Seoul National University
Dept. of Civil and Environmental Engineering

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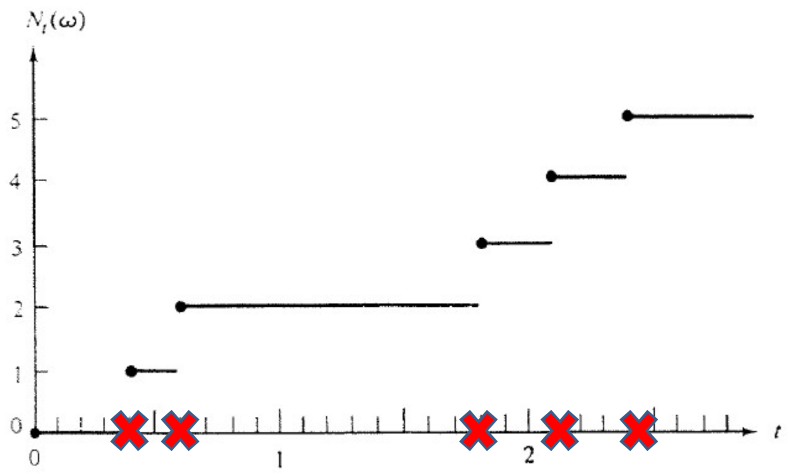
457.643 Structural Random Vibrations
In-Class Material: Class 07

II-1. Random Process (contd.)

◎ Artificial generation of Poisson process



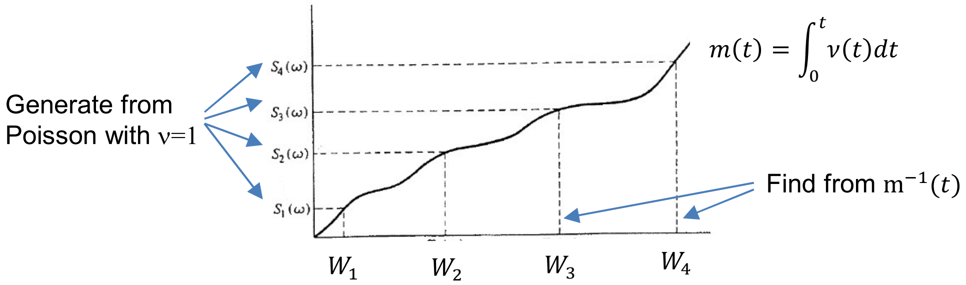
- Actually, we aim to generate waiting times (arrival times) for $n = 1,2,3,\ldots$, i.e. $W_1, W_2, \ldots$
- For a homogeneous Poisson process, we can generate T_n using W_1, and
- T_n and W_1 follows _________ distribution
- In Matlab®, one can generate __________ random variables using `exprnd(μ,M,N)`

  1) μ: mean $= 1/\nu$

  2) M,N: size of the output matrix
- To generate non-homogeneous Poisson process, need to use a theorem,

  $W_i = m^{-1}(S_i), i = 1,2,\ldots$, are arrival times of the non-homogeneous Poisson process with $m(t)$ when $S_i, i = 1,2,\ldots$, are arrival times of the homogeneous Poisson process with $\nu = 1$

Cinlar, E. (1975). *Introduction to Stochastic Processes*, Dover Books (reprinted in 2013)



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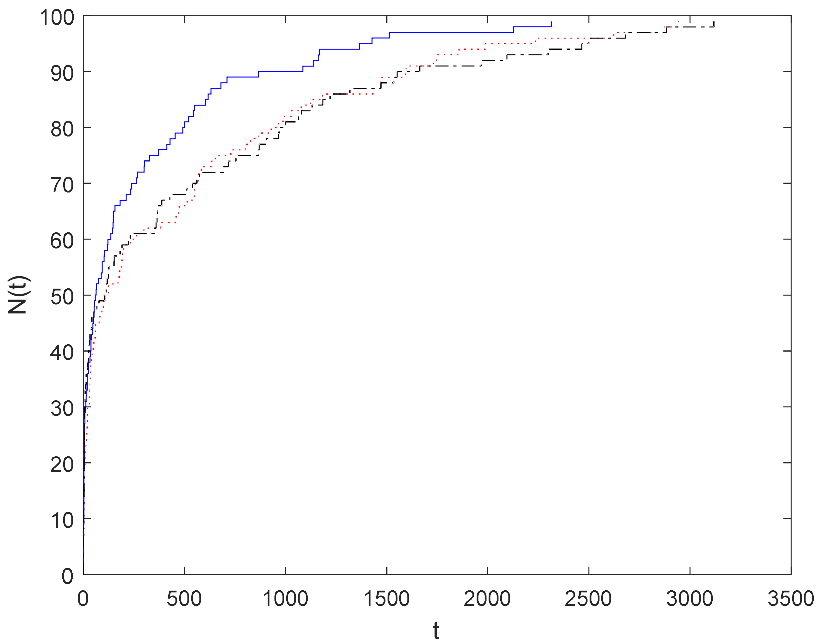
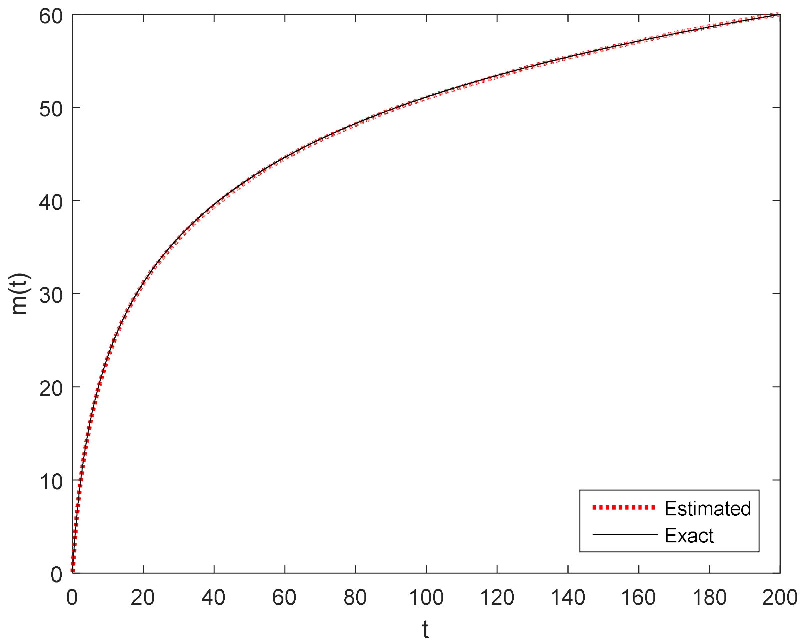
- **Example:** Generating NHPP with $m(t) = 13 \cdot \ln(0.5t + 1)$ (Byun et al. 2017)

Byun J., H.-M. Noh, J. Song (2017). Reliability growth analysis of k-out-of-N systems using matrix-based system reliability method. *Reliability Engineering & System Safety*. Vol. 165, 410-421.

1) Three random samples:



2) Comparison between exact $m(t)$ and estimated one using 1,000 samples



** Check “`NHPoissonGenerationTest.m`” at eTL website for details

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◎ Normal (Gaussian) process (Read L&S 4.10)

$X(t)$ is a Gaussian process
if, for any n, and any $\{t_1, t_2, \cdots, t_n\}$,
the random variables $X(t_1), \cdots, X(t_n)$ are ___________ ____________

- The process is completely defined by specifying ________ for $\forall t$ and _______ for $\forall(t, s)$
- For a Gaussian process, being "weakly stationary" implies stationarity in the ___________ sense
- Any linear function of Gaussian processes is a ___________ process
  e.g. $\dot{X}(t)$ is Gaussian if $X(t)$ is Gaussian (why?)
  e.g. $X(t)$ and $\dot{X}(t)$ are ___________ ____________
- Why useful?
  1) Convenient to handle
  2) _________ __________ theorem
- Hard to justify Gaussian process assumption if
  1) the distribution is not symmetric, or
  2) ___________ is not equal to 3
- Textbook focusing on non-Gaussian processes: M. Grigoriu (1995), *Applied Non-Gaussian Processes*

◎ Jointly Gaussian processes

$X_1(t), X_2(t), \cdots, X_m(t)$ are jointly Gaussian processes
if, for any n, and any $\{t_1, t_2, \cdots, t_n\}$, the random variables
$\{X_1(t_1), \cdots, X_1(t_n), X_2(t_1), \cdots, X_2(t_n), \cdots, X_m(t_1), \cdots, X_m(t_n)\}$ are

___________ ____________

- The processes are completely defined by specifying $\boldsymbol{M}_{\boldsymbol{X}}(t) = \{ \qquad \qquad \}^{\mathrm{T}}$ and $\boldsymbol{\Sigma}_{(\mathrm{t,s})} = [\qquad]$

◎ Other notable processes

Wiener process: Coordinate of a particle doing _________ motion

- Increments $B(t_1) - B(t_2)$ follow zero-mean Gaussian and s.i. (non-overlapping)
- $\mathrm{E}\{[B(t_2) - B(t_1)]^2\} = 2D|t_2 - t_1|$

Non-Gaussian processes:

- Memoryless nonlinear transformation $X(t) = g[Y(t)]$

- Nonlinear transformation with memory

  e.g. Diffusion process $dX(t) = m[X(t), t]dt + \sigma[X(t), t]dB(t)$

Applications of Homogeneous and Non-homogeneous Poisson processes:

- Systems with repairable components

Der Kiureghian, A., O. Ditlevsen, and J. Song (2007). Availability, reliability and downtime of systems with repairable components. *Reliability Engineering & System Safety*, Vol. 92, 231-242.

- Reliability growth of components during the test period

Byun J., H.-M. Noh, J. Song (2017). Reliability growth analysis of k-out-of-N systems using matrix-based system reliability method. *Reliability Engineering & System Safety*. Vol. 165, 410-421.

Renewal processes: generalization of Poisson process

Pandey, M., and van der Weide, J.A.M. (2017). Stochastic renewal process models for estimation of damage cost over the life-cycle of a structure. *Structural Safety*. Vol. 67, 27-38.

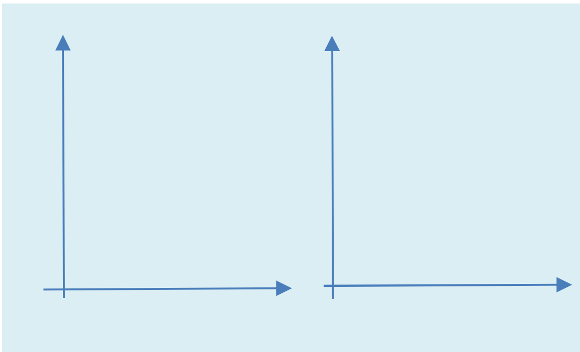
II-2. Stochastic Calculus

Lin, Y.K. (1967) *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, New York, NY.

◎ Motivation

$$\frac{d}{dt}X(t) = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h}$$

The conventional “limit” cannot be applied to random processes



Limit of a random process?

Need to consider the convergence of a sequence of random variables, i.e. $\lim_{n \to \infty} \{X_1, X_2, \cdots, X_n\}$

➔ Converging to the distribution of a random variable (not a particular value)

➔ “____________ Convergence”

◎ Definitions of stochastic convergence

1) Convergence with probability 1 ("almost sure" convergence)

$$P\left(\lim_{n\to\infty} X_n = X\right) =$$

2) Convergence in probability

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = \qquad , \forall \epsilon > 0$$

3) Convergence in distribution

$$\lim_{n\to 0} F_{X_n}(x) =$$

4) ** Convergence in the mean square

$$\lim_{n\to\infty} \mathrm{E}[|X_n - X|^2] =$$

→ requires $\mathrm{E}[X^2] < \infty$, i.e. "   " process

Throughout this course, we use the fourth definition with the notation

$$\mathbf{l.i.m.}_{t\to t_0} \boldsymbol{X(t) = X}$$

to describe "**L**imit **I**n the **M**ean-square"

◎ Two theorems for limit in the mean square

Theorem 1:

If $\mathrm{l.i.m.}_{t\to t_0} X(t) = X$ and $\mathrm{l.i.m.}_{s\to s_0} Y(s) = Y$, then $\lim_{t\to t_0, s\to s_0} \mathrm{E}[X(t)\cdot Y(s)] =$

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Proof:

Using Theorem 1, we can show $\lim_{t \to t_0} \mathrm{E}[X(t)] = \mathrm{E}\left[\underset{t \to t_0}{\mathrm{l.i.m.}} X(t)\right]$

Namely, $E[\cdot]$ and $\mathrm{l.i.m.}$ are c______________ or exchangeable

Proof:

Theorem 2:
$\underset{t \to t_0}{\mathrm{l.i.m.}} X(t) = X$ $\phi_{XX}(t, s)$ is continuous at (t_0, t_0) no matter how (t, s) approaches (t_0, t_0)

- See Ex 4.9 in L&S
- Of course, for "second-order process"

Proof: