 \pm i\omega_1, \sigma_2 \pm i\omega_2 (B^2-4AC<0) \rightarrow$ dynamic instability

Even more higher $q \rightarrow p=0, \pm i\omega_1 (C=0) \rightarrow$ stability boundary

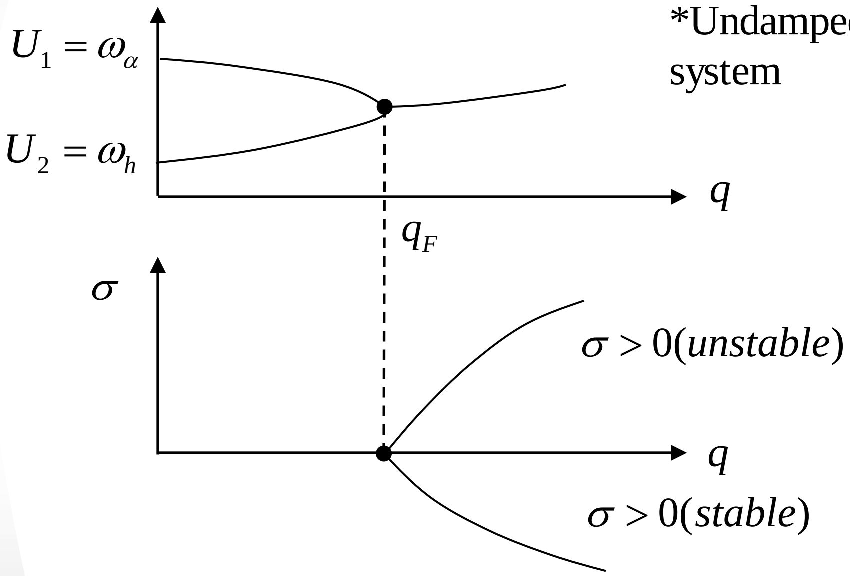
$\therefore$ Flutter condition: $B^2-4AC=0$

Torsional divergence: $C=0$

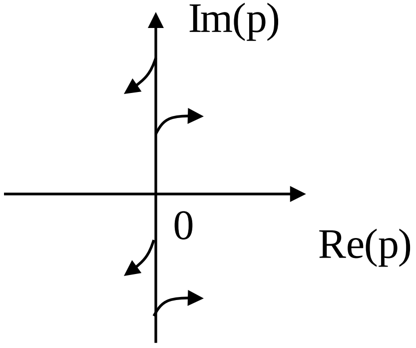
Dynamic aeroelasticity

Graphically,

$$\omega_\alpha^2 = \frac{K_\alpha}{I_\alpha}, \omega_h^2 = \frac{K_h}{m}$$



*Undamped system



- Effect of static unbalance

In Dowell's book, after Pines[1958]

$S_\alpha \le 0 \rightarrow$ avoid flutter, if $S_\alpha = 0, \dfrac{q_F}{q_D} = 1 - \dfrac{\omega_h^2}{\omega_\alpha^2}$

Dynamic aeroelasticity

If $q_D < 0(e < 0)$ $\dfrac{\omega_h}{\omega_\alpha} < 1.0 \Rightarrow q_F < 0$ no flutter

If $q_D > 0$ and $\dfrac{\omega_h}{\omega_\alpha} > 1.0 \Rightarrow$ no flutter

- Inclusion of damping→ "can be a negative damping"

for better accuracy,

$m\ddot{q} + c\dot{q} + Kq = 0,$ where

$$c = \begin{bmatrix} \dfrac{qSC_{L_\alpha}}{U_\infty} & 0 \\ -\dfrac{qSC_{L_\alpha}}{U_\infty} & -qScC_{m_{\dot{\alpha}}} \end{bmatrix}$$

The characteristic equation is now in the form of

$$A_4 p^4 + A_3 p^3 + A_2 p^2 + A_1 p + A_0 = 0$$

Dynamic aeroelasticity

$$A_4 p^4 + A_3 p^3 + A_2 p^2 + A_1 p + A_0 = 0 \cdots *$$

- Routh criteria for stability

; At critical position, the system real part becomes zero, damping becomes zero.

Substitute $p = i\omega$ into (*), we get,

$$\begin{cases} A_4\omega^4 - A_2\omega^2 + A_0 = 0 \\ i(-A_3\omega^3 + A_1\omega) = 0 \end{cases}$$

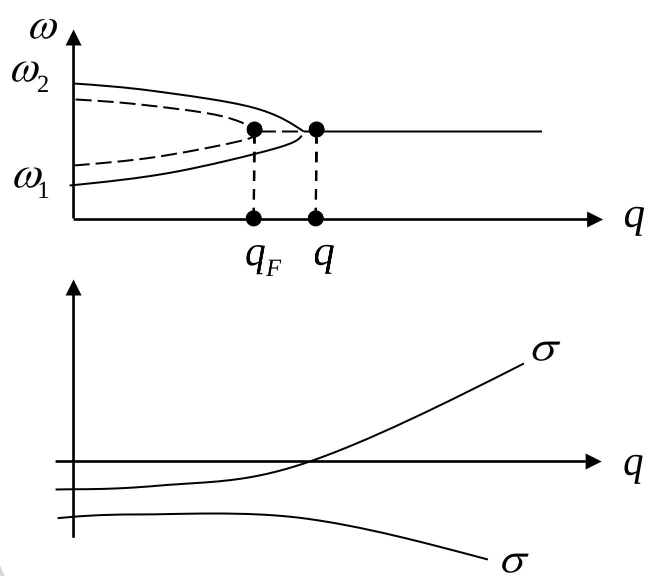
From the second eqn, $\omega^2 = \dfrac{A_1}{A_3}$, substitute into first equation, then,

$$A_4\left(\frac{A_1}{A_3}\right)^2 - A_2\left(\frac{A_1}{A_3}\right) + A_0 = 0$$ or $$A_4 A_1^2 - A_1 A_2 A_3 + A_0 A_3^2 = 0$$

Dynamic aeroelasticity

And, we can examine p as q increases,

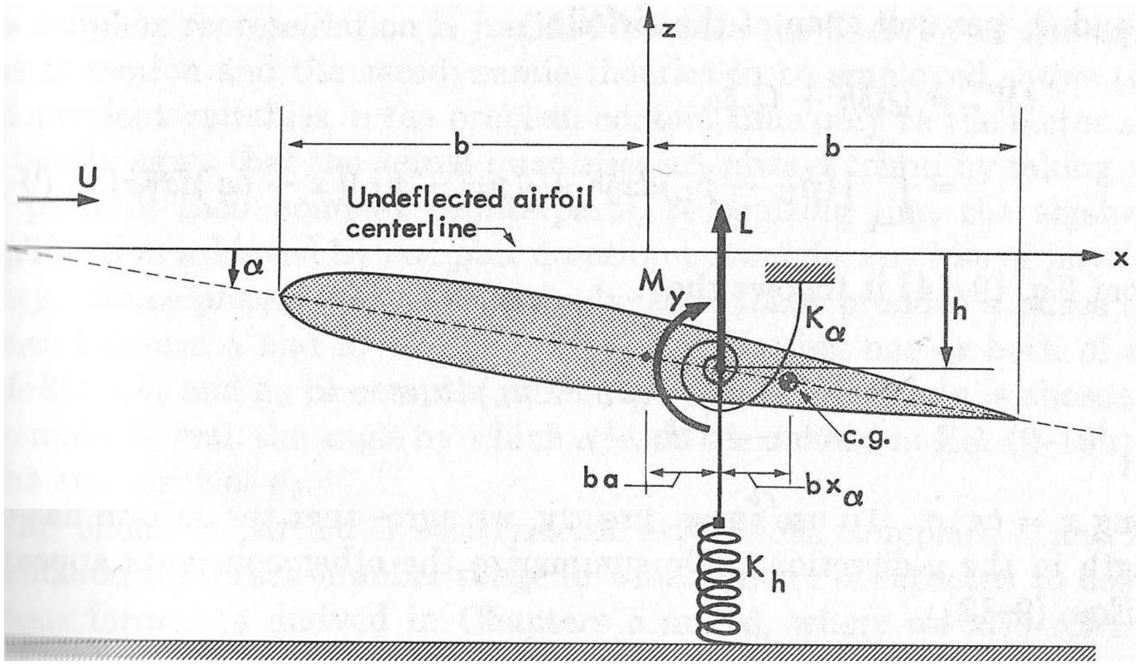
Low	$q \rightarrow p = -\sigma_1 \pm i\omega_1, -\sigma_2 \pm i\omega_2 \rightarrow$ damped natural freq.
Higher	$q \rightarrow p = -\sigma_1 \pm i\omega_1, \pm i\omega_2$
More higher	$q \rightarrow p = -\sigma_1 \pm i\omega_1, \pm\sigma_2 \pm i\omega_2 \rightarrow$ dynamic instability.



- Static instability $\cdots |K| = 0$
- Dynamic instability
  a) frequency coalescence (unsymmetric K)
  b) Negative damping $(C_{ij} < 0)$
  c) Unsymmetric damping (gyroscopic)

Straight Aircraft Wing

Consider disturbance from equilibrium



Using modal method, the displacement $\left(w_{ea}\right)$ and rotation $\left(\theta_{ea}\right)$ at elastic axis can be expressed as

$$\begin{cases} w_{ea} = \sum_{r=1}^{N} h_r(y) q_r(t) \\ \theta_{ea} = \sum_{r=1}^{N} \alpha_r(y) q_r(t) \end{cases}$$

where

$q_r(t)$: generalized (modal) coordinate

$h_r(y), \alpha_r(y)$: mode shape

N: total number of modes

Straight Aircraft Wing

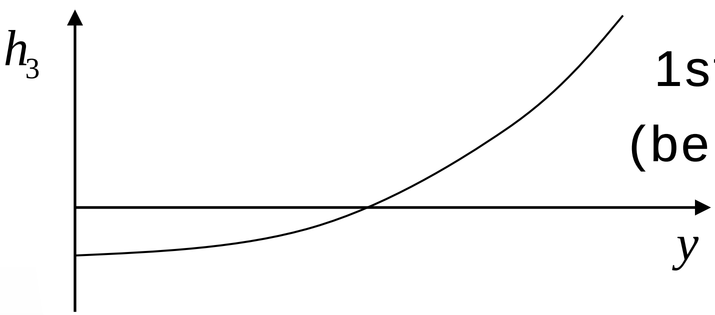
For $N = 4$,

$a)\ h_1 = 1, \alpha_1 = 0:$ rigid translation mode $(\omega_1 = 0)$

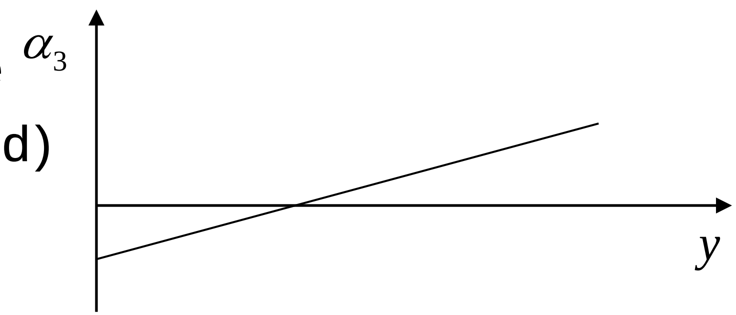
$b)\ h_2 = x_0, \alpha_2 = 0:$ rigid pitch mode about c.g. $(\omega_2 = 0)$

$c)\ h_3(y), \alpha_3(y):$ 1st bending of wing $(\omega_3 \neq 0)$

$d)\ h_4(y), \alpha_4(y):$ 1st torsion of wing $(\omega_4 \neq 0)$



1st bending mode
(bending dominated)



Modes can be assumed, or calculated from mass-spring representation. The displacements and rotations at any point

$$w(x,y,t) = w_{ea} + (x - x_0)\theta_{ea} = \sum_{r=1}^{N} \left[h_r + (x - x_0)\alpha_r \right] q_r(t)$$

$$\theta(x,y,t) = \theta_{ea} = \sum_{r=1}^{N} \alpha_r q_r(t)$$

Straight Aircraft Wing

The kinetic energy (T) is

$$T = \frac{1}{2}\iint_{\frac{1}{2}aircraft} m\left(\dot{w}\right)^2 dxdy$$

$$= \frac{1}{2}\iint m\sum_{r=1}^{N}\left[h_r + \left(x - x_0\right)\alpha_r\right]\dot{q}_r \sum_{s=1}^{N}\left[h_s + \left(x - x_0\right)\alpha_s\right]\dot{q}_s dxdy$$

$$= \frac{1}{2}\sum_{r=1}^{N}\sum_{s=1}^{N} m_{rs}\dot{q}_r\dot{q}_s$$

where, $m_{rs} = \int_0^l \left[M h_r h_s + I_\alpha \alpha_r \alpha_s + S_\alpha\left(h_r\alpha_s + h_s\alpha_r\right)\right]dy$

$M = \int_{LE}^{TE} m dx$: mass/unit span

$S_\alpha = \int_{LE}^{TE}\left(x - x_0\right) m dx$: static unblance/unit span

$I_\alpha = \int_{LE}^{TE}\left(x - x_0\right)^2 m dx$: moment of inertia about E.A./unit span

Straight Aircraft Wing

The potential energy (U) is

$$U = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w_{ea}}{\partial y^2}\right)^2 dy + \frac{1}{2}\int_0^l GJ\left(\frac{\partial \theta_{ea}}{\partial y}\right)^2 dy$$

$$= \frac{1}{2}\int_0^l EI\sum_{r=1}^{N} h_r'' q_r \sum_{s=1}^{N} h_s'' q_s dy + \frac{1}{2}\int_0^l GJ\sum_{r=1}^{N} \alpha_r' q_r \sum_{s=1}^{N} \alpha_s' q_s dy$$

$$= \frac{1}{2}\sum_{r=1}^{N}\sum_{s=1}^{N} K_{rs} q_r q_s$$

where, $K_{rs} = \int_0^l EI h_r'' h_s'' dy + \int_0^l GJ\alpha_r' \alpha' dy$

[Note] $K_{rs} = 0$ for rigid modes 1,2, since $h_1'' = h_2'' = 0$ and $\alpha_1' = \alpha_2' = 0$

Straight Aircraft Wing

Finally, the work done by airloads,

$$\delta W = -\int_0^l L_{ea}\delta w_{ea} dy + \int_0^l M_{ea}\delta\theta_{ea} dy - L_{HT}\delta w_{HT} + M_{HT}\delta\theta_{HT} = \sum_{r=1}^{N} Q_r \delta q_r$$

subscript HT : horizontal tail contribution (rigid fuselage assumption)

where, $Q_r = \int_0^l \left(-h_r L_{ea} + \alpha_r M_{ea}\right) dy - h_{r(HT)} L_{HT} + \alpha_{r(HT)} M_{HT}$

[Note] $r = 1 \rightarrow Q_1 = -\int_0^l L_{ea} dy - L_{HT} = -\frac{1}{2} L_{Total}$

$r = 2 \rightarrow Q_2 = \frac{1}{2} M_{Total}(C.G)$

place T, U, and Q_r into the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial U}{\partial q_r} = Q_r$$

yield the equation of motion

Straight Aircraft Wing

Equation of motion in matrix form

$$[m_{rs}]\{\ddot{q}_r\}+\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&K_{33}&K_{34}\\0&0&K_{43}&K_{44}\end{bmatrix}\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix}=\{Q_r\}$$

zeros are associated with rigid body modes

[Note] If we used normal modes, $w(x,y,t)=\sum_{r=1}^{N}\phi_r(x,y)q_r(t)$

free-free normal mode

The equation of motion would be uncoupled

$$[m_{rs}]\rightarrow\begin{bmatrix}\ddots&&\\&m_{rr}&\\&&\ddots\end{bmatrix},\quad [K_{rs}]\rightarrow\begin{bmatrix}\ddots&&\\&m_{rr}\omega_r^2&\\&&\ddots\end{bmatrix}$$

Straight Aircraft Wing

[Note] Free-free normal mode vs Uncoupled modes

from entire structures

$$M_r \ddot{q}_r + M_r \omega_r^2 q_r = Q_r$$

(more accurate)

for individual components then, combine together by Rayleigh-Ritz method,

$$\sum m_{rs} \ddot{q}_s + \sum k_{rs} q_s = 0$$

(more versatile)

Straight Aircraft Wing

Now, let's introduce the aerodynamic load by considering

2-D, incompressible, strip theory

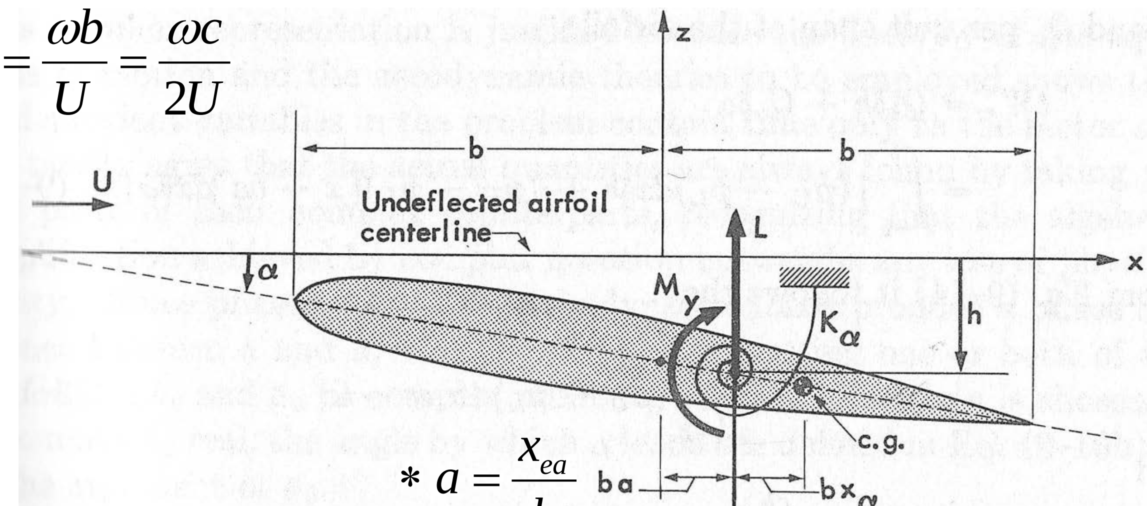
$$L_{ea} = \pi\rho b^2\left[\ddot{w}_{ea} + U\dot{\theta}_{ea} - ba\ddot{\theta}_{ea}\right] + 2\pi\rho UbC(k)\left[\dot{w}_{ea} + U\theta_{ea} - b\left(\tfrac{1}{2}-a\right)\dot{\theta}_{ea}\right]$$

$$M_{ea} = \pi\rho b^3\left[a\ddot{w}_{ea} + U\left(\tfrac{1}{2}-a\right)\dot{\theta}_{ea} - b\left(\tfrac{1}{8}+a^2\right)\ddot{\theta}_{ea}\right]$$

$$+2\pi\rho Ub^2\left(\tfrac{1}{2}+a\right)C(k)\underbrace{\left[\dot{w}_{ea} + U\theta_{ea} - b\left(\tfrac{1}{2}-a\right)\dot{\theta}_{ea}\right]}$$

lift deficiency fn. $\quad \frac{3}{4}c$ airspeed (downwash)

$$* \; k = \frac{\omega b}{U} = \frac{\omega c}{2U}$$



$$* \; a = \frac{x_{ea}}{b}$$

Unsteady Aeroelasticity

- **Unsteady Aeroelasticity in Incompressible Flow (B.A.H p.272 and B.A. p.119)**

  - For incompressible flow $\left(M \ll 1\right)$

    a separation can be made between circulatory and non-circulatory airloads

  - When the airfoil performs chordwise rigid motion.

    the circulatory lift depends only on the downwash at the $\frac{3}{4}c$ station

$$w_{\frac{3}{4}c} = \left[\dot{w}_{ea} + U\theta_{ea} - b\left(\tfrac{1}{2} - a\right)\theta_{ea}\right]:$$ downwash at $\frac{3}{4}c$

$$L_{ea} = \pi\rho b^2\left[\ddot{w}_{ea} + U\dot{\theta}_{ea} - ba\ddot{\theta}_{ea}\right] + 2\pi\rho UbC(k)\left[\dot{w}_{ea} + U\theta_{ea} - b\left(\tfrac{1}{2} - a\right)\theta_{ea}\right]$$

"always acts at $\frac{1}{4}c$ "

lift deficiency fn.

Unsteady Aeroelasticity

However, $\begin{cases} w_{ea} = \sum_s h_s q_s \\ \theta_{ea} = \sum_s \alpha_s q_s \end{cases}$

and placing these into L_{ea}, M_{ea} yields

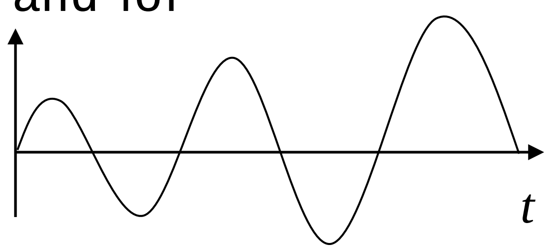
$$Q_r = \int_0^l \left(-h_r L_{ea} + \alpha_r M_{ea}\right) dy + H.O.T = Q_r\left(q_s, \dot{q}_s, \ddot{q}_s\right)$$

coupled set of homogeneous differential equations.

For stability analysis, assume $q_r(t) = \bar{q}_r e^{pt}$

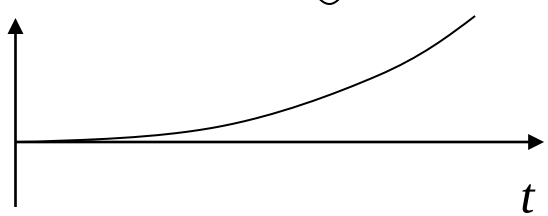
where $p = \sigma + i\omega$, and for

$a) + \sigma, \omega \neq 0$



"flutter"

$b) + \sigma, \omega = 0$



"divergence"

Solutions of the Aeroelastic E.O.M

- **Solutions of the Aeroelastic Equations of Motion (Dowell pp.100~106)**

  - Two Groups: a) Time domain and b) Frequency domain

    a) Time domain: fundamentally, a step by step solution for the time history

    · Direct integration method

    ① equilibrium satisfied at discrete time t

    ② assumed variation of variables $(q, \dot{q}, \ddot{q})$ within the time interval Δt

    · Examples of methods

    ① central difference

    ② Newmark

    ③ Houbolt

    Ref. Bathe, “Finite Element Procedures”, Chap. 9

Solutions of the Aeroelastic E.O.M

- When selecting a method, three main issues to be aware

  ① efficient scheme

  ② numerical stability

  conditionally stable – dependent on Δt

  unconditionally stable

  ③ numerical accuracy

  amplitude decay

  period elongation

- Advantage and disadvantage of time domain analysis

  ① Advantage: straight forward method

  ② Disadvantage: aerodynamic loads may be a problem

  $\rightarrow$ theories are not well-developed

  $\rightarrow$ intensive numerical calculation

  for small number of frequency (k)

Solutions of the Aeroelastic E.O.M

b) Frequency domain: most popular approach

- Main issue: aerodynamic loads are well developed for simple harmonic motion
- consider simple harmonic motion $q_r(t) = \bar{q}_r e^{i\omega t}$

and corresponding lift and moment, (Ref. Drela, last page)

$$L_{ea} = \bar{L}_{ea} e^{i\omega t}$$
$$M_{ea} = \bar{M}_{ea} e^{i\omega t}$$

where,

$$\bar{L}_{ea} = \pi \rho b^3 \omega^2 \left[l_h(k, M_\infty) \frac{\bar{w}_{ea}}{b} + l_\alpha(k, M_\infty) \bar{\theta} \right]$$
$$\bar{M}_{ea} = \pi \rho b^4 \omega^2 \left[m_h(k, M_\infty) \frac{\bar{w}_{ea}}{b} + m_\alpha(k, M_\infty) \bar{\theta} \right]$$

$l_h, l_\alpha, m_h, m_\alpha$ are dimensionless complex fn. of (k, M_∞)

(Refs. Dowell, p.116 and B.A. pp. 103~114)

Solutions of the Aeroelastic E.O.M

- Then, the governing equation becomes

$$-\omega^2[M]\{\overline{q}\}+[K]\{\overline{q}\}+\omega^2\left[A(k,M_\infty)\right]\{\overline{q}\}=0$$

aerodynamic operator (aero. mass matrix)

It is presumed that the following parameters are known.

$$\underbrace{M, S_\alpha, I_\alpha,}_{\text{inertia}} \quad \underbrace{\omega_h, \omega_\alpha,}_{\text{stiffness}} \quad b\left(=\tfrac{1}{2}c\right)$$

The unknown quantities are

$$\underbrace{\overline{q}, \omega,}_{\text{determined by } p} \quad \rho, M_\infty, k\left(=\frac{\omega b}{U}\right)$$

Solutions of the Aeroelastic E.O.M

I) k-method (V-g method)

- consider a system with just the right amount of structural damping, so the motion is simple harmonic

$$-\omega^2[M]\{\bar{q}\}+(1+\underset{\substack{\uparrow\\ \text{structural damping coefficient}}}{ig})[K]\{\bar{q}\}+\omega^2[A]\{\bar{q}\}=0$$

[Note] structural damping – restoring force in phase with velocity, but proportional to displacement

$$F_0=-g\underbrace{\left(\dot{q}/|\dot{q}|\right)}_{\text{phase}}\underbrace{|q|}_{\text{displacement}}$$

$g_{required} > g_{available}$: unstable

$g_{required} = g_{available}$: neutral

$g_{required} < g_{available}$: stable

* viscous damping - $F_c=-c\dot{q}$

- Rewrite equation

$$[M-A]\{\bar{q}\}=\underbrace{(1+ig)/\omega^2}_{\Lambda,\ \ \mathrm{Re}[\Lambda]=1/\omega^2,\ \ \mathrm{Im}[\Lambda]=g/\omega^2}[K]\{\bar{q}\}$$

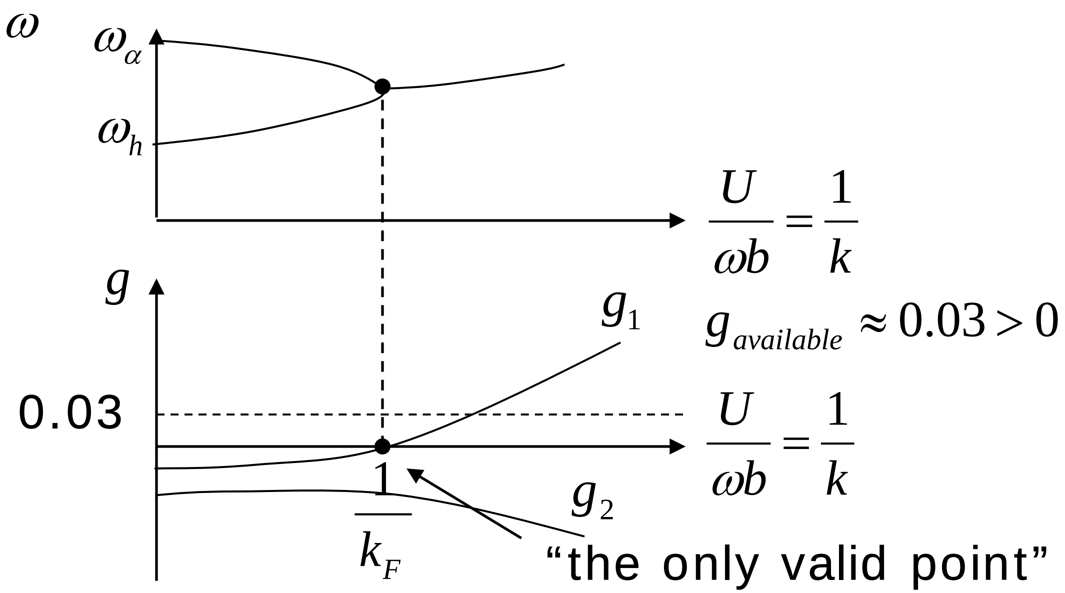
Solutions of the Aeroelastic E.O.M

- Solution process

① Given $M, S_\alpha, I_\alpha, \omega_h/\omega_\alpha, b$

② Assume ρ (fix altitude), $M_\infty = U/a_\infty$

③ For a set of k values, solve for eigenvalues for Λ



④ for $g_1 = 0 \;\rightarrow\; \omega = \omega_F \left(k_F = b\omega_F / U_F\right)$

⑤ matching problem

$$U_F \;\rightarrow\; M_F = M_\infty$$

Solutions of the Aeroelastic E.O.M

I) k-method (V-g method) (Dowell, p.106)

- Structural damping is introduced by multiplying at $\omega_h^2, \omega_\alpha^2$
  $\times(1+ig)$, $g:$ structural damping coefficient
  pure sinusoidal motion is assumed $\rightarrow \omega \equiv \omega_R, \omega_I \equiv 0$
  for a given U , the g required to sustain pure sinusoidal motion is determined
- Advantage – the aero. force need to be determined for real frequencies
- Disadvantage – loss of physical sight, only at $U = U_F\left(\omega = \omega_R, \omega_I = 0\right)$ the mathematical solution will be meaningful
- Following parameters are prescribed

$$M, S_\alpha, I_\alpha, \omega_h/\omega_\alpha, k, m/2\rho_\infty bS$$

then, the characteristic equation becomes a complex polynomial in unknowns $(\omega_\alpha/\omega)(1+ig)$

Solutions of the Aeroelastic E.O.M

I) k-method (V-g method) (Dowell, p.106)

· A complex roots are obtained for ω_α / ω and g

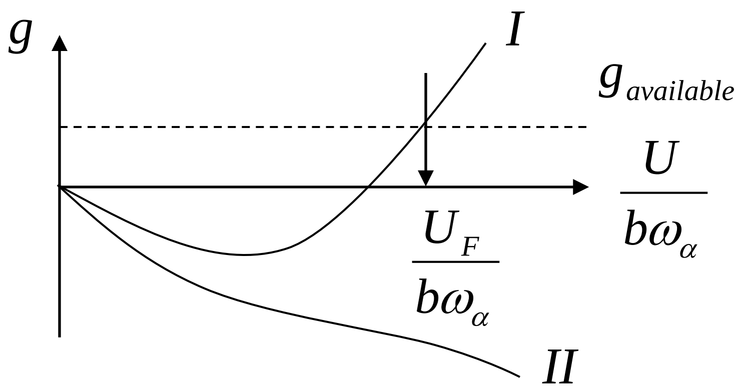
From ω_α / ω and the previously selected $k \equiv \omega b / U$,

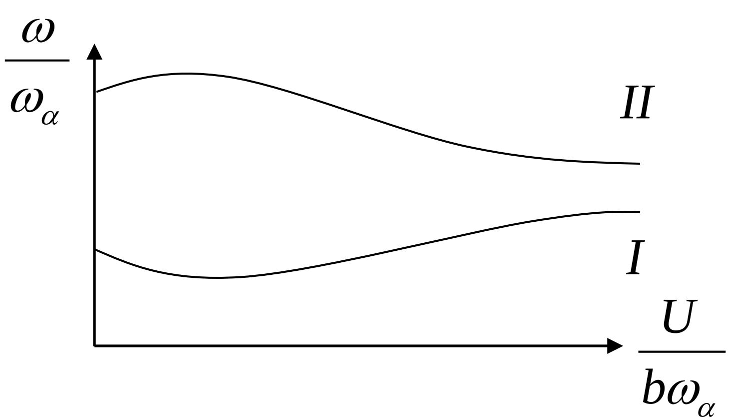
$$\frac{\omega_\alpha b}{U_\infty} = \frac{\omega_\alpha}{\omega} k$$

Then, plot g vs $U_\infty / b\omega_\alpha$ (typical plot for two d.o.f system below)

g : value of structural damping required to sustain neutral stability

→ If the actual damping is $g_{available}$, then flutter occurs when $g = g_{available}$





If $g < g_{available}$, $U < U_F \rightarrow$ no flutter will occur

Solutions of the Aeroelastic E.O.M

I) k-method (V-g method) (Dowell, p.106)

- Uncertainty about $g_{available}$ in a real physical system, flutter speed is defined as minimum value of $U_F / b\omega_\alpha$ for any $g > 0$

Solutions of the Aeroelastic E.O.M

II) p-method – time dependent solution $q = \bar{q}e^{pt}, \quad p = \sigma + i\omega$

- The equation,

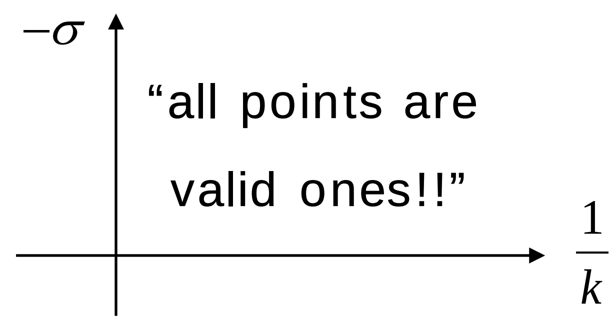
$$p^2[M]\{\bar{q}\} + [K]\{\bar{q}\} = [A(p, M)]\{\bar{q}\}$$

Now the aero becomes more approximate

i) quasi-steady aero

ii) induced lift function

iii) flow Eigen solution



[Note] I) k-method (V-g method), $q_r = \bar{q}_r e^{i\omega t}$

only valid for single harmonic motion – $k \sim \omega$

II) p-method, $q = \bar{q}e^{pt}, \quad p = \sigma + i\omega$

$[M]\{\ddot{q}\} + [K]\{\bar{q}\} = [A(p, M)]$ - "true damping" (H. Hassig)

Solutions of the Aeroelastic E.O.M

III) p-k method

- The solution is assumed arbitrary (as in p-method)

  However, the aero. is assumed to be

$$A(p, M) \cong A(k, M)$$

  Then, the eqn. becomes:

$$\left\{p^2[M]+[K]-\left[A(k, M)\right]\right\}\{\bar{q}\}=0$$

- Solution process

① specify k_i, M_i

② solve for $p_0 = \sigma_0 + \underbrace{i\omega_0}_{k_0}$

③ check for double matching

$$k_0 = k_i$$

$$M_F = M_i$$

Solutions of the Aeroelastic E.O.M

[Note] p-k method usually requires handful of iteration to converge.
It is more expensive than k-method.

· Alternative: p-k method (Dowell)

$h, \alpha \sim e^{pt}$ is assumed, $p = \sigma + i\omega$

in aero. terms, only a $k \equiv \omega b/U$ is assumed

The eigenvalues p are computed $\rightarrow$ new $\omega \rightarrow$ new k $\rightarrow$ new aero. terms – iteration continues until the process converges

For small σ, i.e., $|\sigma| << |\omega|$, $\sigma \sim$ true damping solution

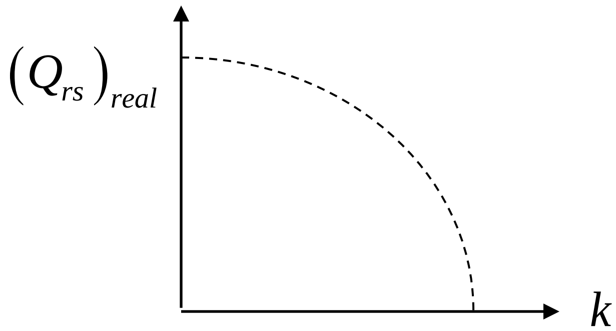
Padé Approximation Method

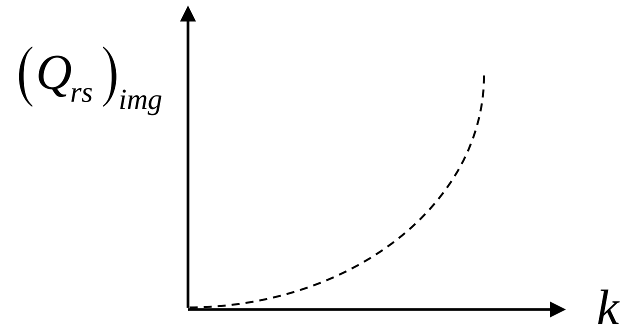
The generalized forces Q_r are computed for harmonic motion

$$Q_r = \tfrac{1}{2}\rho U^2 Q_{rs}\bar{q}_s e^{i\omega t}$$

$$\left(\pi\rho\omega^2 A_{rs}\bar{q}_s e^{i\omega t}\right)$$

where $Q_{rs} = \left(Q_{rs}\right)_{real} + i\left(Q_{rs}\right)_{img}$: complex function of reduced frequency





one can fit above by Padé Approximation

in Laplace transform domain p of from

$$Q_r = \frac{1}{2}\rho U^2\left[\underset{\substack{\uparrow\\ \text{mass}}}{A_2\left(b/U\right)^2 p^2} + \underset{\substack{\uparrow\\ \text{damping}}}{A_1\left(b/U\right)p} + \underset{\substack{\uparrow\\ \text{stiffness}}}{A_0} + A_3\frac{\left(b/U\right)p}{\underset{\substack{\uparrow\\ \text{lag}}}{\left(b/U\right)p+\beta_1}}\right]q_s$$

Padé Approximation Method

For harmonic motion $p = i\omega$

$$Q_r = \frac{1}{2}\rho U^2 \left[\underbrace{\left(-A_2 + A_0 + A_3 \frac{k^2}{k^2+\beta_1} \right)}_{(Q_{rs})_{real}} + i \underbrace{\left(A_1 k - A_3 \frac{\beta_1 k}{k^2+\beta_1^3} \right)}_{(Q_{rs})_{img}} \right] q_s$$

and then evaluate coefficients $A_2, A_1, A_0, A_3, \beta_1$ to fit Q_{rs}

over certain range of k , $0 \le k \le 2 \ \left(k \equiv \omega b / U \right)$

[Note] for better fit, use more lag terms,

$$Q_r = \frac{1}{2}\rho U^2 \left[A_2 \left(b/U\right)^2 p^2 + A_1 \left(b/U\right) p + A_0 + \sum_{m=3}^{N} A_m \frac{\left(b/U\right) p}{\left(b/U\right) p + \beta_{m-2}} \right] q_s$$

Padé Approximation Method

Next, introduce new augmented state variables y_s , defined as

$$y_s = \frac{(b/U)\,p}{(b/U)\,p + \beta_s} q_s = \frac{p}{p + (U/b)\,\beta_s} q_s$$

$$p y_s + (U/b)\,\beta_s y_s = p q_s$$

Return to time domain,

$$Q_r = \tfrac{1}{2}\rho U^2 \left[A_2 (b/U)^2 \ddot{q}_s + A_1 (b/U)\,\dot{q}_s + A_0 q_s + A_3 y_s \right]$$

$$\dot{y}_s + (U/b)\,\beta_s y_s = \dot{q}_s$$

and governing equation,

$$M\ddot{q} + C\dot{q} + Kq = \tfrac{1}{2}\rho U^2 \left[A_2 (b/U)^2 \ddot{q}_s + A_1 (b/U)\,\dot{q}_s + A_0 q_s + A_3 y_s \right]$$

$$\dot{y}_s + \begin{bmatrix} \ddots & & \\ & U\beta/b & \\ & & \ddots \end{bmatrix} y_s = \dot{q}_s$$

Padé Approximation Method

or $$\begin{bmatrix} M^* & 0 & 0 \\ 0 & M^* & 0 \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \ddot{q} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} 0 & -M^* & 0 \\ K^* & C^* & G \\ 0 & -I & H \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \\ y \end{Bmatrix} = 0$$

where $$\begin{cases} M^* = M - \frac{1}{2}\rho b^2 A_2 \\ C^* = C - \frac{1}{2}\rho b A_1 \\ K^* = K - \frac{1}{2}\rho U^2 A_0 \\ G = \frac{1}{2}\rho U^2 A_3 \\ H = \begin{bmatrix} \ddots & & \\ & U\beta/b & \\ & & \ddots \end{bmatrix} \end{cases}$$

and then, $$\begin{Bmatrix} \dot{q} \\ \ddot{q} \\ \dot{y} \end{Bmatrix} = [A] \begin{Bmatrix} q \\ \dot{q} \\ y \end{Bmatrix} \quad \rightarrow \quad \dot{\mathbf{X}} = \mathbf{AX}$$

Ref.: Karpel, minimum-state (1991)

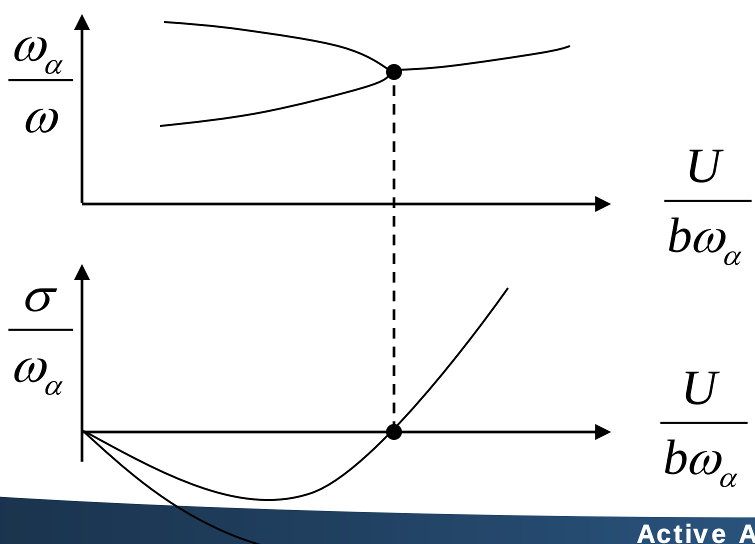
Types of Flutter

I) "Coalescence" or "Merging frequency" flutter

- coupled-mode, bending-torsion flutter (2 d.o.f flutter)
- for $U > U_F$, one of $\omega_I \rightarrow (+)$ and large (stable pole)

  the other $\omega_I \rightarrow (-)$ and large (unstable pole)

  ω_R remain nearly the same
- although $\left\{ \begin{matrix} \text{torsion mode being unstable} \\ \text{bending mode being stable} \end{matrix} \right\}$ the airfoil is undergoing on oscillation composed of both



→ torsional mode usually goes unstable

→ flutter mode contains significant contributions of both bending and torsion

Types of Flutter (Dowell. P.103)

I) "Coalescence" or "Merging frequency" flutter

· the "out-of-phase" (damping) force are not qualitatively important

→ may neglect structural damping entirely and use a quasi-steady or even a quasi-static aerodynamic assumption

→ simplified analysis

Out-of-Phase Force (BAH p.528)

- 2-D rigid airfoil with a torsional spring (1 d.o.f system)

$$I_\alpha \ddot{\alpha} + K_\alpha \alpha = M_y$$

by assuming

$$\alpha = \bar{\alpha}_o e^{i\omega t}$$

$$\frac{I_\alpha}{\pi \rho b^4}\left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2\right] + m_y = 0$$

where

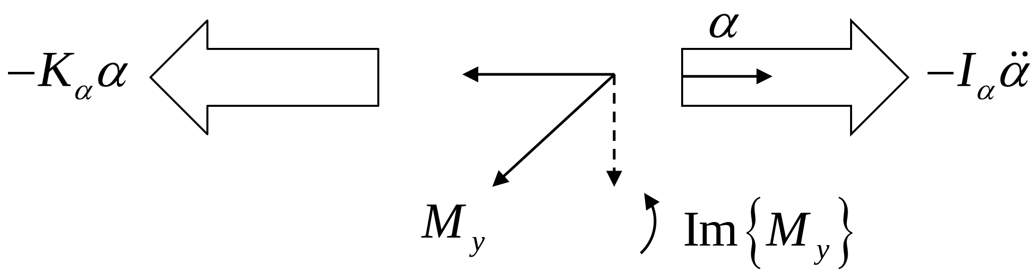
$$m_y = \frac{M_y}{\pi \rho b^4 \omega^2 \bar{\alpha}_o e^{i\omega t}},$$ function of only $k = \omega b / U$

Substituting into (1), flutter occurs when the out-of-phase aerodynamic damping component vanish.

- Rotating complex vector diagram

Out-of-Phase Force (BAH p.528)

- Rotating complex vector diagram



$U < U_F$	$U > U_F$
$\mathrm{Im}\{M_y\} < 0$	$\mathrm{Im}\{M_y\} > 0$

$$\mathrm{Im}\{M_y\} = 0$$

M_y which lags that motion ($\mathrm{Im}\{M_y\} < 0$), removes energy from the oscillation, providing damping.

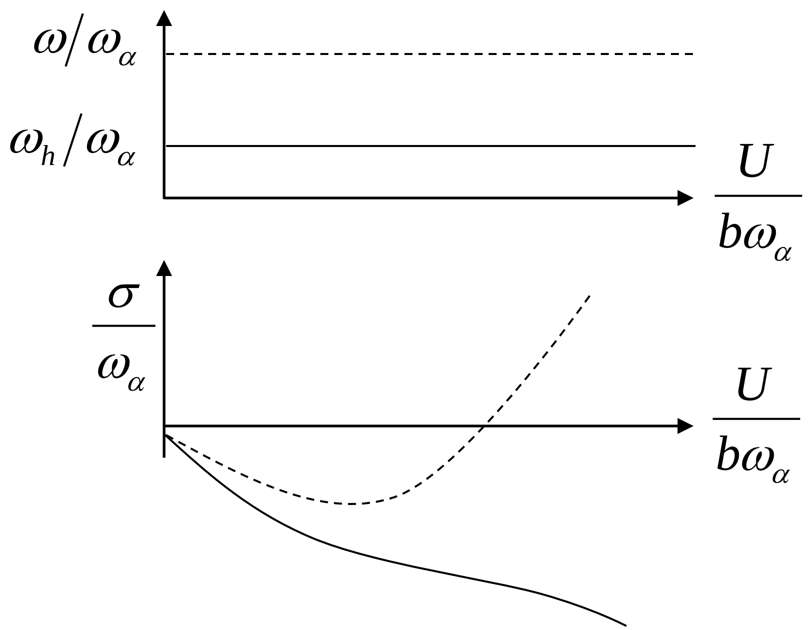
This out-of-phase component, $\mathrm{Im}\{M_y\}$, is the only source of damping or instability from the system.

Types of Flutter

II) Single d.o.f. flutter

- frequency of mode almost independent of reduced velocity
- results from negative damping
- out-of-phase part of aerodynamic operator is very important
- typical of systems with large mass ratio at large reduced velocity

  (e.g. turbo machinery, bridge, …)



Types of Flutter (Dowell. P.103)

II) Single d.o.f. flutter

· frequencies, ω_R , independent of the airspeed $(U/b\omega_\alpha)$ variation

· true damping, ω_I , also moderate change with airspeed

· one of the mode (usually torsion) becomes slightly $(-)$ at U_F

$\rightarrow$ very sensitive to structural and aerodynamic damping forces

$\rightarrow$ since those forces are less precisely described,

analysis gives less reliable results

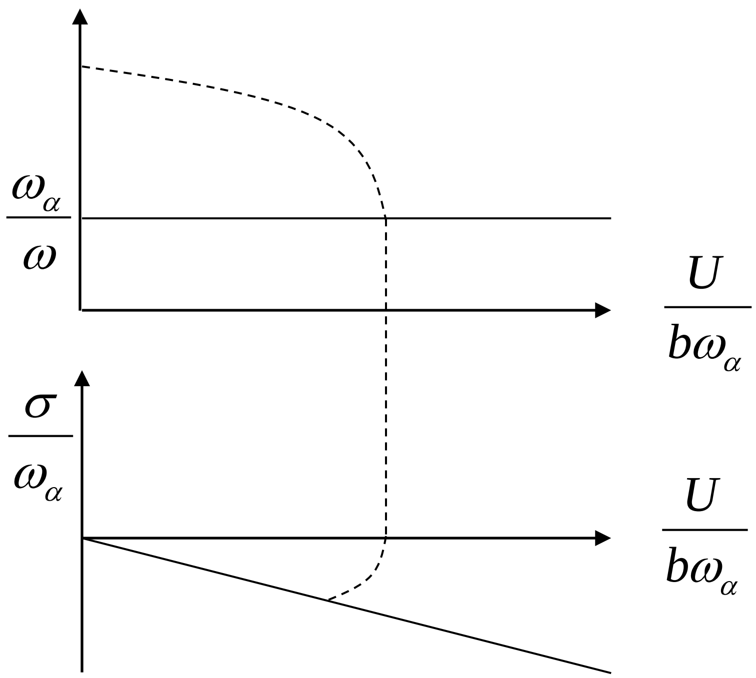
· Since the flutter mode is virtually the same as that of the system at zero airspeed, the flutter mode and frequency are predicted rather accurately (mass ratio < 10)

· Airfoil blades in turbo machinery and bridges in a wind.

Types of Flutter

III) Divergence

- flutter at zero frequency
- very special type of single d.o.f. flutter
- out-of-phase forces unimportant
- analysis reliable



Parameter Effects on Wing Flutter

- When one non-dimensionalizes the flutter determinant (2D), 5 parameters will appear:

$$
\begin{cases}
\mu = \dfrac{m}{\pi\rho b^2} = \text{mass ratio} \\[2ex]
x_\alpha = \dfrac{S_\alpha}{mb} = \dfrac{\text{distance CG aft of EA}}{b} \\[2ex]
\gamma_\alpha = \sqrt{\dfrac{I_\alpha}{mb^2}} = \dfrac{\text{radius of gyration about EA}}{b} \\[2ex]
a = \dfrac{e}{b} = \dfrac{\text{distance EA aft of midchord}}{b} \\[2ex]
\dfrac{\omega_h}{\omega_\alpha} = \text{uncoupled bending to torsion frequency ratio}
\end{cases}
$$

Parameter Effects on Wing Flutter

[additional]

$\omega_\alpha t =$ nondimensional time

$M =$ Mach. No. (compressibility effect)

$$K_\alpha = \frac{\omega_\alpha b}{U} = \text{reduced frequency}$$

$$= \frac{1}{\text{reduced velocity}}$$

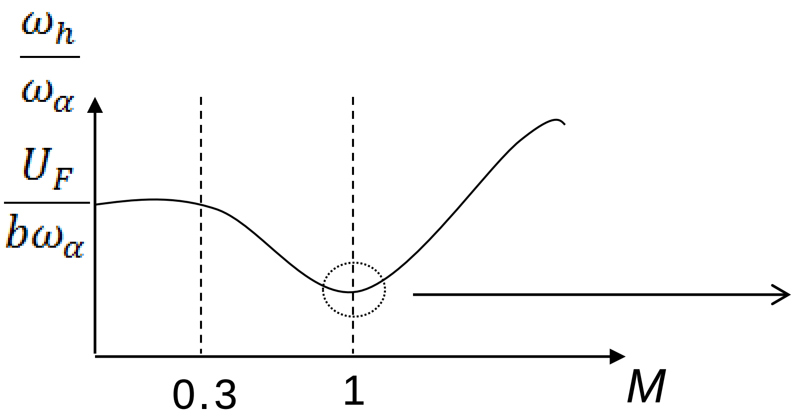
$$\frac{U_F}{b\omega_\alpha} = f\left(\mu, x_\alpha, \gamma_\alpha, a, \frac{\omega_h}{\omega_\alpha}, M\right)$$

Flutter Parameters Trends

The trends are:

a) $x_\alpha < 0$, (CG. Ahead of EA) - frequently no flutter

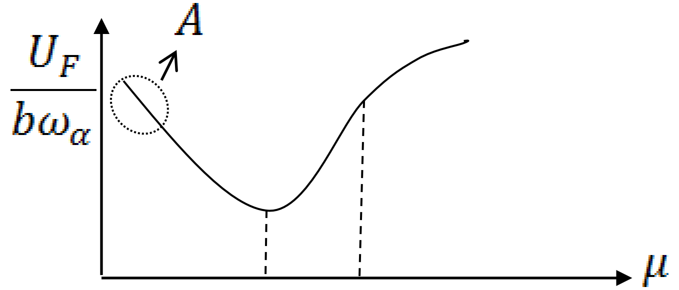
b)



dip can be quite severe and approach to zero

- Structural damping can remove dip completely

c)



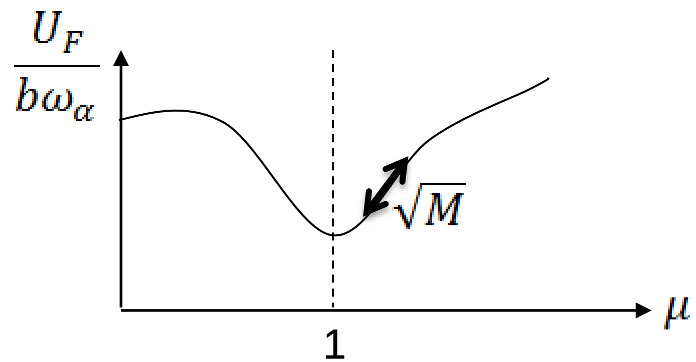
10 → helicopter rotor blade $\mu \geq 10$

2-D airfoil theory
$A, M \equiv 0$

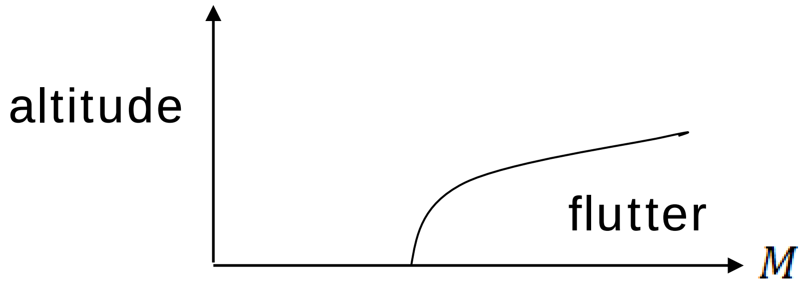
- For large μ, q_F constant, for small μ, U_F constant (dashed line)

Flutter Parameters Trends

d)

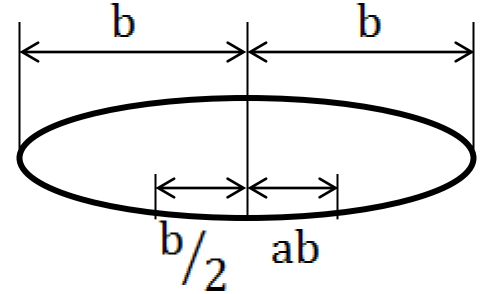


Various $\rho, \mu, \frac{a_\infty}{b\omega_\alpha}$



Flutter Approximate Formula

An approximate formula was obtained by Theodorsen and Garrick for small $\frac{\omega_h}{\omega_\alpha}$ large μ.



$$\frac{U_F}{b\omega_n}\frac{1}{\sqrt{\mu}} \cong \sqrt{\frac{{\gamma_\alpha}^2}{2\underbrace{(\frac{1}{2}+a+x_\alpha)}}}$$

Distance (non-dimensional)
between AC and CG (B.A.H. 9-22)

Recall divergence: $q_D = \frac{K_\alpha}{\rho c C_{l\alpha}} = \frac{1}{2}e{U_D}^2$

$$\frac{U_D}{b\omega_\alpha}\frac{1}{\sqrt{\mu}} \cong \sqrt{\frac{{\gamma_\alpha}^2}{2(\frac{1}{2}+a)}}$$

Flutter Parameters Trends

. non dimensionalize the typical section equation of motion

$$\frac{h}{b} = F_1(\omega_\alpha t : \frac{S_\alpha}{mb}, \frac{I_\alpha}{mb^2}, \frac{m}{\rho(2b)^2}, \frac{e}{b}, \frac{\omega_h}{\omega_\alpha}, M, \frac{U}{b\omega_a})$$

$$\alpha = F_2(\omega_\alpha t : \frac{S_\alpha}{mb}, \frac{I_\alpha}{mb^2}, \frac{m}{\rho(2b)^2}, \frac{e}{b}, \frac{\omega_h}{\omega_\alpha}, M, \frac{U}{b\omega_a})$$

- Choice of non-dimensional parameters:

. not unique, but a matter of convenience

i) non dimensional dynamic pressure, or 'aeroelastic stiffness No.'

$\lambda \equiv \frac{1}{\mu K_\alpha{}^2} = \frac{4\rho U^2}{m\omega_\alpha{}^2}$ instead of a non dimensional velocity, $\frac{U}{b\omega_\alpha{}^2}$

Flutter Parameters Trends

ii) $\omega_\alpha t$ nondimensional time

$\gamma_\alpha \equiv \dfrac{S_\alpha}{mb}$ static unbalance

${\gamma_\alpha}^2 \equiv \dfrac{I_\alpha}{mb^2}$ radius of gyration (squared)

$\mu \equiv \dfrac{m}{\rho(2b)^2}$ mass ratio

$a \equiv \dfrac{e}{b}$ location of e.a measured from a.c or mid-chord

$\dfrac{\omega_h}{\omega_\alpha}$ frequency ratio

M Mach number

$k_a = \dfrac{\omega_\alpha b}{U}$ Reduced frequency

Flutter Parameters Trends

- For some combinations of parameters, the airfoil will be dynamically unstable ('flutter')

- Alternative parametric representation

  Assume harmonic motion $h = \bar{h} e^{i\omega t}, \alpha = \bar{\alpha} e^{i\omega t}$

  Eigenvalues $\omega = \omega_R + i\omega_I$

$$\frac{\omega_R}{\omega_\alpha} = G_R(x_\alpha, r_\alpha, \mu, a, \frac{\omega_h}{\omega_a}, M, \frac{U}{b\omega_\alpha}) \quad , \quad \frac{\omega_I}{\omega_\alpha} = G_I(x_\alpha, r_\alpha, \mu, a, \frac{\omega_h}{\omega_a}, M, \frac{U}{b\omega_\alpha})$$

- For some combinations, $\omega_I < 0$, the system flutters.

Flutter Parameters Trends

— the coalescence flutter , conventional flow condition (no shock oscillation and no stall)

I) Static unbalance, x_α ... if $x_\alpha < 0$, frequently no flutter occurs

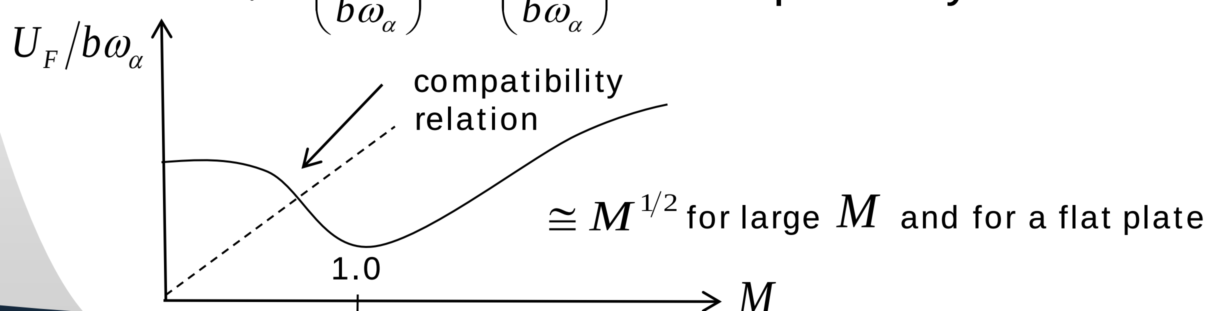
II) Frequency ration $\dfrac{\omega_h}{\omega_\alpha}$ ... $U_F / b\omega_\alpha$ is a minimum when $\dfrac{\omega_h}{\omega_\alpha} \approx 1$

III) Mach No. M ... aero pressure on an airfoil is normally greatest near $M = 1 \rightarrow$ flutter speed tends to be a minimum

For $M >> 1$, from aero piston theory, $p \approx \rho \dfrac{U^2}{M}$

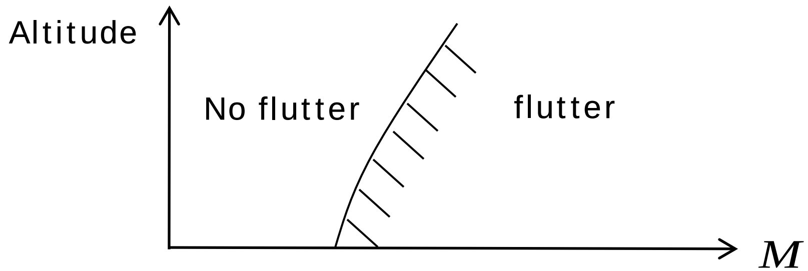
For $M \geq 1$ and constant μ, $U_F \approx M^{1/2}$

— for flight at constant altitude, ρ (hence μ) and α_∞ (speed of sound) fixed. $U = M\alpha_\infty \rightarrow \left(\dfrac{U}{b\omega_\alpha}\right) = M\left(\dfrac{\alpha_\infty}{b\omega_\alpha}\right) \rightarrow$ compatibility relation



Flutter Parameters Trends

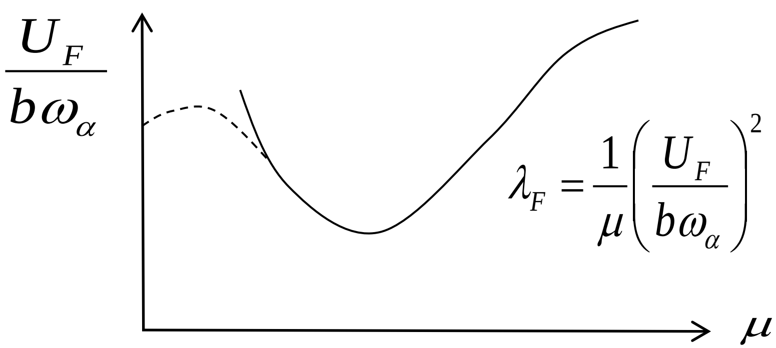
– by repeating flutter calculation for various altitudes (various ρ, α_∞, various μ and $\alpha_\infty / b\omega_\alpha$)



IV) Mass ratio μ... For large $\mu \rightarrow$ constant flutter dynamic pressure

For small $\mu \rightarrow$ constant flutter velocity (dashed line)

for $M \equiv 0$ and 2-D airfoil theory $\rightarrow U_F \rightarrow \infty$ for some small but finite μ (solid line)



$$\lambda_F = \frac{1}{\mu}\left(\frac{U_F}{b\omega_\alpha}\right)^2 \cong \text{constant for large } \mu$$

Flutter Prevention

- Flutter Prevention

  - add mass or redistribute the mass $\longrightarrow$ $x_\alpha < 0$ ("mass balance")

  - increase ω_α

  - move $\dfrac{\omega_h}{\omega_\alpha}$ away from 1

  - add damping, mainly for single D.O.F flutter

  - use composite materials
    - couple bending and torsion
    - shift ω_α away from ω_h

  - limit flight envelope by "fly slower"

Physical Explanation of Flutter (BA p. 258)

- Purely rotational motion of the typical section

$$I_\alpha \ddot{\alpha} + K_\alpha \alpha = M_y$$

— Approximate form: $\left[I_\alpha + \frac{\pi}{2}\rho_0 b^3 S\left(\frac{1}{8} + a^2\right)\right]\ddot{\alpha} - \frac{\partial M_y}{\partial \dot{\alpha}}\dot{\alpha} + \left[K_\alpha - \frac{\partial M_y}{\partial \alpha}\right]\alpha = 0$

if $\frac{\partial M_y}{\partial \dot{\alpha}}, \frac{\partial M_y}{\partial \alpha}$ are known, $\rightarrow$ second-order, damped-parameter system with 1DOF

— Laplace transform variable p, characteristic polynomial $a_0 p^2 + a_1 p + a_2$
two possible ways of instability

I) α coeff. $(+) \rightarrow (-), a_2 \leq 0$ in Routh's criterion $\rightarrow$ "torsional divergence" ...negative "aerodynamic spring" about E.A. overpowers K_α

II) $\frac{\partial M_y}{\partial \dot{\alpha}}$ $(-) \rightarrow (+), a_1 \leq 0$ in Routh's criterion $\rightarrow$ dynamic instability
entirely aerodynamic "negative" damping $I_m\{M_y\} = 0$

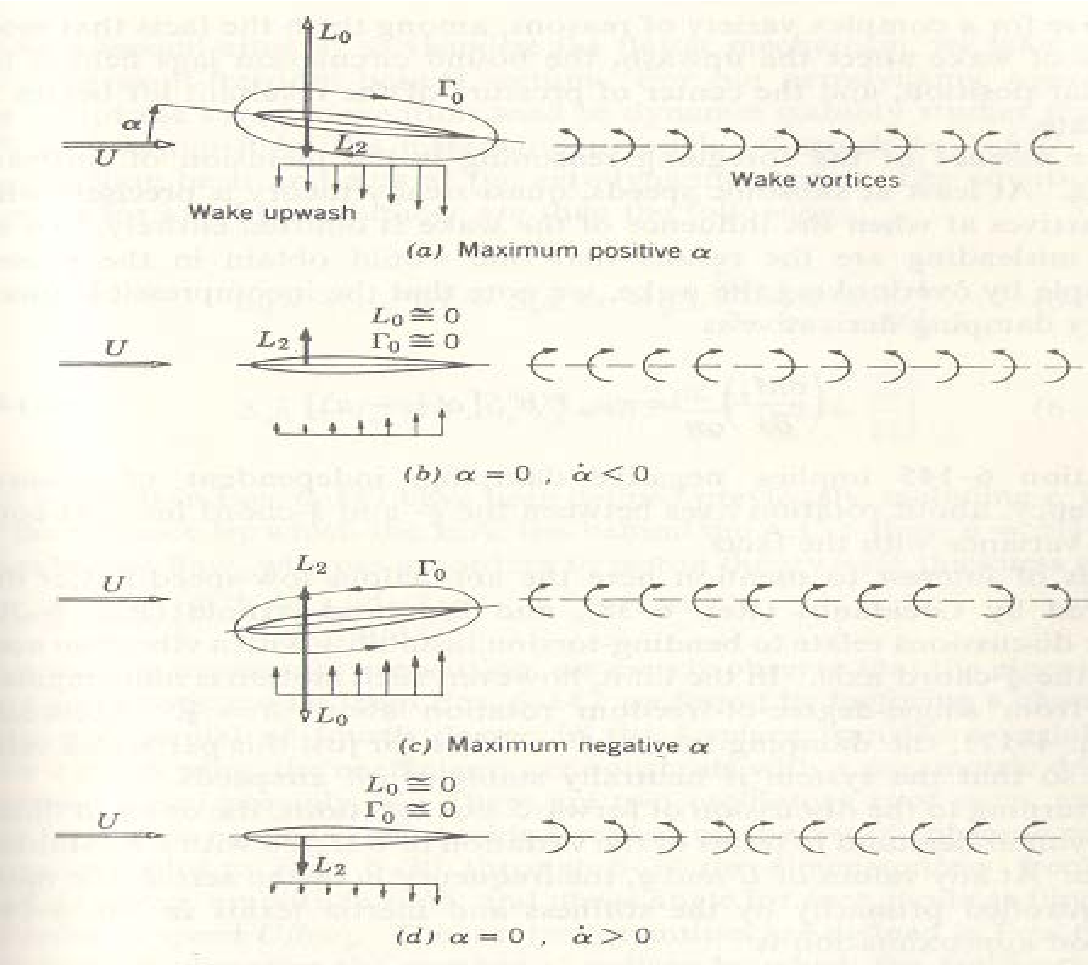
Physical Explanation of Flutter

- Qualitative explanation of negative damping
  - principal part L_o... due to the incremental a.o.a, in phase with α, e.a. at ¼ chord
    $\rightarrow$ adding to the torsional spring K_α when $a < -\frac{1}{2}$
  - bound circulation Γ_o ... changing with time. Since the total circulation is const., countervortices strength are induced
    shed from the trailing edge $\rightarrow$ wake vortex sheet
    $\rightarrow$ out-of-phase loading is induced (upwash) at low k
  - upwash... produces additional lift L_2
    $\Rightarrow$ when e.a. lies ahead of ¼ chord, the moment due to L_2 is in the same sense of $\dot{\alpha} \rightarrow$ net positive work per cycle of the wing "negative damping"
  - at higher k, damping becomes $(+)$
    more cycles of wake effects upwash, bound circulation lags behind α , center of pressure of lift oscillates

Physical Explanation of Flutter (BA p.258)

i) Pure rotational system (1 D.O.F)

$$I_\alpha \ddot{\alpha} + K_\alpha \alpha = M_y$$



2 D. O. F. system

$$m\left[\dot{h}+\omega_h^2 h\right]+S_\alpha\ddot{\alpha}=-qS\frac{\partial G}{\partial\alpha}\left[\alpha+\frac{\dot{h}}{U}\right]$$

$$S_\alpha\ddot{h}+I_\alpha\left[\ddot{\alpha}+\omega_\alpha^2\alpha\right]=qS\frac{\partial G}{\partial\alpha}e\left[\alpha+\frac{\dot{h}}{U}\right]$$

$$e=\begin{cases} b\left[\dfrac{1}{2}+a\right] \\ b\left[a+(\dfrac{\gamma+1}{4})M\dfrac{Aw}{2b^2}\right] & \text{Piston theory}\end{cases}$$

— Dimensionless frequency and damping

I) $U=0\approx 1/2$ critical $U/b\omega_\alpha$ ... mode shape remains the same as for free vibration, involving pure rotation about an axis

II) rotation axis moves forward, as indicated by falling amplitude of bending

III) gradual suppression of h... caused by lift variation due to torsion, lift, in phase with α, drives the bending freedom at ω greater ω_h
$\rightarrow$ response to it has a maximum downward amplitude at the instant of maximum upward force

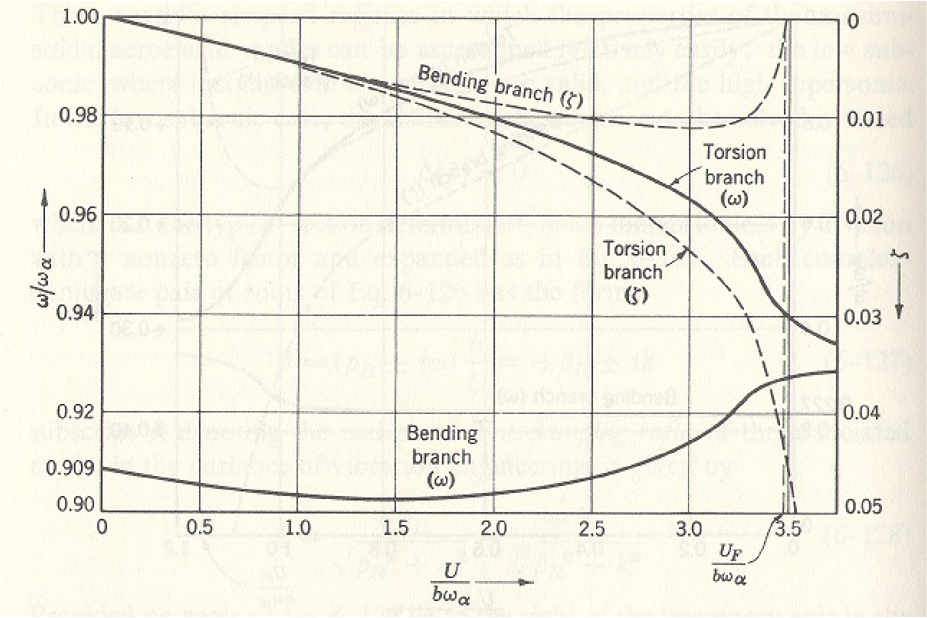
2 D. O. F. system

—Dimensionless frequency and damping

IV) simultaneously ω drops… lift constitutes a negative "aerodynamic spring" on the torsional freedom with "spring constant" ~ dynamic pressure

V) small advances in φ_h… due to lift, due to $\dot{h}$

VI) flutter occurrence … bending amplitude =0, only pure rotational oscillation about E.A., no damping acts



Flutter of a simple system 2 D.O.F
(BAH p. 532)

– flutter from coupling between the bending and torsional motions

the most dangerous but not the most frequently encountered

– Equations of motions

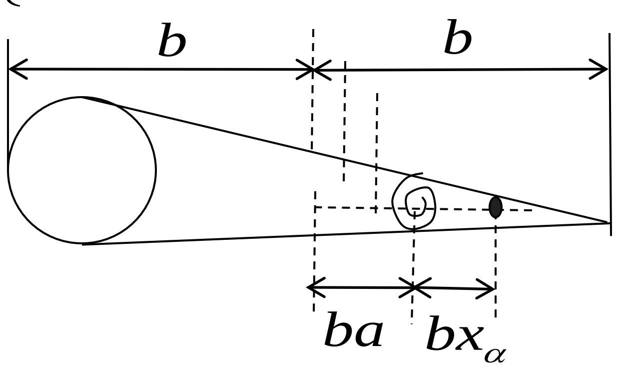
$$\begin{cases} m\ddot{h} + S_\alpha \ddot{\alpha} + m\omega_h^2 h = Q_h = -L \\ S_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + I_\alpha \omega_\alpha^2 \alpha = Q_\alpha = M_y \end{cases}$$

– Simple harmonic motion

b

$$h = \bar{h}_0 e^{i\omega t}, \alpha = \alpha_0 e^{i(\omega t + \varphi)} = \bar{\alpha}_0 e^{i\omega t}$$

$$\Rightarrow \begin{cases} -\omega^2 mh - \omega^2 S_\alpha \alpha + \omega h^2 mh = -L \\ -\omega^2 S_\alpha h - \omega^2 I_\alpha \alpha + \omega^2 I_\alpha \alpha = M_y \end{cases}$$



Flutter of a simple system 2 D.O.F

– Aerodynamic operator

$$L = -\pi\rho b^2\omega^2\left\{L_h\frac{h}{b} + \left[L_\alpha - L_h(\frac{1}{2}+a)\right]\alpha\right\}$$

$$M_y = -\pi\rho b^2\omega^2\left\{\left[M_h - L_h(\frac{1}{2}+a)\right]\frac{h}{b} + \left[M_\alpha - (L_\alpha + M_h)(\frac{1}{2}+a) + L_h(\frac{1}{2}+a)^2\right]\alpha\right\}$$

function of L_h, L_α, M_α (incompressible) $K, M_\alpha \ldots 1/2$

Plugging the aerodynamic operator, and set the coefficient determinant to zero

- characteristic eqn. for ω_α/ω... implicitly dependent on the 5 dimensionless system parameters

  a : axis location

  ω_h/ω_α : bending-torsion frequency ratio

  $x_\alpha = S_\alpha/mb$: dimensionless static unbalance

  $r_\alpha = \sqrt{I_\alpha/mb^2}$: radius of gyration

  $m/\pi\rho b^2$: density ratio

- parametric trends of U_F in terms of 5 parameters

Flutter of a simple system 2 D.O.F

– Divergence speed U_p

$$U_p = \sqrt{\frac{2K_\alpha}{\rho e c c_{l\alpha}}} = \sqrt{\frac{K_\alpha}{2\pi\rho b^2\left[\frac{1}{2}+a\right]}}$$

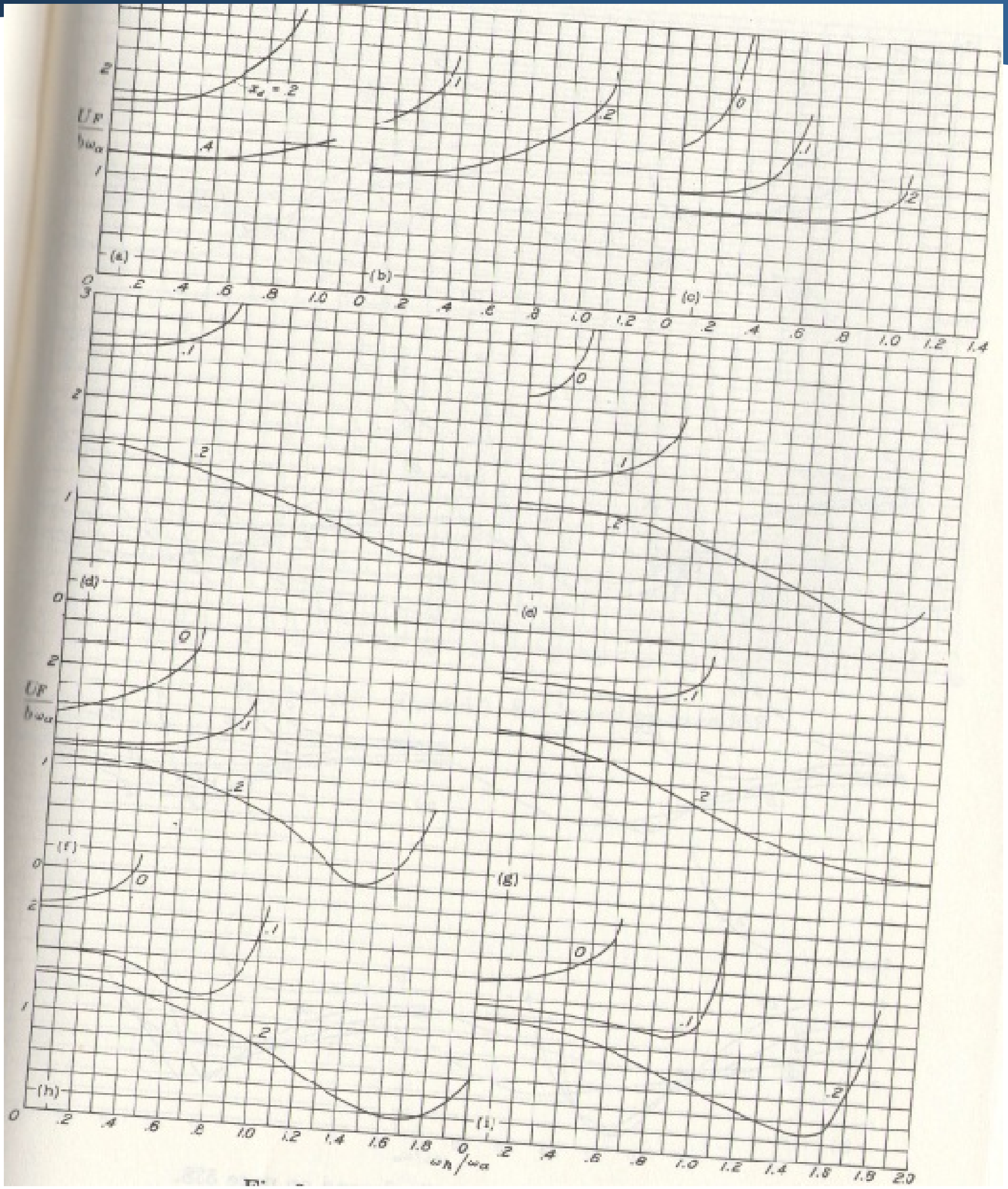
$b(\frac{1}{2}+a)$ from a.c. to e.a.

$$\frac{U_D}{b\omega_\alpha} = \frac{U_p}{b\omega_\alpha} = \frac{1}{b\omega_\alpha}\sqrt{\frac{K_\alpha b^2}{I_\alpha}}\sqrt{\frac{I_\alpha}{mb^2}\frac{m}{\pi\rho b^2[1+2a]}} = \sqrt{\frac{m}{\pi\rho b^2}\frac{r_\alpha^2}{[1+2a]}}$$

both U_D above and the flutter speeds in Fig 9-5 from the 2-D aerodynamic strip theory $\rightarrow$ the predicted U_F will not exceed U_D

Flutter of a simple system 2 D.O.F

- Fig. 9-5 (A)



Flutter of a simple system 2 D.O.F

- Fig. 9-5 (B)

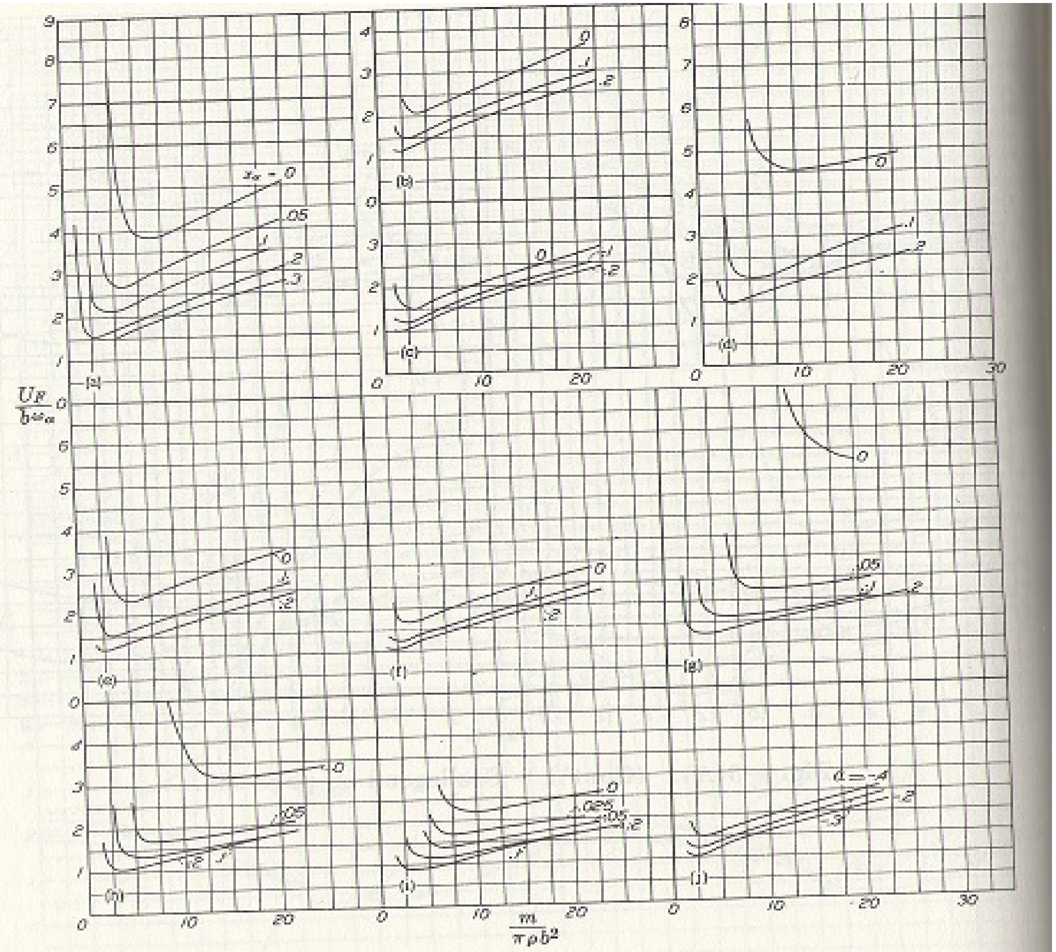


Fig. 9-5(B). Dimensionless flutter speed $U_F/b\omega_\alpha$ plotted against density ratio $m/\pi\rho b^2$ for various values of static unbalance x_α; $r_\alpha^2 = \frac{1}{4}$.

Flutter of a simple system 2 D.O.F

- Fig. 9-5 (C)

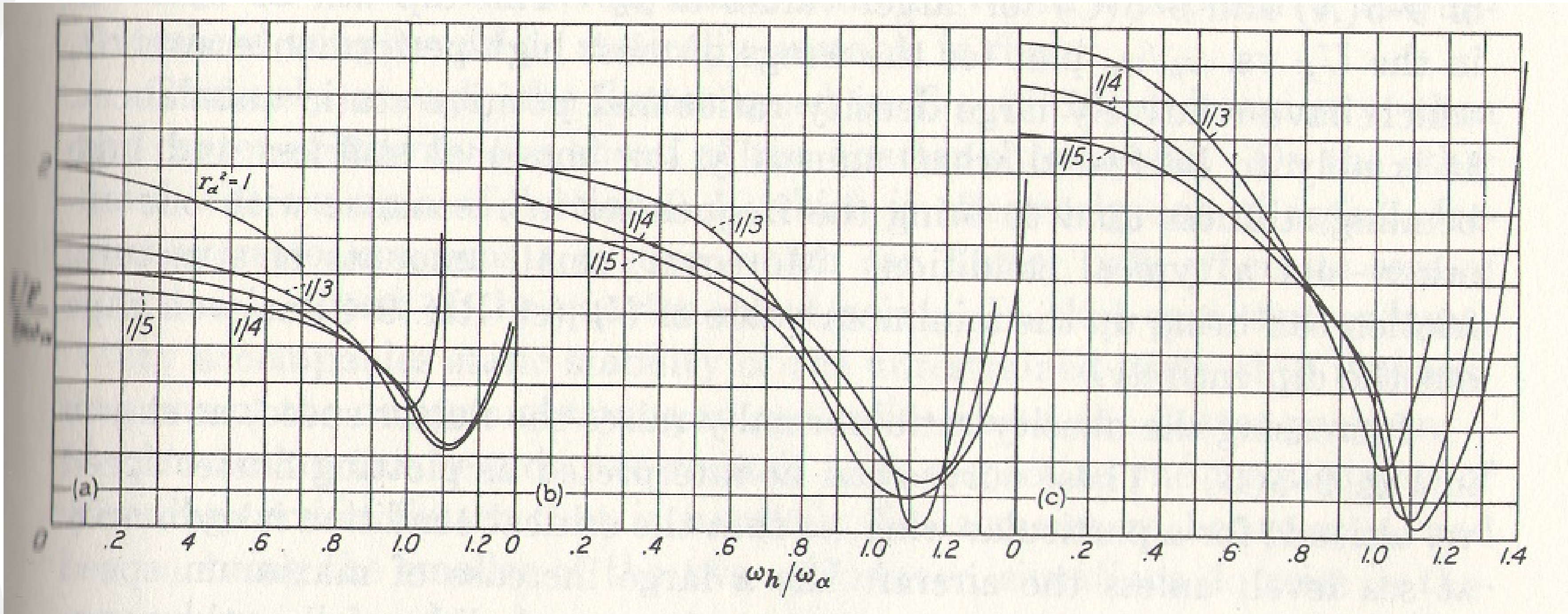


Fig. 9–5(C). Dimensionless flutter speed $U_F/b\omega_\alpha$ plotted against frequency ratio ω_h/ω_α for various values of radius of gyration r_α^2; $a = -0.2$, $x_\alpha = 0.1$.

Flutter of a simple system 2 D.O.F

- Fig. 9-5 (D)

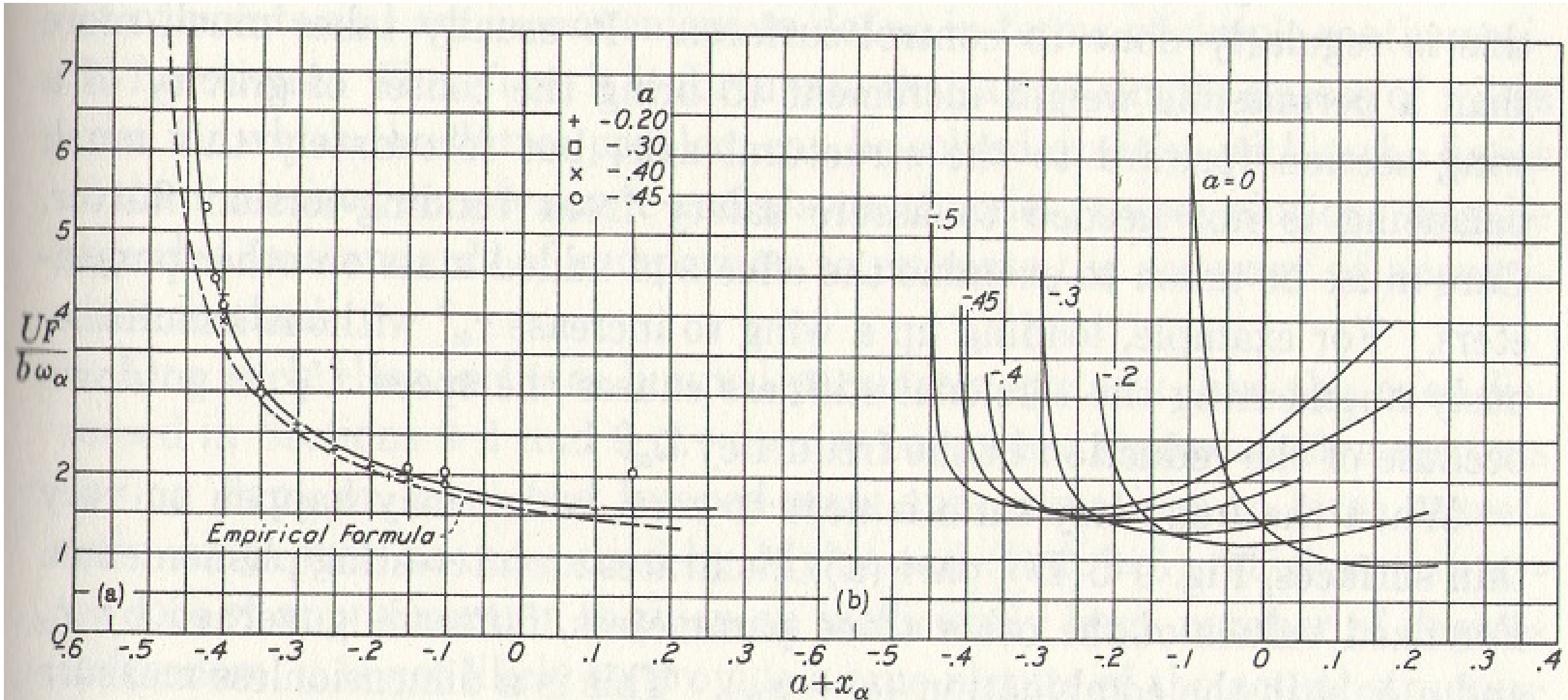


Fig. 9–5(D). Dimensionless flutter speed $U_F/b\omega_\alpha$ plotted against center-of-gravity location $a + x_\alpha$ for various values of axis location a; $r_\alpha^2 = \frac{1}{4}$.

Flutter of a simple system 2 D.O.F

Fig. 9-5(A),(C)... dip near $\omega_h/\omega_\alpha \cong 1 \rightarrow$ can bring up with small amounts of structural friction

(B)... density ratio increase $\rightarrow$ raise flutter speed

(flutter speed vs. altitude)

"mass balancing"... flutter speed is more sensitive to a change of x_α

$\rightarrow$ Not much balancing is needed to assure safety form bending-torsion flutter

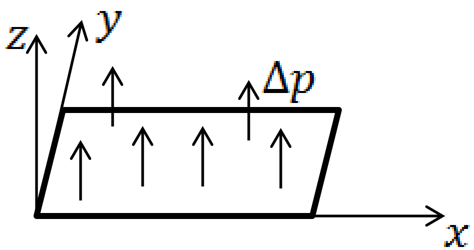
Fig. 9-5(D)... flutter is governed by $(a + x_\alpha)$ chordwise c.g.

Garrick and Theodorsen (1940):

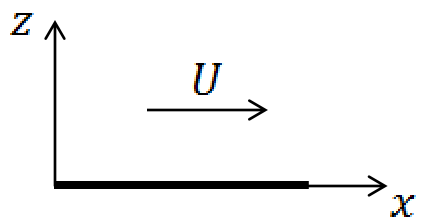
$$\frac{U_F}{b\omega_\alpha} \approx \sqrt{\frac{m}{\pi\rho b^2}\frac{r_\alpha^2}{\underbrace{\left[1+2(a+x_\alpha)^2\right]}_{\text{From a.c. to c.g.}}}}$$

Panel Flutter

- Panel Flutter:
  - Self-excited oscillation of the external skin of a flight vehicle when exposed to airflow on that side (supersonic flow)



simplify



- For simplicity, consider a 2-D simply supported panel in supersonic flow; for a linear panel flutter analysis, the equation of motion is:

$$D\frac{\partial^4 w}{\partial x^4} + m\ddot{w} = P_A$$, where $$D = \frac{Eh^3}{12\left(1-\nu^2\right)}$$ (isotropic, plate stiffness)

$m =$ mass/unit, h : thickness

P_A = aerodynamic pressure

Panel Flutter

$$For\ M > 1.6, P_A \approx \frac{-\rho V^2}{\sqrt{M^2-1}}\left\{\frac{\partial w}{\partial x} + \frac{M^2-2}{M^2-1}\frac{1}{V}\frac{\partial w}{\partial t}\right\}$$

- Putting all together, the governing equation becomes:

$$D\frac{\partial^4 w}{\partial x^4} + \frac{\rho V^2}{\sqrt{M^2-1}}w' + \frac{\rho V}{\sqrt{M^2-1}}\frac{M^2-2}{M^2-1}\dot{w} + \ m\ddot{w}$$

- It is subject to :

$$w(0,t) = w(a,t) = 0$$

$$w"(0,t) = w"(a,t) = 0$$

- They are the simply supported B.C
- Using Galerkin Method $\Rightarrow$ $w(x,t) = \sum_{j=1}^{n} sinj\frac{\pi x}{a}q_j(t)$
- satisfies all the B.C.'s

Panel Flutter

- By setting: $q_j(t) = \bar{q}_j e^{\bar{p}t}$
- we get:

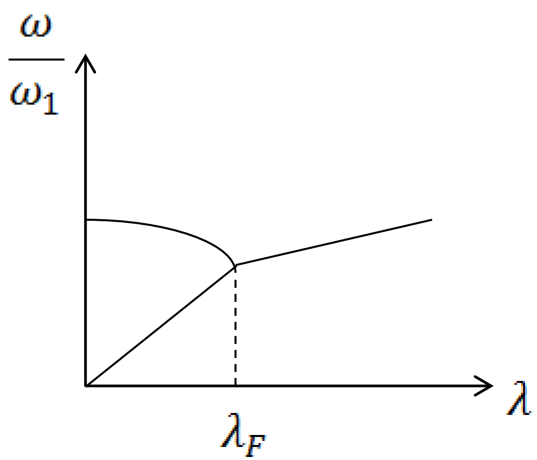
$$\underbrace{\begin{bmatrix} (p^2 + a_\infty p + {\omega_1}^2) & -\dfrac{8{\omega_1}^2}{3\pi^2}\lambda_F \\ \dfrac{8{\omega_1}^2}{3\pi^2}\lambda & (p^2 + a_\infty p + 16{\omega_1}^2) \end{bmatrix}}_{\text{Anti-symmetric}} = 0$$

- where a_∞ : speed of sound, $\lambda \equiv \dfrac{\rho V^2 a^3}{D\sqrt{M^2 - 1}}$: critical speed param.

$\omega_1 = \pi^2 \sqrt{\dfrac{D}{\pi a^4}}$: lowest natural frequency

Panel Flutter

- A typical result :



[Note] $\lambda_F = \dfrac{\rho {U_F}^2 a^3}{D\sqrt{M^2 - 1}}$

If λ_F constant ⟹ $E \uparrow \rightarrow D \uparrow \rightarrow \; q_F \uparrow$

$h \uparrow \rightarrow D \uparrow \rightarrow \; q_F \uparrow$

$a \downarrow \rightarrow \; q_F \uparrow$

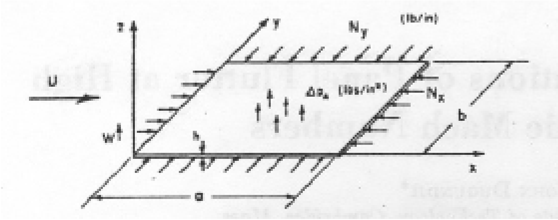
$\dfrac{a}{b} \uparrow \rightarrow \lambda_F \uparrow$

Panel Flutter

Theoretical considerations of panel flutter at high supersonic mach numbers (AIAA J, 1966)

- Basic Panel Flutter Eqn. and its Sol.
- A rectangular panel simply supported on all 4 edges and subject to a supersonic flow over one side, midplane compressive force Nx, Ny, elastic foundation K structural damping G_s

$$D\Delta^4 w = \Delta p_A - \rho_M h \frac{\partial^2 w}{\partial t^2} - Nx\frac{\partial^2 w}{\partial x^2} - Ny\frac{\partial^2 w}{\partial y^2} - Kw - G_s\frac{\partial w}{\partial t} \quad (1)$$



- Aerodynamic pressure for high supersonic Mach No.

$$\Delta p_A \approx -\left[\frac{\rho_A U^2}{(M^2-1)^2}\right]\cdot\left[\frac{\partial w}{\partial x} + \frac{1}{U}\frac{\partial w}{\partial t}\,\frac{M^2-2}{M^2-1}\right] \quad (2)$$

(1)+(2) : non-dimensional coordinates introduced ζ, η, τ

$$\frac{\partial^4 w}{\partial \zeta^4} + 2\left(\frac{a}{b}\right)^2\frac{\partial^4 w}{\partial \zeta^2 \partial \eta^2} + \left(\frac{a}{b}\right)^4\frac{\partial^4 w}{\partial \eta^4} + \lambda\frac{\partial w}{\partial \zeta} + \pi^4 g + \frac{\partial w}{\partial \tau} + \pi^4\frac{\partial^2 w}{\partial \tau^2}$$
$$+ \pi^4 kw + \pi^2\gamma_x\frac{\partial^2 w}{\partial \zeta^2} + \pi^2\gamma_y\left(\frac{a}{b}\right)^2\frac{\partial^2 w}{\partial \eta^2} = 0$$

Panel Flutter

- Additional non-dimensional parameters

$$\lambda = \frac{\rho_A U^2 a^3}{D(M^2-1)} : dynamic\ pressure\ parameter$$

$$g_T = g_A + g_S : total\ damping\ parameter$$

$$g_A = a335\left\{M(M^2-2)(M^2-1)^{\frac{3}{2}}\right\} * \left(\frac{\rho_A}{\rho_M}\right)\left(\frac{c_A}{c_M}\right)\left(\frac{a}{h}\right)^2 : \text{aerodynamic damping coefficient}$$

$$g_s = \frac{g_i \omega_i}{\omega_0} : \text{effective structural damping coefficient}$$

$$\frac{a}{b} = aspect\ ratio$$

$$k = \frac{k a^4}{\pi^4 D} : foundation\ parameter$$

$$\gamma_x = \frac{N_x a^2}{\pi^2 D} : longitudinal\ compression\ parameter$$

$$\gamma_y = \frac{N_y a^2}{\pi^2 D} : lateral\ compression\ parameter$$

Panel Flutter

- Simply supported B.C's

$$\text{At } \eta = 0{,}1;\; w = 0, \frac{\partial^2 w}{\partial y^2} = 0$$

- Solution procedure

$$w(\zeta,\eta,\tau) = \overline{w}(\zeta)[\sin m\pi\eta]e^{\bar{\theta}\tau}$$

$$\bar{\theta} = \bar{\alpha} + i\,\overline{w}$$

O.D.E

$$\frac{d^4\overline{w}}{d\zeta^4} + C\frac{d^2\overline{w}}{d\zeta^2} + A\frac{d\overline{w}}{d\zeta} + (B_R + iB_I)\overline{w} = 0$$

$$C = \pi^2\left[-z\left(\frac{ma}{b^2}\right) + \gamma_x\right]$$

$$A = \lambda$$

$$B = B_R + iB_I = \pi^4[\left(\frac{ma}{b^2}\right) + k - \left(\frac{ma}{b}\right)\gamma_y{}^2 + g_T\bar{\theta} + \bar{\theta}^2]$$

Panel Flutter

General solution of O.D.E

$$\bar{w}(\zeta) = C_1 e^{z_1 \zeta} + C_2 e^{z_2 \zeta} + C_3 e^{z_3 \zeta} + C_4 e^{z_4 \zeta}$$

This along with the B.C, the determinant must be:

- Equal to zero for nontrivial solutions

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ {z_1}^2 & {z_2}^2 & {z_3}^2 & {z_4}^2 \\ e^{z_1} & e^{z_2} & e^{z_3} & e^{z_4} \\ {z_1}^2 e^{z_1} & {z_2}^2 e^{z_2} & {z_3}^2 e^{z_3} & {z_4}^2 e^{z_4} \end{vmatrix} = 0$$

- For low values of the determinant, the eigenvalues are real. $B_I = 0$

- Above a certain value of A, they become imaginary. $B_I \neq 0$

Panel Flutter

Complete panel behavior

- Plotting $\bar{\theta} = \bar{\alpha} + i\ \bar{\omega}, \omega, \gamma, t, dynamic\ pressure$

- The Frequency coalescence: Instability occurs at

  $\bar{\alpha} = 0$

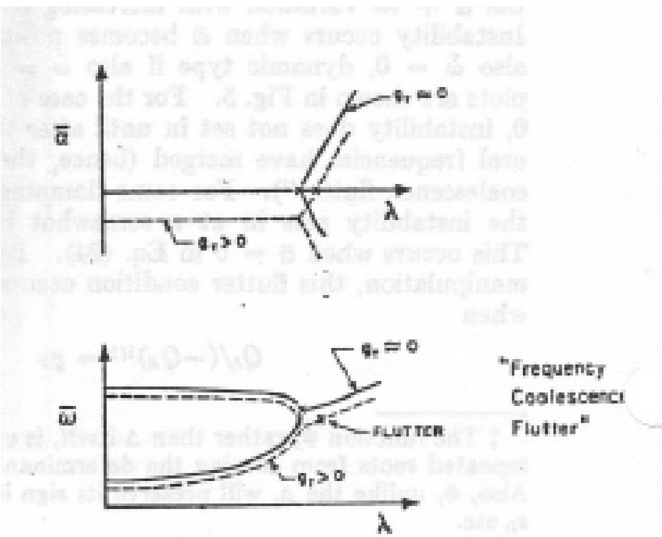
$$\frac{Q_I}{(-Q_R)^{\frac{1}{2}}} = g_T$$

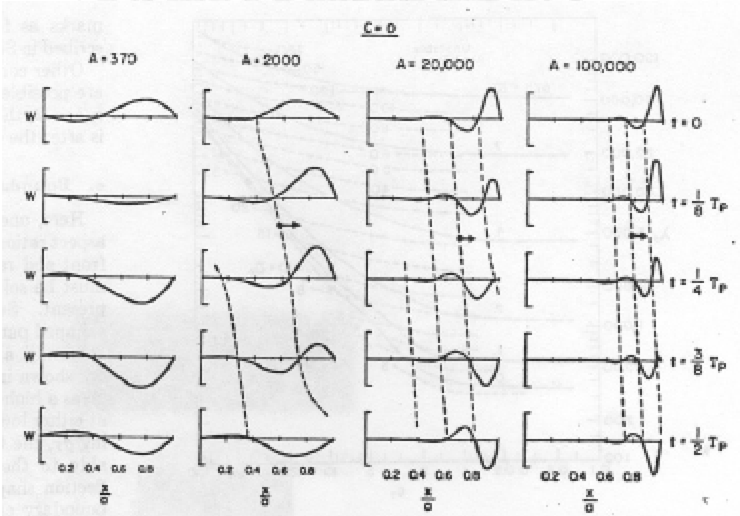
- Flutter Frequency:

$$\overline{\omega_F} = (-Q_R)^{\frac{1}{2}} = \omega_F/\omega_0$$

- Deflection shapes

  - Simple sine shape standing-wave type for A=0
  - Standing-wave type at low A
  - Traveling-wave type at high values of A





Computational Aeroelasticity

Computational Aeroelasticity

Structure + Aerodynamics

- With the abundance of computational resources and algorithms, there has been a great development in two areas:

- CFD: Computational Fluid Dynamics

- CSD: Computational Structural Dynamics
    - CAE: Computational Aeroelasticity

Computational Aeroelasticity

- Difficulties arise from the nature of the two methods.
  - CFD: Finite difference discretization procedure based on Eulerian (spatial) description
  - CSD: finite element method based on Lagrangian (material) description.
- Define the nature of the coupling when combining the two numerical schemes.

Computational Aeroelasticity

i) Tightly (or closely) coupled analysis:

- Most popular
- Interaction between CFD and CSD codes occurs at every time step
- Guarantee of convergence and stability

ii) Loosely coupled analysis:

- CFD and CSD are solved alternatively
- Occasional interaction only

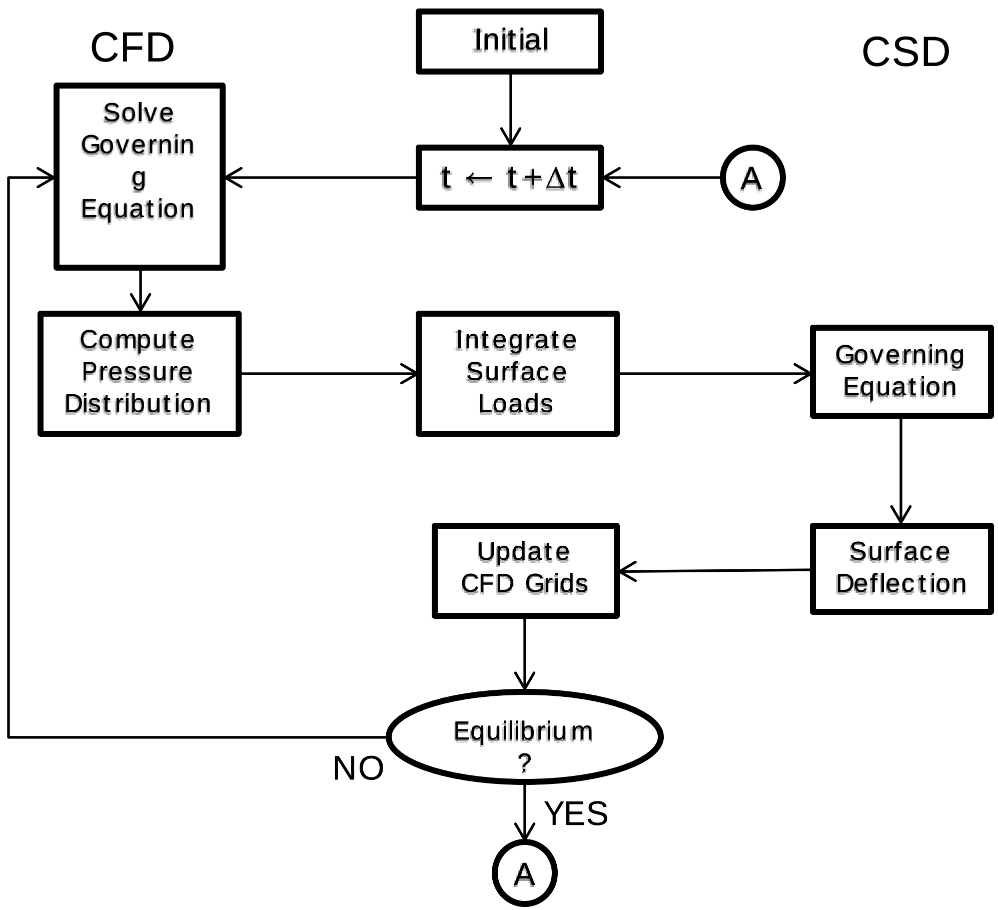
=> Difficulties in convergence

iii) Intimately coupled (unified) analysis:

- The governing equations are re-formulated and solved together

Computational Aeroelasticity

i) – Tightly (or closely) coupled analysis:



End of Chapter III