Seoul National University
Dept. of Civil and Environmental Engineering

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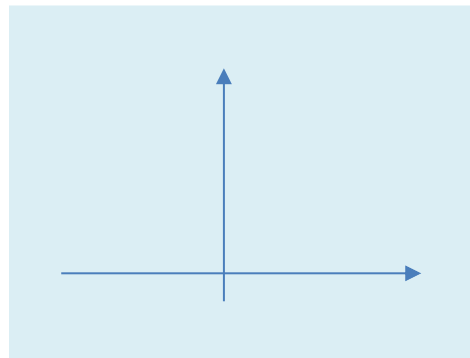
457.643 Structural Random Vibrations
In-Class Material: Class 18

III-2. Random Vibration Analysis of Linear Structures (contd.)

◎ Stationary response of standard SDOF oscillator to "white noise"

Useful for linear random vibration analysis of MDOF systems using modal combination, i.e. each mode is represented by a standard SDOF oscillator (will be shown later)

$\Phi_{FF}(\omega) = \Phi_0$



PSD of the stationary response

$$\Phi_{XX}(\omega) = \Phi_0 |H(\omega)|^2 = \frac{\Phi_0}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2}$$

Thus,

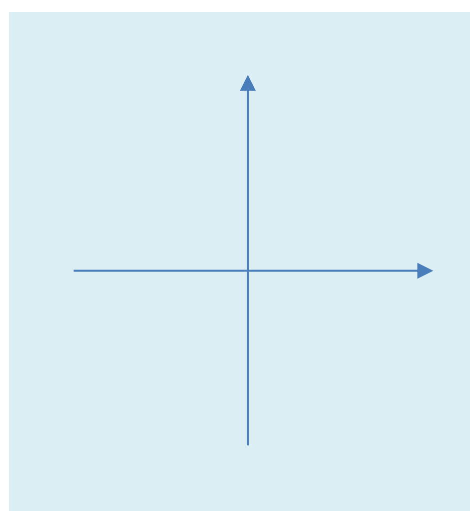
$$\Gamma_{XX}(\tau) = \int_{-\infty}^{\infty} \Phi_{XX}(\omega) e^{i\omega\tau} d\omega$$

$$= \Phi_0 \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2\omega^2} d\omega$$

How? We can use ____________ theorem. Poles?

$$f(z) = \frac{\Phi_0 e^{iz\tau}}{(\omega_0^2 - z^2)^2 + 4\xi^2\omega_0^2 z^2} = \frac{\Phi_0 e^{iz\tau}}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}$$

Solve $(\omega_0^2 - z^2)^2 + 4\xi^2\omega_0^2 z^2 = 0$ for z



$\omega_0^2 - z^2 = \pm 2i\xi\omega_0 z$

$z^2 \pm 2i\xi\omega_0 z - \omega_0^2 = 0$

- $z_1 = \omega_D - i\xi\omega_0$
- $z_2 = -\omega_D - i\xi\omega_0$
- $z_3 = \omega_D + i\xi\omega_0$
- $z_4 = -\omega_D + i\xi\omega_0$

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First consider $\tau > 0$, note $z = a + ib$

$$e^{iz\tau} = e^{i(a+ib)\tau} = e^{ia\tau} \cdot e^{-b\tau}$$

Therefore, the function inside the integral vanishes as $r \to \infty$ if $b \quad 0$.

That is, we should use upper/lower half-plane for the residue theorem.

$$\int_{-\infty}^{\infty} f(\omega)d\omega = \oint_C f(z)dz$$

$$= 2\pi i\left(\text{Res}\, f(\quad) + \text{Res}\, f(\quad)\right)$$

$$= 2\pi i\left(\frac{\Phi_0 e^{i\quad\tau}}{(\quad -z\quad)(\quad -z\quad)(\quad -z\quad)} + \frac{\Phi_0 e^{i\quad\tau}}{(\quad -z\quad)(\quad -z\quad)(\quad -z\quad)}\right)$$

As a result,

$$\Gamma_{XX}(\tau) = \frac{\pi\Phi_0}{2\xi\omega_0^3} e^{-\xi\omega_0\tau}\left(\cos\omega_D\tau + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D\tau\right), \tau > 0$$

From ___________ property of $\Gamma_{XX}(\tau)$, for $\forall\tau$

$$\boxed{\Gamma_{XX}(\tau) = \frac{\pi\Phi_0}{2\xi\omega_0^3} e^{-\xi\omega_0|\tau|}\left(\cos\omega_D\tau + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D|\tau|\right)}$$

1) Variance of the stationary response of standard SDOF oscillator to “white noise”

$$\boxed{\sigma_X^2 = \Gamma_{XX}(\quad) = \frac{\pi\Phi_0}{2\xi\omega_0^3}}$$

2) Exist in the mean-square sense?

3) Mean-square differentiable?

4) Cross-covariance of the response and its time derivative

$$\Gamma_{\dot{X}X}(\tau) = -\Gamma_{X\dot{X}}(\tau) = \frac{d\Gamma_{XX}(\tau)}{d\tau} = -\frac{\pi\Phi_0}{2\xi\omega_0\omega_D} e^{-\xi\omega_0\tau}\sin\omega_D\tau \quad \text{for } \tau > 0$$

Therefore, for $\forall\tau$,

$$\boxed{\Gamma_{\dot{X}X}(\tau) = -\frac{\pi\Phi_0}{2\xi\omega_0\omega_D} e^{-\xi\omega_0|\tau|}\sin\omega_D\tau}$$

5) Auto-covariance of the time derivative

$$\boxed{\Gamma_{\dot{X}\dot{X}}(\tau) = -\frac{d^2\Gamma_{XX}(\tau)}{d\tau^2} = \frac{\pi\Phi_0}{2\xi\omega_0} e^{-\xi\omega_0|\tau|}\left(\cos\omega_D\tau - \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_D|\tau|\right)}$$

6) Note that the time derivative of the SDOF response to white noise is _______ differentiable (in the mean-square sense).

Setting $Y(t) = \dot{X}(t)$, $\Gamma_{YY}(\tau) = \Gamma_{\dot{X}\dot{X}}(\tau)$

$\frac{d\Gamma_{YY}(\tau)}{d\tau}$ does not exist at $\tau = 0$

7) PSD of the time derivative (velocity)

$$\boxed{\begin{aligned}\Phi_{\dot{X}\dot{X}}(\omega) &= \omega^2\Phi_{XX}(\omega) \\ &= \frac{\Phi_0\omega^2}{(\omega_0^2-\omega^2)^2 + 4\xi^2\omega_0^2\omega^2}\end{aligned}}$$

8) Note

$$\Gamma_{\dot{X}\dot{X}}(\tau) = \Phi_0 \int_{-\infty}^{\infty} \frac{\omega^2 e^{i\omega\tau}}{(\omega_0^2-\omega^2)^2 + 4\xi^2\omega_0^2\omega^2} d\omega = \text{Result in (5)}$$

The term inside the integral is $o(\omega^2)/o(\omega^4)$: decays faster than $1/\omega$

How about $\ddot{X}(t)$? $\Phi_{\ddot{X}\ddot{X}}(\omega) = \omega^4\Phi_{XX}(\omega)$

Therefore, $\Gamma_{\ddot{X}\ddot{X}}(\tau)$ is the integral of the term proportional to $o(\omega^4)/o(\omega^4)$: does NOT decay faster than $1/\omega$. Thus $\Gamma_{\ddot{X}\ddot{X}}(\tau) \to \infty$

9) Variance of the time derivative $\dot{X}(t)$

$$\boxed{\sigma_{\dot{X}}^2 = \Gamma_{\dot{X}\dot{X}}(\quad) = \frac{\pi\Phi_0}{2\xi\omega_0}}$$

◎ Non-stationary response of standard SDOF oscillator to "white noise"

$$\kappa_{XX}(t_1, t_2) = \int_{-\infty}^{\infty} \Phi_0 \mathcal{H}(\omega, t_1)\mathcal{H}^*(\omega, t_2) e^{i\omega(t_1-t_2)} d\omega$$

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◎ Supplementary Materials: SDOF responses to WN

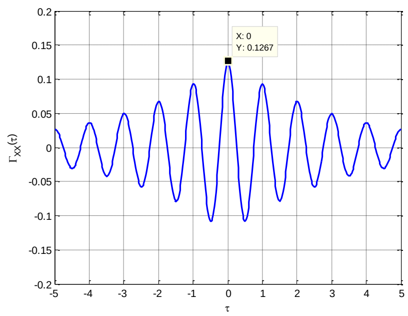
I. "Stationary" responses of the standard SDOF oscillator to white noise: $X(t)$ and its derivative $Y(t) = dX(t)/dt$

(Plots generated for $\omega_0 = 2\pi$, $\zeta = 0.05$ and $\Phi_0 = 1.0$)

(1) Autocovariance function of $X(t)$:

$$\Gamma_{XX}(\tau) = \frac{\pi\Phi_0}{2\zeta\omega_0^3} e^{-\zeta\omega_0|\tau|}\left(\cos\omega_D\tau + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D|\tau|\right)$$

$$\Gamma_{XX}(0) = \sigma_X^2 = \frac{\pi\Phi_0}{2\zeta\omega_0^3}$$

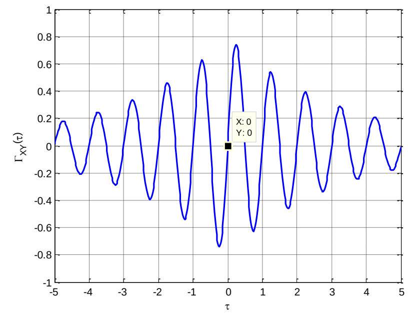


(2) Crosscovariance function of $X(t)$ and $Y(t)$:

$$\Gamma_{XY}(\tau) = -\frac{d\Gamma_{XX}(\tau)}{d\tau} = \frac{\pi\Phi_0}{2\zeta\omega_0\omega_D} e^{-\zeta\omega_0|\tau|}\sin\omega_D\tau$$

$$\Gamma_{XY}(0) = 0$$

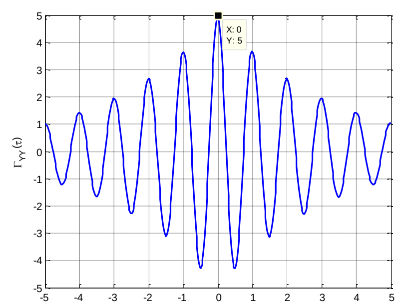
(Note: A stationary r.p. and its derivative are always orthogonal, i.e. $R_{XY}(0) = 0$.)



(3) Autocovariance function of $Y(t)$:

$$\Gamma_{YY}(\tau) = -\frac{d^2\Gamma_{XX}(\tau)}{d\tau^2} = \frac{\pi\Phi_0}{2\zeta\omega_0} e^{-\zeta\omega_0|\tau|}\left(\cos\omega_D\tau - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D|\tau|\right)$$

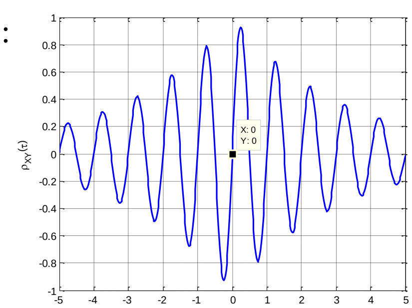
$$\Gamma_{YY}(0) = \sigma_Y^2 = \frac{\pi\Phi_0}{2\zeta\omega_0}$$



(4) Crosscorrelation coefficient function of $X(t)$ and $Y(t)$:

$$\rho_{XY}(\tau) = \frac{\Gamma_{XY}(\tau)}{\sqrt{\Gamma_{XX}(0)\Gamma_{YY}(0)}} = e^{-\zeta\omega_0|\tau|}\frac{1}{\sqrt{1-\zeta^2}}\sin\omega_D\tau$$

$$\rho_{XY}(0) = 0$$



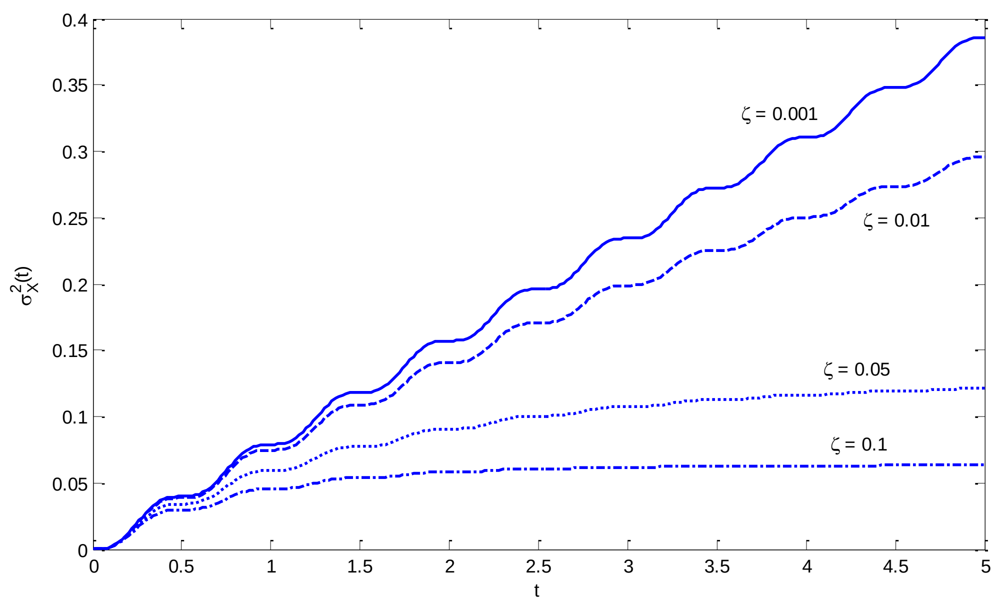
II. "Nonstationary" responses of the standard SDOF oscillator to white noise: $X(t)$ and its derivative $Y(t) = dX(t)/dt$

(1) Autocovariance function of $X(t)$:

$$\kappa_{XX}(t_1,t_2) = \int_{-\infty}^{\infty} \Phi_0 H(\omega,t_1)H^*(\omega,t_2)e^{i\omega(t_1-t_2)}d\omega = 2\pi\Phi_0 \int_0^{\min(t_1,t_2)} h(t_1-\tau)h(t_2-\tau)d\tau$$

$$= \frac{\pi\Phi_0}{2\zeta\omega_0^3}\left\{\begin{array}{l} e^{-\zeta\omega_0|t_1-t_2|}\left[\cos\omega_D(t_1-t_2)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D|t_1-t_2|\right]- \\ e^{-\zeta\omega_0(t_1+t_2)}\left[\frac{1}{1-\zeta^2}\cos\omega_D(t_1-t_2)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D(t_1+t_2)-\frac{\zeta^2}{1-\zeta^2}\cos\omega_D(t_1+t_2)\right]\end{array}\right\}$$

$$\kappa_{XX}(t,t) = \sigma_X^2(t) = \frac{\pi\Phi_0}{2\zeta\omega_0^3}\left\{1-e^{-2\zeta\omega_0 t}\left[\frac{1}{1-\zeta^2}+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin 2\omega_D t-\frac{\zeta^2}{1-\zeta^2}\cos 2\omega_D t\right]\right\}$$



Note:

1) Eventually approaches to the variance of the stationary response, $\frac{\pi\Phi_0}{2\zeta\omega_0^3}$
2) The "sufficient" time to achieve stationarity depends on $\zeta\omega_0$.

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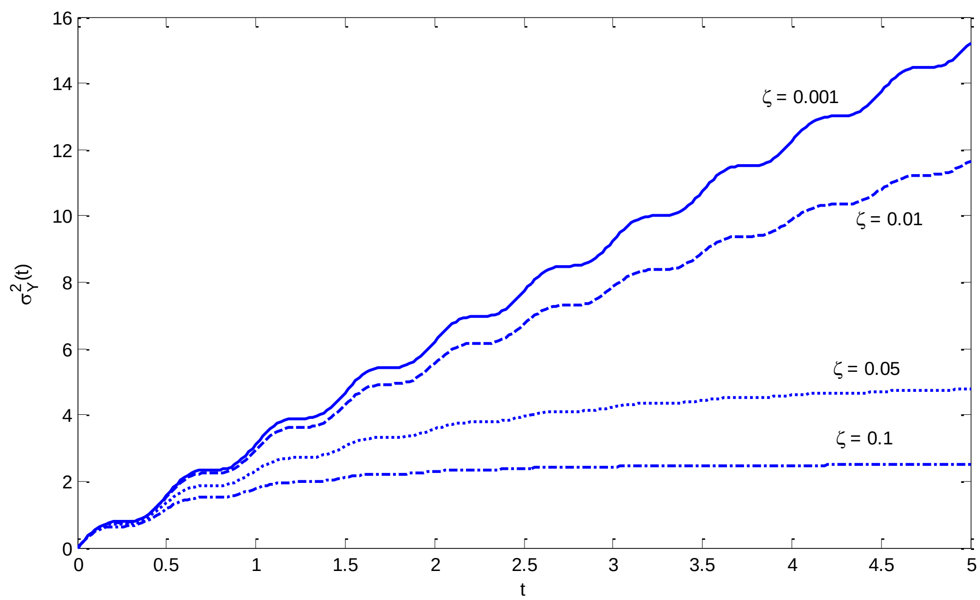
(2) Autocovariance function of $Y(t)$:

$$\kappa_{YY}(t_1,t_2)=\frac{\partial^2\kappa_{XX}(t_1,t_2)}{\partial t_1\partial t_2}$$

$$=\frac{\pi\Phi_0}{2\zeta\omega_0}\left\{\begin{array}{l} e^{-\zeta\omega_0|t_1-t_2|}\left[\cos\omega_D(t_1-t_2)-\dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D|t_1-t_2|\right]- \\ e^{-\zeta\omega_0(t_1+t_2)}\left[\dfrac{1}{1-\zeta^2}\cos\omega_D(t_1-t_2)-\dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D(t_1+t_2)-\dfrac{\zeta^2}{1-\zeta^2}\cos\omega_D(t_1+t_2)\right]\end{array}\right\}$$

$$\kappa_{YY}(t,t)=\sigma_Y^2(t)$$

$$=\frac{\pi\Phi_0}{2\zeta\omega_0}\left\{1-e^{-2\zeta\omega_0 t}\left[\frac{1}{1-\zeta^2}-\frac{\zeta}{\sqrt{1-\zeta^2}}\sin 2\omega_D t-\frac{\zeta^2}{1-\zeta^2}\cos 2\omega_D t\right]\right\}$$



Note:

1) Eventually approaches to the variance of the stationary response, $\dfrac{\pi\Phi_0}{2\zeta\omega_0}$

2) The “sufficient” time to achieve stationarity depends on $\zeta\omega_0$.

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(3) Crosscovariance of $X(t)$ and $Y(t)$:

$$\kappa_{XY}(t_1,t_2)=\frac{\partial\kappa_{XX}(t_1,t_2)}{\partial t_2}$$

$$=\frac{\pi\Phi_0}{2\zeta\omega_0\omega_D}\left\{\begin{aligned}&e^{-\zeta\omega_0|t_1-t_2|}\sin\omega_D|t_1-t_2|-\\&e^{-\zeta\omega_0(t_1+t_2)}\left[\sin\omega_D|t_1-t_2|-\frac{\zeta}{\sqrt{1-\zeta^2}}\cos\omega_D(t_1-t_2)+\frac{\zeta}{\sqrt{1-\zeta^2}}\cos\omega_D(t_1+t_2)\right]\end{aligned}\right\}$$

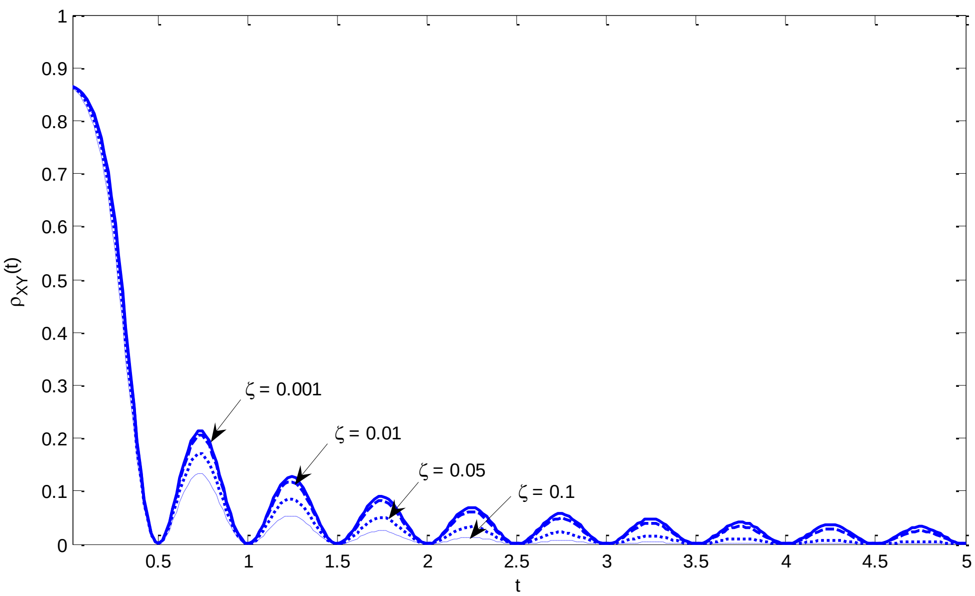
(4) Crosscorreltion coefficient function of $X(t)$ and $Y(t)$:

$$\rho_{XY}(t_1,t_2)=\frac{\kappa_{XY}(t_1,t_2)}{\sqrt{\kappa_{XX}(t_1,t_1)\kappa_{YY}(t_2,t_2)}}$$

$$\rho_{XY}(t,t)=\frac{\kappa_{XY}(t,t)}{\sqrt{\kappa_{XX}(t,t)\kappa_{YY}(t,t)}}$$

in which

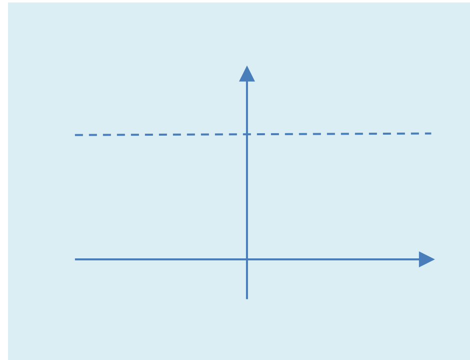
$$\kappa_{XY}(t,t)=\frac{\pi\Phi_0}{2\omega_D^2}e^{-2\zeta\omega_0 t}\left(1-\cos 2\omega_D t\right)$$



Note:

1) $\rho_{XY}(t)$ is not zero due to the nonstationarity.
2) Therefore, $\rho_{XY}(t)$ can be used as a criterion for checking stationarity.

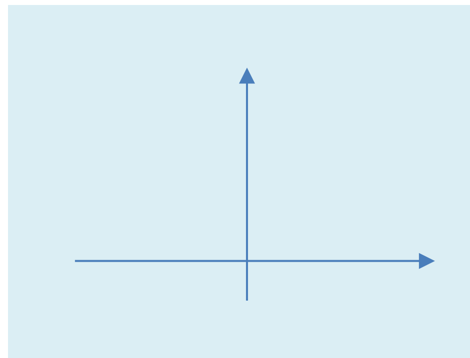
◎ Stationary response of standard SDOF oscillator to wide-band inputs: approximation by "white-noise" response



$$\Phi_{XX}(\omega) = \Phi_{FF}(\omega)|H(\omega)|^2$$
$$\cong \Phi_0 |H(\omega)|^2$$

where $\Phi_0 = \Phi_{FF}(\omega_0)$

The accuracy of the WN approximation depends on



- ξ: Bandwidth of $|H(\omega)|^2$ (accurate if it is narrow band, i.e. $\xi \cong 0$)
- Bandwidth parameter of $F(t)$ (e.g. δ, s, ξ_g) accurate if it is wideband, e.g. $\xi_g \gg 0$
- $\frac{\omega_0}{\omega_g} \cong 1$

◎ Spectral moments of stationary response of SDOF to white noise input

$$\lambda_m = \int_0^\infty \omega^m G_{XX}(\omega) d\omega$$
$$= 2\Phi_0 \int_0^\infty \omega^m |H(\omega)|^2 d\omega$$

- $\lambda_0 = \mathrm{E}\left[X^2\right] = \sigma_X^2 = \frac{\pi\Phi_0}{2\xi\omega_0^3}$
- $\lambda_1 = \frac{\pi\Phi_0}{2\xi\omega_0^2} \times \frac{2}{\pi\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \cong \frac{\pi\Phi_0}{2\xi\omega^2} \times \left(1 - \frac{2\xi}{\pi}\right)$ for $\xi \cong 0$

  ➔ Useful for identifying the bandwidth of the process, e.g. $\delta = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0\lambda_2}}$

- $\lambda_2 = \mathrm{E}\left[\dot{X}^2(t)\right] = \sigma_{\dot{X}}^2 = \frac{\pi\Phi_0}{2\xi\omega_0}$
- $\lambda_m \to \infty$ if $m > 3$ (Why?)