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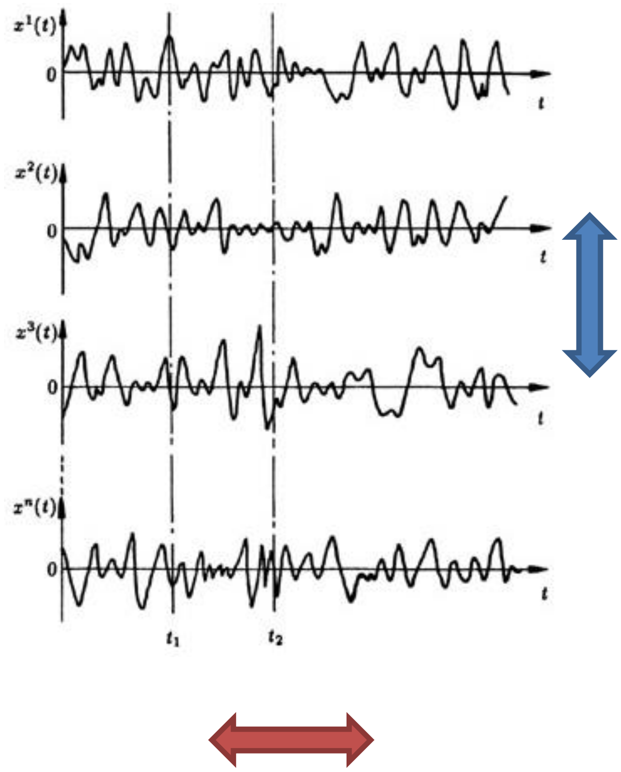
457.643 Structural Random Vibrations
In-Class Material: Class 13

II-2. Stochastic Calculus (contd.)

◎ Ergodicity



Ensemble Average: $\mathrm{E}[X(t)] =$

Temporal Average: $m_T =$

Average over the time domain is actually a _________ variable. Consider

$$M_T = \frac{1}{T}\int_0^T X(t)dt = \langle X(t) \rangle$$

$$\phi_T(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} X(t+\tau)X(t)dt$$

These are random variable and function because the results depend on random outcome (selection) of a time history.

1) "Ergodic" process: If $X(t)$ is an ergodic process, one can use the temporal average from a time history $x(t)$ as a substitute for an ___________ expectation.

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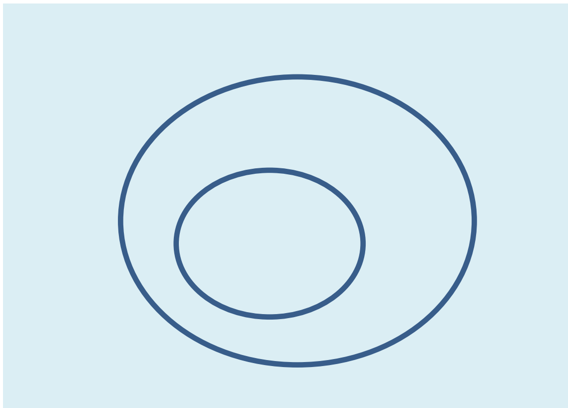
2) Basic __________ condition for ergodicity: stationarity

Stationary Ergodic



What if $X(t)$ is NOT stationary? $\mu(t_1) \neq \mu(t_2)$

3) Condition for ergodicity in the mean M_T

- $\lim_{T\to\infty} \mathrm{E}[M_T] = \mu_X(t) = \mathrm{E}[X(t)]$ ~ automatically satisfied for _________ process

- $\lim_{T\to\infty} \mathrm{Var}[M_T] =$

$$\mathrm{Var}[M_T] = \mathrm{E}\left\{\left[\frac{1}{T}\int_0^T X(t)\,dt - \mu_X\right]^2\right\}$$

$$= \mathrm{E}\left\{\left[\frac{1}{T}\int_0^T (X(t) - \mu_X)\,dt\right]^2\right\}$$

$$= \frac{1}{T^2}\int_0^T \int_0^T \Gamma_{XX}(t_1 - t_2)dt_1 dt_2$$

$$= \frac{2}{T}\int_0^T \left(1 - \frac{\tau}{T}\right)\Gamma_{XX}(\tau)d\tau$$

← **"Flip and Rotate" trick used**

$\therefore \lim_{T\to\infty} \mathrm{Var}[M_T] = 0 \leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_0^T \Gamma_{XX}(\tau)d\tau = 0$ (See Lin 1967, p. 64)

This is the condition for "ergodicity in the mean"

4) Condition for ergodicity in the correlation function $\phi_T(\tau)$

- $\lim_{T\to\infty} \mathrm{E}[\phi_T(\tau)] = R_{XX}(\tau)$ ~ automatically satisfied for _________ process

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- $\lim_{T\to\infty} \mathrm{Var}[\phi_T(\tau)] = 0$

The latter is equivalent to (Lin 1967, p. 65)

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{E}\{[X(t+\tau)X(t) - R_{XX}(\tau)][X(t+\tau+u)X(t+u) - R_{XX}(\tau)]\}du = 0$$

Example: Telegraph Random Process

$\Gamma_{XX}(\tau) = \sigma^2 \exp(-2\nu|\tau|)$

Is the process ergodic in the mean?

Example: $Y(t) = A + X(t)$ where $X(t)$ is the random telegraph process, and $\mathrm{E}[A] = 0$ and $\mathrm{Var}[A] = \sigma^2$. Is $Y(t)$ ergodic in the mean?

III. Random Vibration of Linear Structures

◎ Stochastic response of "linear" structural system

Recall the system equation introduced in **0. Introduction**

$$\mathcal{D}[\mathcal{X}(x,t)] = \mathcal{Y}(x,t), \qquad t \geq 0, \qquad x \in D \subset \mathcal{R}^d$$

1. Deterministic systems and input (457.516 Dynamics of Structures)
2. Deterministic systems and stochastic input **(457.643 Structural Random Vibrations)**
3. Stochastic systems and deterministic input (457.646 Topics in Structural Reliability)
4. Stochastic systems and input

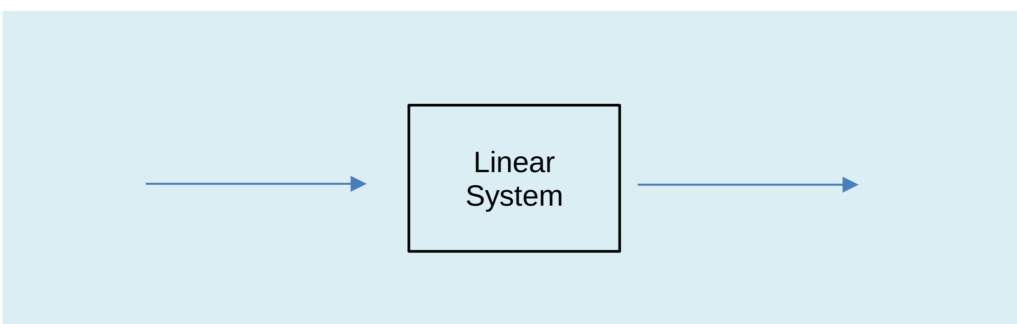
We consider the second case in this course. Note that the system is "linear" when the differential equation is linear, i.e. s___________ principle works.

e.g. if $x_1(t)$ is the response to $y_1(t)$, and $x_2(t)$ is the response to the input $y_2(t)$, the response to $y_1(t) + y_2(t)$ is _______________

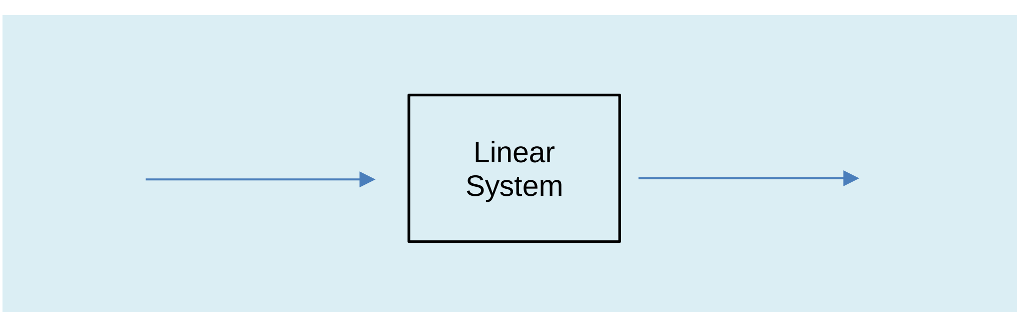
III-1. Response Functions of Structural Systems

◎ Characterization of linear systems

1) Time-domain: **"Impulse Response Function"**



2) Frequency-domain: **"Frequency Response Function"**



◎ Impulse response function of a linear system

Consider the differential equation (DE) of a general linear system

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = p(t)$$

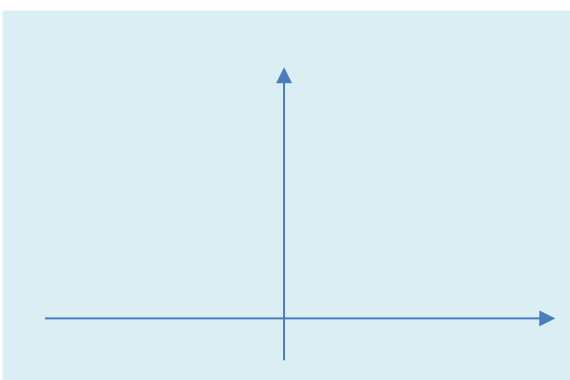
Impulse response function of a linear system $h(t)$ is the solution of the DE when $p(t) = \delta(t)$, i.e.

$$a_n \frac{d^n h}{dt^n} + a_{n-1} \frac{d^{n-1} h}{dt^{n-1}} + \cdots + a_1 \frac{dh}{dt} + a_0 h = \delta(t) \ \cdots \ (*)$$

$$h(t) = h_h(t) + h_p(t)$$

where $h_h(t)$: homogeneous solution and $h_p(t)$: particular solution

Strategy: Model the dirac delta input function by a triangular function for $t \in (-\epsilon, \epsilon)$. Then, obtain $h(t) = h_h(t)$ with the initial conditions caused by the impulse.



From (*), it is clear that only the ________est term can be dirac delta function because if an non-highest-order term is dirac delta, the higher-order-terms will blow up.

$\therefore a_n \frac{d^n h}{dt^n} = \delta(t)$, $0^- < t < 0^+$

$$a_n \left.\frac{d^{n-1} h}{dt^{n-1}}\right|_{0^-}^{0^+} = \int_{0^-}^{0^+} \delta(t) dt = \qquad \rightarrow a_n \left.\frac{d^{n-1} h}{dt^{n-1}}\right|_{t=0^+} =$$

Therefore, the initial conditions at $t = 0^+$ are

$$\frac{d^{n-1} h}{dt^{n-1}} = \qquad , \qquad \frac{d^{n-2} h}{dt^{n-2}} = \cdots = h =$$