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457.643 Structural Random Vibrations
In-Class Material: Class 17

III-2. Random Vibration Analysis of Linear Structures (contd.)

◎ Response of a linear system to weakly stationary input

$\kappa_{FF}(t_1, t_2) = \Gamma_{FF}(\tau)$ where $\tau = t_1 - t_2$

Assuming zero initial conditions,

$$\kappa_{XX}(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} \kappa_{FF}(\tau_1, \tau_2) h(t_1 - \tau_1) h(t_2 - \tau_2) d\tau_2 d\tau_1$$
$$= \int_0^{t_1}\int_0^{t_2} \Gamma_{FF}(\tau) h(t_1 - \tau_1) h(t_2 - \tau_2) d\tau_2 d\tau_1$$

where $\tau = \tau_1 - \tau_2$

Note $\Gamma_{FF}(\tau) = \int_{-\infty}^{\infty} \Phi_{FF}(\omega) e^{i\omega\tau} d\omega$

Thus,

$$\kappa_{XX}(t_1, t_2) = \int_{-\infty}^{\infty}\int_0^{t_1}\int_0^{t_2} \Phi_{FF}(\omega) h(t_1 - \tau_1) h(t_2 - \tau_2) e^{i\omega\tau} d\tau_2 d\tau_1 \, d\omega$$
$$= \int_{-\infty}^{\infty}\int_0^{t_1}\int_0^{t_2} \Phi_{FF}(\omega) h(t_1 - \tau_1) h(t_2 - \tau_2) e^{-i\omega(t_1-\tau_1)} e^{i\omega(t_2-\tau_2)} e^{i\omega(t_1-t_2)} d\tau_2 d\tau_1 \, d\omega$$

By changing variable $u = t_1 - \tau_1$, one can show

$$\int_0^{t_1} h(t_1 - \tau_1) e^{-i\omega(t_1-\tau_1)} d\tau_1 = \int_0^{t_1} h(u) e^{-i\omega u} du$$
$$\boxed{= \int_{-\infty}^{t_1} h(u) e^{-i\omega u} du = \mathcal{H}(\omega, t_1)}$$

This is so-called "incomplete" Fourier transform of the impulse response function.

cf. "complete" FT of IRF gives the FRF

$$\boxed{H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt}$$

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Therefore, $\kappa_{XX}(t_1, t_2)$ for a weakly stationary input $F(t)$ is expressed as

$$\kappa_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} \Phi_{FF}(\omega)\,\mathcal{H}(\omega, t_1)\mathcal{H}^*(\omega, t_2)e^{i\omega\tau}d\omega$$

Note:

- The response of a linear system to a stationary input is ___________ stationary necessarily.

- However, as $t_1, t_2 \to \infty$, the incomplete FTs becomes independent of t_1 and t_2, Therefore, $\kappa_{XX}(t_1, t_2)$ depends only on $\tau = t_1 - t_2$

Observations:

1. $\lim_{t\to\infty} \mathcal{H}(\omega, t) = H(\omega)$ for a "stable" system

   Therefore, the response of a linear system to a stationary input becomes

   ____________ e__________

2. $\kappa_{XX}(0,0)$ must be ________ and it means $\sigma_X^2(0) =$. This makes sense because we assumed _________ IC's

3. For the stationary response, i.e. $t_1, t_2 \to \infty$

$$\kappa_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} \Phi_{FF}(\omega)\,\mathcal{H}(\omega, t_1)\mathcal{H}^*(\omega, t_2)e^{i\omega\tau}d\omega$$

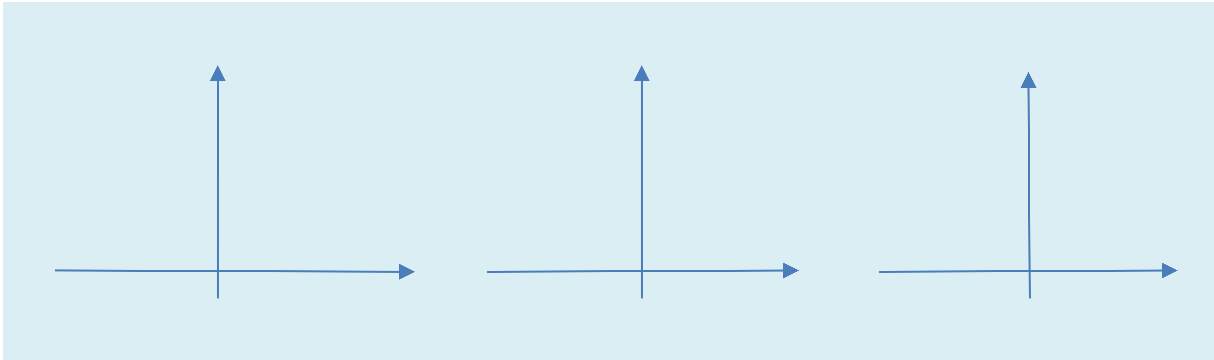
$$= \int_{-\infty}^{\infty} \Phi_{FF}(\omega)\,|H(\omega)|^2 e^{i\omega\tau}d\omega$$

   That is,

$$\Gamma_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{FF}(\omega)\,|H(\omega)|^2 e^{i\omega\tau}d\omega$$

4. From this result, for a <u>stationary</u> response, it is found that

$$\Phi_{XX}(\omega) =$$



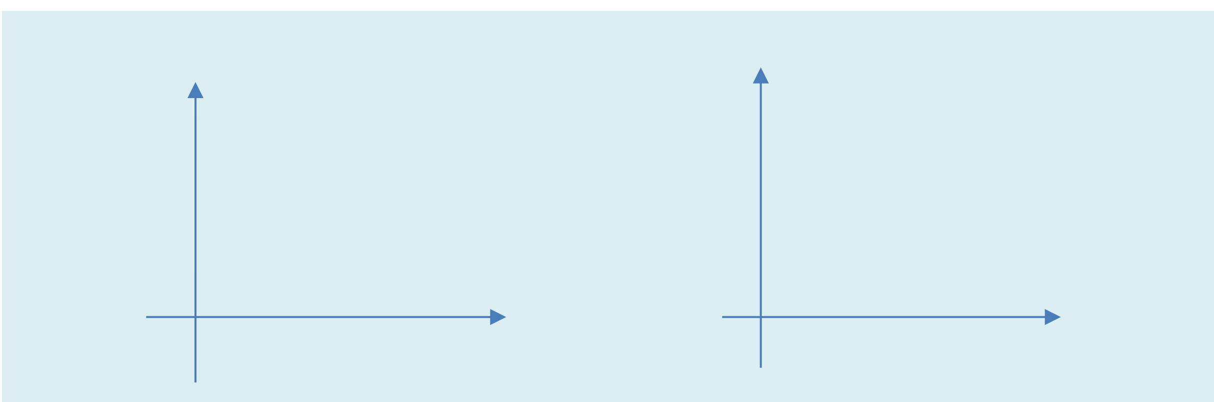
For example, let us consider…

◎ $\mathcal{H}(\omega, t)$ and $H(\omega)$ of standard SDOF oscillator

Recall

- $h(t) = \frac{1}{\omega_D} e^{-\xi\omega_0 t} \sin\omega_D t$
- $H(\omega) = \frac{1}{\omega_0^2 - \omega^2 + 2i\xi\omega_0\omega} = \frac{\omega_0^2 - \omega^2 - 2i\xi\omega_0\omega}{(\omega_0^2 - \omega^2)^2 + (2\xi\omega_0\omega)^2}$

$$|H(\omega)|^2 = \frac{1}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2}$$



$$\mathcal{H}(\omega, t) = \int_{-\infty}^{t} \frac{1}{\omega_D} e^{-\xi\omega_0\tau} \sin\omega_D\tau \cdot U(\tau) \cdot e^{-i\omega\tau} d\tau$$

$$= H(\omega)\left[1 - \left(\cos\omega_D t + \frac{\xi\omega_0 + i\omega}{\omega_D}\sin\omega_D t\right) \cdot e^{-\xi\omega_0 t} \cdot e^{-i\omega t}\right]$$

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From this result, the terms in ( ) has the same order as 1, and $e^{-i\omega t}$ oscillates. Therefore, the rate of the convergence of the terms in [ ] to _____ is determined by _______

In other words, "sufficient" time to achieve stationarity depends on $\xi\omega_0 t = \xi \frac{2\pi}{T_0} t$

Suppose we set $\xi\omega_0 t = 2\pi\xi \frac{t}{T_0} = \pi$ (note $e^{-\pi} = 4\%$) and solve it for t, i.e. time to make the exponentially decaying term as 4%, $t_{4\%} = \frac{T_0}{2\xi}$

e.g. $\xi = 0.1$ → $t_{4\%} \cong 5T_0$, $\xi = 0.05$ → $t_{4\%} \cong 10T_0$

※ Alternative (empirical) method:

Wang, Z., and Song, J. (2017) Equivalent linearization method using Gaussian mixture (GM-ELM) for nonlinear random vibration analysis, *Structural Safety*, Vol. 64, 9-19. (http://dx.doi.org/10.1016/j.strusafe.2016.08.005)

3.1.1. Remark 1: Selecting sample points

One issue in selecting sample points in the aforementioned algorithm is that the nonlinear response takes a certain amount of time to achieve stationarity, thus using the whole time series including a nonstationary part will introduce errors to the estimated PDF. To reduce this error, for each of the M response histories obtained from the first step of the algorithm, we need to select $\bar{N}$ stationary response values as the sample points.

Here we provide a method to crudely estimate the time that the system would take to achieve stationarity. To begin with, the standard deviation of the response at a sequence of time points, denoted as $\mathrm{std}[Z(j\Delta t)]$, in which $j = 1, 2, \ldots$ and Δt is the time step of the nonlinear analysis, is estimated using the recorded M response histories, and then a sigmoid function expressed as

$$f_{fit}(j) = \frac{1}{1 + e^{-aj\Delta t + b}} \tag{12}$$

is employed to fit the $\mathrm{std}[Z(j\Delta t)]$ curve. Note that $f_{fit}(\cdot) \in (0, 1)$, thus the $\mathrm{std}[Z(j\Delta t)]$ curve should be scaled by a factor $J / \sum_{j=1}^{J} \mathrm{std}[Z(j\Delta t)]$ ($J\Delta t$ is the duration of the excitation) so that it approximately ranges from 0 to 1. The parameters a and b in Eq. (12) can be determined from a least-square regression analysis. A typical scaled $\mathrm{std}[Z(j\Delta t)]$ curve and its corresponding fitting function $f_{fit}(\cdot)$ is illustrated in Fig. 3. With $f_{fit}(t)$ available, the time the system takes to achieve stationarity, denoted by $j_{ns}\Delta t$, can be estimated via

$$j_{ns} = argmin\{j | 1 - f(j\Delta t) \leqslant Tol, j = 1, 2, \ldots\} \tag{13}$$

where Tol denotes a specified tolerance. With j_{ns} determined, for each of the M response histories, $\bar{N} = J - j_{ns}$ time points corresponding to the stationary responses are selected to be the sample points, and the total number of sample points is $N = M \cdot \bar{N} = M \cdot (J - j_{ns})$.

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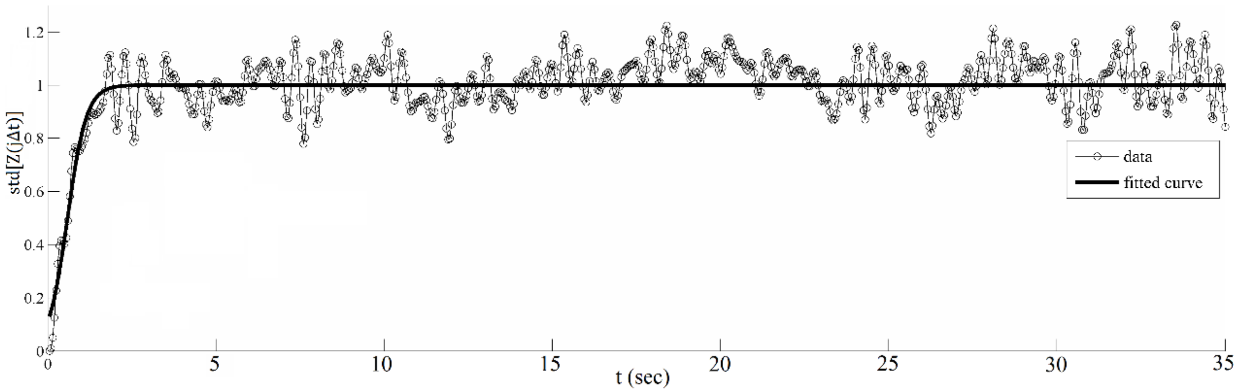
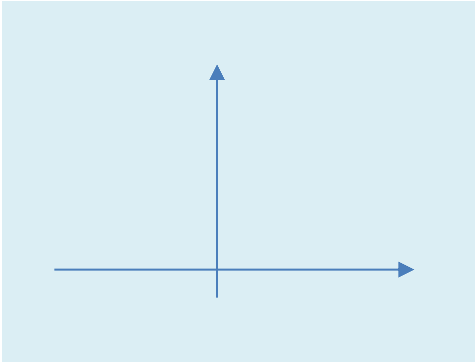


Figure 3. A typical scaled $\mathrm{std}[Z(j\Delta t)]$ curve and the fitting function

◎ Stationary response of standard SDOF oscillator to "white noise"

Useful for linear random vibration analysis of MDOF systems using modal combination, i.e. each mode is represented by a standard SDOF oscillator (will be shown later)

$\Phi_{FF}(\omega) = \Phi_0$



PSD of the stationary response

$$\Phi_{XX}(\omega) = \Phi_0 |H(\omega)|^2$$
$$= \frac{\Phi_0}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2}$$

Thus,

$$\Gamma_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega) e^{i\omega\tau} d\omega$$
$$= \Phi_0 \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2} d\omega$$

How? We can use ____________ theorem

(to be continued…)