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457.643 Structural Random Vibrations
In-Class Material: Class 10

II-2. Stochastic Calculus (contd.)

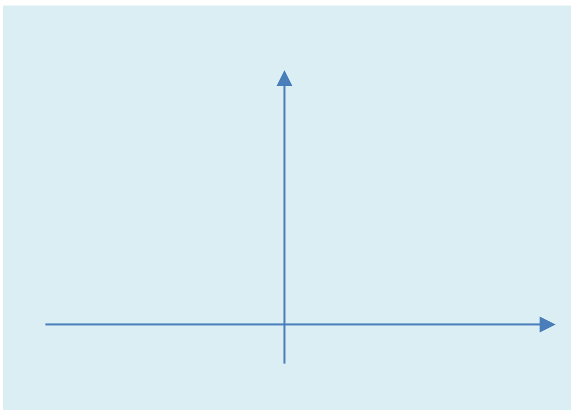
◎ Power spectral density (PSD) function of a stationary process $X(t)$

Fourier pair involving the power spectral density function and auto-correlation function:

$$\Phi_{XX}(\omega) \equiv \lim_{T\to\infty} \frac{2\pi}{T} \mathrm{E}[|\bar{X}(\omega,T)|^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-\mathrm{i}\omega\tau)\,d\tau$$

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega)\exp(\mathrm{i}\omega\tau)\,d\omega$$

- This is often called "Wiener-Khintchine formula"
- The PSD exists when the auto-correlation function is absolutely integrable, i.e. $\int_{-\infty}^{\infty}|R_{XX}(\tau)|d\tau < \infty$
- $R_{XX}(0) = \int_{-\infty}^{\infty} \qquad d\omega = \mathrm{E}[\quad]$
  This indicates that the auto PSD $\Phi_{XX}(\omega)$ describes the distribution of "   " process, i.e. distribution of X^2 over __________ domain. This is why it is called "power spectral density function"



◎ Properties of PSD $\Phi_{XX}(\omega)$

1) Non-negative

   $\because\ \Phi_{XX}(\omega) \propto \mathrm{E}[|\quad|^2]$

   ($R_{XX}(\tau)$ is _________________)

2) Symmetric and Real

$\Phi_{XX}(-\omega) =$

$\because R_{XX}(\tau)$ is _____________ and $\Phi_{XX}(\omega) \propto \mathrm{E}[| \quad |^2]$

3) Tail behavior of PSD tells us about whether the process is a 2nd order process or not.

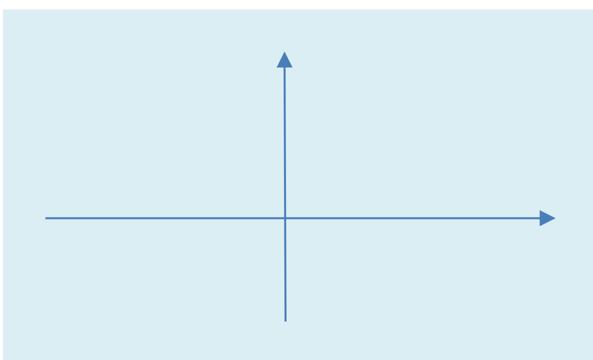
If $\lim_{|\omega|\to\infty} |\omega| \cdot \Phi_{XX}(\omega) = 0$, the integral $\int_{-\infty}^{\infty} \Phi_{XX}(\omega) d\omega = \mathrm{E}[\quad]$ is ___________, thus $X(t)$ is a 2nd order process.

4) Behavior of PSD at $\omega = 0$: Note that

$$\Phi_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau) d\tau$$

Therefore, $\Phi_{XX}(0)$ diverges if $\lim_{\tau\to\infty} R_{XX}(\tau) \neq 0$

If the process has non-zero mean or includes periodic component, the PSD diverges at $\omega = 0$



※ Alternative definition of PSD (e.g. L&S)

$$\Phi_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_{XX}(\tau) \exp(-i\omega\tau)\, d\tau$$

(Reasoning of the alternative definition)

Since $\lim_{|\tau|\to\infty} R_{XX}(\tau) = \mu^2$ (even without periodic components in the process), $\Phi_{XX}(0)$ diverges in general. By contrast, if $\Gamma_{XX}(\tau)$ is used in the definition of PSD, $\Phi_{XX}(0)$ may not diverge even if the process has non-zero mean μ.

Of course, there is no problem if $\mu = 0$ (since $R_{XX}(\tau) = \Gamma_{XX}(\tau)$)

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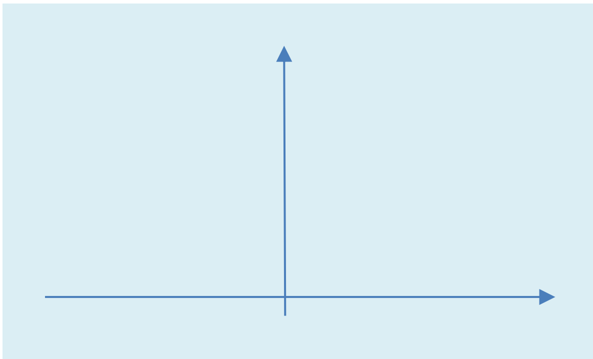
◎ One-sided PSD (Using symmetry of PSD)

$G_{XX}(\omega) = 2\Phi_{XX}(\omega),\ \omega \geq 0$



Note:

$$\Phi_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-\mathrm{i}\omega\tau)d\tau$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)(\cos\omega\tau - \mathrm{i}\cdot\sin\omega\tau)d\tau$$
$$= \frac{1}{\pi}\int_{0}^{\infty} R_{XX}(\tau)\cos\omega\tau\, d\tau$$

Therefore,

$$G_{XX}(\omega) = \frac{2}{\pi}\int_{0}^{\infty} R_{XX}(\tau)\cos\omega\tau\, d\tau$$

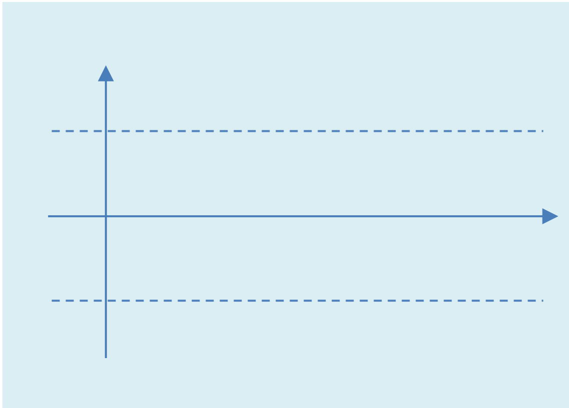
Inversely,

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega)\exp(\mathrm{i}\omega\tau)\, d\omega$$
$$= \int_{0}^{\infty} \qquad\qquad \cos(\omega\tau)\, d\omega$$

◎ Example: "Random Telegraph Process"

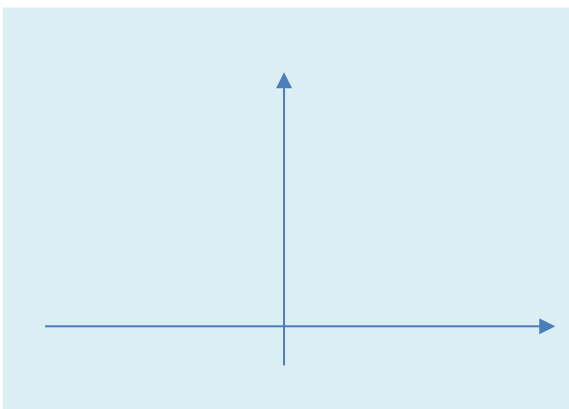
$X(t) = X_0 \cdot (-1)^{N(t)}$

where $X_0 \sim N(0, \sigma^2)$ and $N(t)$ is a homogeneous Poisson process with the mean occurrence rate ν



One can show the auto-correlation function of a random telegraph process is

$R_{XX}(\tau) = \sigma^2 \cdot e^{-2\nu|\tau|}$



The PSD of $X(t)$ is derived as follows:

$$\Phi_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-\mathrm{i}\omega\tau)\,d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma^2 \cdot e^{-2\nu|\tau|} \cdot \exp(-\mathrm{i}\omega\tau)\,d\tau$$

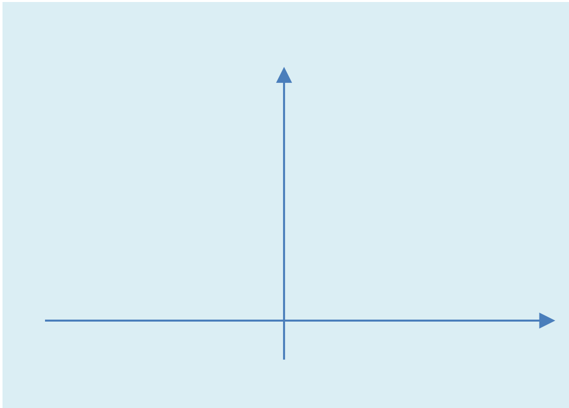
$$= \frac{1}{\pi}\int_{0}^{\infty} \qquad\qquad\qquad d\tau$$

$$= \frac{\sigma^2}{\pi}\left[\frac{e^{-2\nu\tau}}{4\nu^2+\omega^2}(-2\nu\cos\omega\tau + \omega\sin\omega\tau)\right]_{\tau=0}^{\infty}$$

$$= \underline{\qquad\qquad\qquad}, -\infty < \omega < \infty$$

Note:

$\int e^{cx}\cos bx\,dx = \frac{e^{cx}}{(c^2+b^2)}(c\cdot\cos bx + b\cdot\sin bx)$



One-sided PSD:

$G_{XX}(\omega) =$ __________ , $\omega \geq 0$

◎ Estimating PSD from sample time histories of a stationary process using Matlab®

One possible way to estimate the auto PSD is to get an ensemble average of the *periodogram*, i.e.

$$\Phi_{XX}(\omega) \equiv \lim_{T\to\infty} \frac{2\pi}{T} \mathrm{E}[|\bar{X}(\omega,T)|^2] \approx \frac{1}{N_{st}} \sum_{i=1}^{N_{st}} \frac{2\pi}{t_d} \left|\bar{X}^{(i)}(\omega,t_d)\right|^2$$

where t_d is the duration of each of the N_{st} sample time histories.

The Fourier Transform definitions in our note and Matlab® command 'fft()' are respectively,

$$\bar{X}_k = \frac{1}{2\pi} \sum_{j=1}^{n} X_j \, e^{-i\omega_k t_j} \cdot \Delta t$$

$$\bar{X}_{k,FFT} = \sum_{j=1}^{n} X_j e^{\left(-\frac{2\pi i}{n}(j-1)(k-1)\right)}$$

where $X_j = X(t_j)$ and $\bar{X}_k = \bar{X}(\omega_k, t_d)$. Therefore, our FT can be obtained by

$$\bar{X}_k = \frac{1}{2\pi} \bar{X}_{k,FFT} \Delta t$$

The equality $\omega_k t_j = \frac{2\pi}{n}(j-1)(k-1)$ can be satisfied by defining t_j and ω_k as $t_j = (j-1)\cdot\Delta t$ and $\omega_k = \frac{2\pi}{n\Delta t}\cdot(k-1)$, i.e. $t_j \in \{0, \Delta t, 2\Delta t, \ldots, (n-1)\cdot\Delta t\}$ and $\omega_k \in \{0, \Delta\omega, 2\Delta\omega, \ldots, (n-1)\cdot\Delta\omega\}$ where $\Delta\omega = 2\pi/n\Delta t = 2\pi/t_d$

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Note: Because of the periodicity of Fourier Transforms, the PSD estimates for the second half of the frequency range should be considered as the estimates for the negative frequency values (or can be discarded since we know the auto PSD is symmetric).