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457.643 Structural Random Vibrations
In-Class Material: Class 01

0. Introduction

Grigoriu, M. (2004) Research Perspective in Stochastic Mechanics. *Engineering Design Reliability Handbook*, edited by E. Nikolaidis, D.M. Ghiocel, and S. Singhal, CRC Press, Boca Raton, FL., Chap. 6

The response and evolution ($\mathcal{X}(x,t)$) of mechanical, hydrological/hydraulic, biological, and other systems subjected to an input $\mathcal{Y}(x,t)$ can be characterized by an equation of the form

$$\mathcal{D}[\mathcal{X}(x,t)] = \mathcal{Y}(x,t), \qquad t \geq 0, \qquad x \in D \subset \mathcal{R}^d$$

where

$\mathcal{D}$: algebraic, integral, or differential operator with random or deterministic coefficients

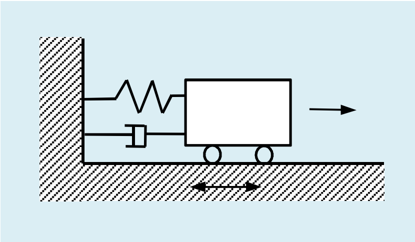
$\mathcal{Y}(x,t)$: random or deterministic input function

$\mathcal{X}(x,t)$: random or deterministic output (response) function

The above equation can represent four classes of problems:

1. Deterministic systems and input (457.516 Dynamics of Structures)
2. Deterministic systems and stochastic input **(457.643 Structural Random Vibrations)**
3. Stochastic systems and deterministic input (457.646 Topics in Structural Reliability)
4. Stochastic systems and input

For example, consider an SDOF linear oscillator subject to earthquake ground motion:



E.O.M.:

Some results of "random vibration analysis":

- Mean and variance of $X(t)$:
- Instantaneous failure probability:
- First-passage failure probability:

See "Syllabus and Course Outline" handout for course objectives and contents.

I. Basic Elements

◎ Review on basic theories of probability

Self-review of "II. Basic Theory of Probability and Statistics" part of the course "457.646 Topics in Structural Reliability" is required.

Additional basic topics to review for this course:

◎ Characteristic function (L&S Chapter 3)

Alternative (complete/incomplete) description of random variable X

$M_X(\theta) \equiv \mathrm{E}_X[\exp(i\theta X)] = \int \qquad\qquad dx$ ______________ transform of ______

Therefore,

$f_X(x) = \frac{\quad}{\quad}\int \qquad\qquad d\theta$ _________ __________ transform of ______

※ See Appendix B of L&S for a brief review of Fourier transform (if necessary)

Note:

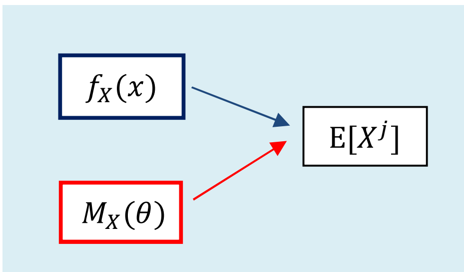
1) $M_X(\theta)$ always exists because the condition for the existence of a Fourier transform is $\int_{-\infty}^{\infty} |f_X(x)|dx < \infty$ ("absolutely integrable"), and we know that $\int_{-\infty}^{\infty} |f_X(x)|dx = \quad$.
2) Why use $M_X(\theta)$?
   - Useful for analytical development or proof (will be shown later in the course)
   - Especially useful for generating ____________

◎ M________ generating property of characteristic function

Remember $M_X(\theta) = \mathrm{E}_X[\exp(i\theta X)] = \int_{-\infty}^{\infty} \exp(i\theta X)\, f_X(x)dx$

$$\frac{d^j}{d\theta^j} M_X(\theta) = i^j \mathrm{E}_X[X^j \exp(i\theta X)]$$

$$\left.\frac{d^j}{d\theta^j} M_X(\theta)\right|_{\theta=0} = i^j \mathrm{E}_X[X^j]$$



Therefore,

$$\left.\frac{1}{i^j}\frac{d^j}{d\theta^j} M_X(\theta)\right|_{\theta=0} = \mathrm{E}_X[X^j] = \int$$

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※ McLauren series of $M_X(\theta)$

$$M_X(\theta) = \sum_{k=0}^{\infty} \frac{d^k M_X(\theta)}{d\theta^k}\bigg|_{\theta=0} \frac{\theta^k}{k!}$$
$$= \sum_{k=0}^{\infty} i^k \mathrm{E}_X[X^k] \frac{\theta^k}{k!}$$
$$= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} \mathrm{E}_X[X^k]$$

- We could approximate the characteristic function using low-order moments?

※ "Moment generating function" $\mathrm{E}_X[\exp(-rX)] =$

- L________ transform
- Moment generating equation more simple (because real-valued)
- May not exist mathematically for some probability density function (p. 86 L&S)

Example

1) Derive the characteristic function of $X \sim N(\mu, \sigma^2)$

2) Generate the first and second moment of X using the characteristic function to confirm.

◎ Log-characteristic function

$L_X(\theta) \equiv \ln M_X(\theta)$

n^{th} order cumulant function $\kappa_n(X) \equiv \frac{1}{i^n}\frac{d^n L_X(\theta)}{d\theta^n}\Big|_{\theta=0}$

- $\kappa_1(X) = \frac{1}{i}\frac{dL_X(\theta)}{d\theta}\Big|_{\theta=0} =$
- $\kappa_2(X) = \mathrm{E}_X[(X-\mu)^2] =$
- $\kappa_3(X) = \mathrm{E}_X[(X-\mu)^3] =$
- $\kappa_4(X) = \mathrm{E}_X[(X-\mu)^4] - 3\sigma^4 =$

For Gaussian, $\kappa =$ and thus $\kappa_4(X) =$. For $n \geq 3$, $\kappa_n(X) =$

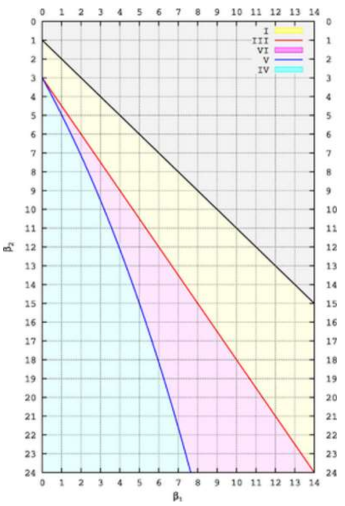
- ➢ Cumulants are useful since they are related to "c_______" moments
- ➢ Another merit: $\kappa_n(X) \cong 0$ for higher order, so easier to approximate PDF (through log-characteristic function)

$$L_X(\theta) = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}\kappa_n(X)$$

Note: $\kappa_0(X) = 0$ (check by yourself)

◎ Importance of moment analysis (L&S 3.8)

"In many random variable problems (and in much of the analysis of stochastic processes), one performs detailed analysis of only the first and second moments of the various quantities, with occasional consideration of skewness and/or kurtosis. One reason for this is surely the fact that analysis of mean, variance, or mean squared value is generally much easier than analysis of probability distributions. Furthermore, in many problems, one has some idea of the shape of the probability density functions, so knowledge of moment information may allow evaluation of the parameters in that shape, thereby giving an estimate of the complete probability distribution. If the shape has only two parameters to be chosen, in particular, then knowledge of mean and variance will generally suffice for this procedure. In addition to these pragmatic reasons, though, the results in Eqs. 3.31, 3.32, 3.35, and 3.36 (i.e. McLauren series expansion of characteristic function and log-characteristic functions) give a theoretical justification for focusing attention on the low-order moments. Specifically, mean, variance, skewness, kurtosis, and so forth, in that order, are the first items in an infinite sequence of information that would give a complete description of the problem. In most situations, it is impossible to achieve the complete description, but it is certainly logical for us to focus our attention on the first items in the sequence."



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For example, if one assumes a random quantity follows a Pearson distribution, the type is determined by the square of the skewness (β_1 in the left figure) and the kurtosis (β_2). The first four moments completely describe the parameters of the distribution.