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Dept. of Civil and Environmental Engineering

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457.212 Statistics for Civil & Environmental Engineers

In-Class Material: Class 13

Multiple Random Variables – Part I (A&T: 3.3)

<< Complete description of random variable(s) >>

- Single random variable, X
  - PMF (discrete) $p_X(x) =$
  - PDF (continuous) $f_X(x) =$
  - CDF (discrete/continuous) $F_X(x) =$
- $P(X \le 20)$?
- Random variables, X (weight) and Y (height):
  - $P(X \le x_{0.1} \cap Y \le y_{0.1})$?
  - $P(X \le x_{0.1} \cup Y > y_{0.9})$?
- Complete description of multiple random variables: "joint" PMF, PDF, and CDF

1. **Joint** cumulative distribution function (**CDF**) ~ continuous/discrete r.v.

$$F_{XY}(x, y) = P(\qquad \cap \qquad) = P(\qquad , \qquad)$$

(a) $F_{XY}(-\infty, -\infty) =$ $\qquad$, $F_{XY}(\infty, \infty) =$

(b) $F_{XY}(-\infty, y) =$ $\qquad$, $F_{XY}(\infty, y) =$

$F_{XY}(x, -\infty) =$ $\qquad$, $F_{XY}(x, \infty) =$

(c) $F_{XY}(x, y)$ is non-negative, non-decreasing function of x and y.

2. **Joint** probability mass function (**PMF**) ~ discrete r.v.

$$p_{XY}(x, y) = P(\qquad , \qquad)$$

(a) $F_{XY}(x, y) = \sum_{\{x_i \le x, y_i \le y\}} p_{XY}(x_i, y_i)$

(b) **Conditional Probability Mass Function**

$$p_{X|Y}(x \mid y) = P(\qquad \mid \qquad) = \frac{P(\qquad)}{P(\qquad)} = \underline{\qquad}$$

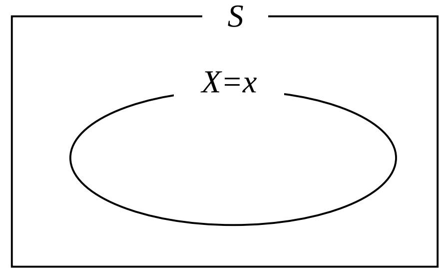
$$p_{Y|X}(y \mid x) = \underline{\qquad}$$

(c) $p_X(x)$ from $p_{XY}(x,y)$ ("marginal" from "joint")

$$p_X(x) = \sum_{\text{all } y_i} P(X = x \mid \qquad)P(\qquad)$$

$$= \sum_{\text{all } y_i} P(\qquad\qquad)$$

$$= \sum$$

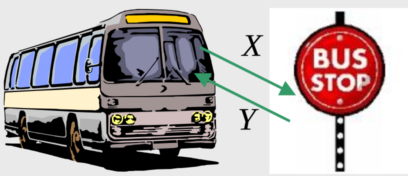


$p_Y(y) = \sum$ (marginalization rule)

(d) If X and Y are statistically independent, $p_{X|Y}(x \mid y) =$

Thus, $p_{XY}(x,y) =$

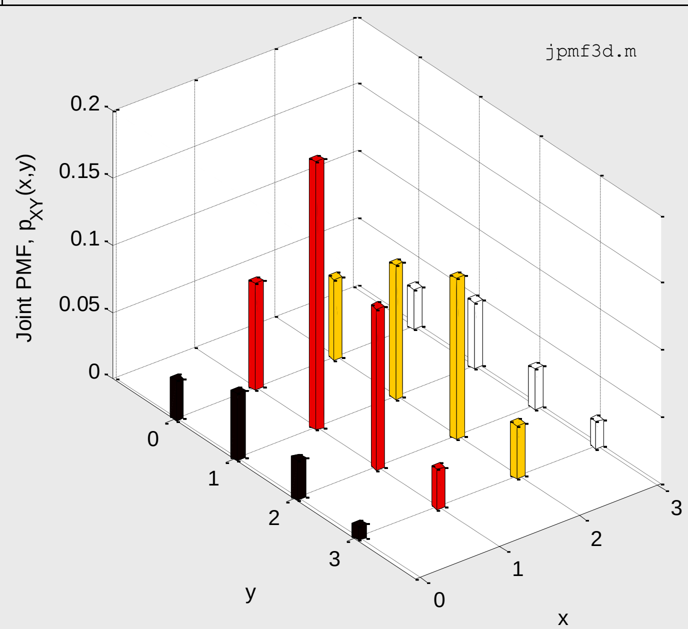
Example 1: Uncertain number of passengers getting off/on at bust stop A



X : number of passengers getting off at bus stop A
Y : number of passengers getting on at bus stop A

Joint PMF of X and Y, $P_{XY}(x,y) = P(X = x \cap Y = y)$ is known as

y \ x	0	1	2	3	$p_Y(y)$
0	0.03	0.08	0.06	0.03	
1	0.05	0.20	0.10	0.05	
2	0.03	0.12	0.12	0.03	
3	0.01	0.03	0.04		
$p_X(x)$					



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```
x=rep(0:3,each=4)
y=rep(0:3,4)
p=c(0.03, 0.05, 0.03, 0.01, 0.08, 0.20, 0.12, 0.03, 0.06, 0.10, 0.12,
   0.04, 0.03, 0.05, 0.03, 0.02)
CM13_EX1 = data.frame(x,y,p)

install.packages("latticeExtra")
library(latticeExtra)

cloud(p~x+y, CM13_EX1, panel.3d.cloud=panel.3dbars, col.facet='grey',
      xlim=c(-0.5,3.5), ylim=c(-0.5,3.5), xbase=0.3, ybase=0.3,
      scales=list(arrows=FALSE))
# cloud: generic function to draw 3D plots
# panel.3d.cloud=panel.3dbars : 3D bar plot
# col.facet : bar color
# xbase & ybase : bar size
# scales=list(arrows=FALSE)) : display coordinate values instead of arrows
```

(1) Obtain the marginal PMF's of X and Y from the joint PMF

(2) Find the value of $p_{XY}(3,3)$

(3) Suppose there are two passengers waiting for the bus to arrive. What's the probability that one passenger will get off the bus?

(4) Are $X = x$ and $Y = y$ statistically independent?

For your information, the following table shows $p_X(x) \times p_Y(y)$.

$y \backslash x$	0	1	2	3
0	0.024	0.086	0.064	0.026
1	0.048	0.172	0.128	0.052
2	0.036	0.129	0.096	0.039
3	0.012	0.043	0.032	0.013

Another way to check is to compare the marginal PMF's with the conditional PMF's. For example, compare the following conditional PMF's $p_{Y|X}(1|x)$ with the marginal PMF $p_Y(1) = 0.40$.

$y \backslash x$	0	1	2	3
1	0.417	0.465	0.313	0.385

(5) Are $X = 3$ and $Y = 2$ mutually exclusive?

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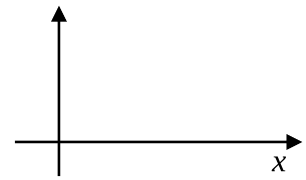
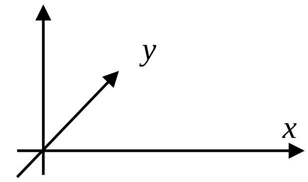
3. **Joint** probability density (**PDF**) ~ continuous r.v.



- Marginal PDF: $f_X(x) = \lim_{\Delta x \to 0} \frac{P(x < X \le x + \Delta x)}{\Delta x}$

  ~ Probability over unit (   )

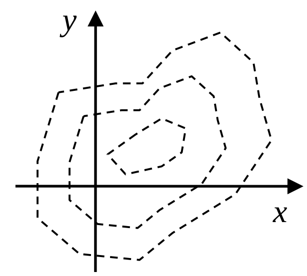


- Joint PDF:

  $$f_{XY}(x, y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{P(x < X \le x + \Delta x \cap y < Y \le y + \Delta y)}{\Delta x \Delta y}$$

  ~ Probability over unit (   )

(a) Relationship with joint CDF

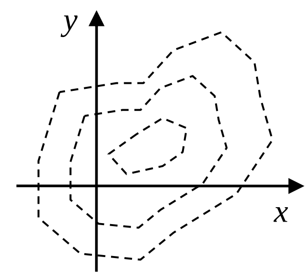


$F_{XY}(x, y) =$

$f_{XY}(x, y) = \overline{\qquad\qquad}$

(b) $P(a < X \le b, c < Y \le d) =$

(c) **Conditional Probability Density Function**



~ Density of a random variable when the outcome of the other r.v. is known

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

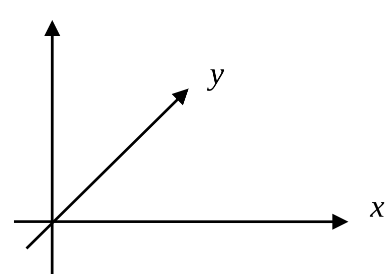
Multiplication rule:

$$f_{XY}(x, y) =$$
$$=$$

If, X and Y are statistically independent,

(d) Joint PDF → marginal PDF

(Recall ~ Discrete case: marginalization rule – sum over the other random variable)



$f_X(x) =$

$f_Y(y) =$

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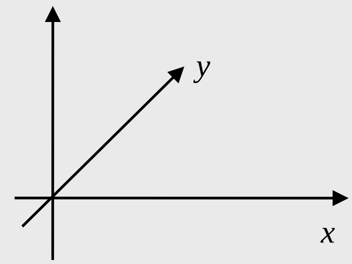
Example 2: Consider non-negative random variables X and Y. The likelihood of the pair (x, y) is proportional to $\exp[-3(x+y)]$.

(a) Joint PDF?

(a-2) Sketch of joint PDF?

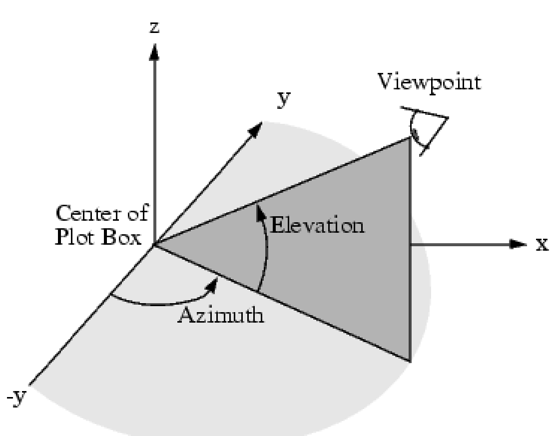



```
install.packages("plot3D")
library(plot3D)

CM13_EX2 = function(x,y) {z=9*exp(-3*(x+y))}
x = seq(0,1,length=20)
y = seq(0,1,length=20)
grid = mesh(x,y)
# construct a 2D plane with x and y coordinates
z = CM13_EX2(grid$x, grid$y) # Tie z with (x,y)

contour2D(z,x,y,levels=seq(0,10,0.5),colkey=TRUE)
# draw 2D contour plot
image2D(z,x,y,levels=seq(0,10,0.5),rasterImage=TRUE,lighting=FALSE,contour
   =TRUE)
# image 2D plot

persp3D(x,y,z, border='black', facets=TRUE, colkey=TRUE,
   ticktype='detailed', bty='b2') # draw 3D surface plot
plotdev(theta=40, phi=60)
# perspective (theta = azimuth (default 40), phi=altitude (default 40))
persp3D(x,y,z, border='black', facets=TRUE, colkey=TRUE,
   ticktype='detailed', bty='b2', contour=list(side=c("zmax","z"),
   color='red'))
# draw 3D surface plot + contour plot
```



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(b) Joint CDF?

(c) Check the result in (b)? How?

(d) Marginal PDF's?

(e) Check

$F_{XY}(-\infty,-\infty)$
$F_{XY}(\infty,\infty)$
$F_{XY}(-\infty,y)$
$F_{XY}(\infty,y)$

(f) Conditional PDF's?

(g) $P(1 \le X \le 2, 1 \le Y \le 2) =$