Seoul National University
Dept. of Civil and Environmental Engineering

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457.212 Statistics for Civil & Environmental Engineers

In-Class Material: Class 23

Testing Validity of Distributions: (1) Probability Papers (A&T: 7.2)

Given: Sample data set $\{x_1, x_2, ..., x_n\}$

Question: Does it follow a certain type of distribution or not? (e.g. Normal, Lognormal…)

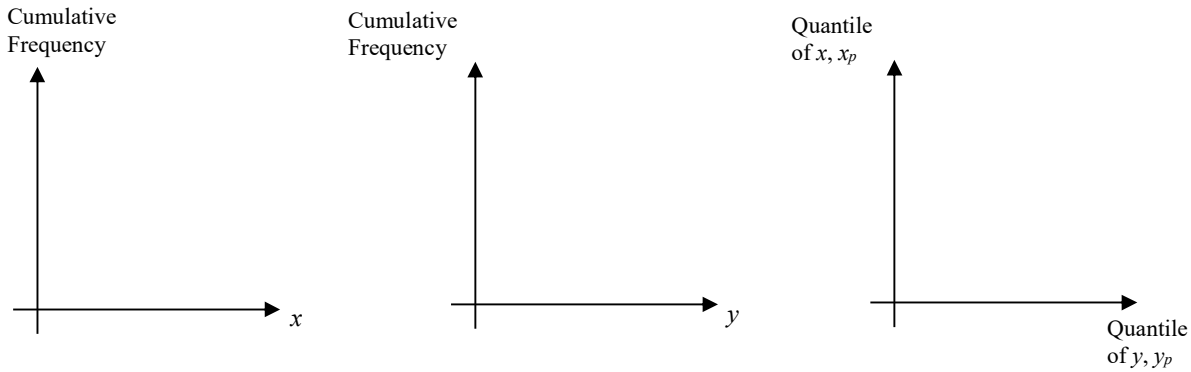
"Goodness-of-Fit" Test

(1) Visual: by probability paper (plot)

(2) Numerical: Chi-square test, K-S test, A-D test

1. **Q-Q (Quantile-Quantile) plot:**

Two samples $\{x_1, x_2, ..., x_n\}$ and $\{y_1, y_2, ..., y_m\}$: Following the same distribution?



Linear pattern (i.e. scaling and shifting) indicates that the two sets follow the same type of distribution.

[R practice] Q-Q plots of three data sets generated from Gaussian distributions and one data set from a Gamma distribution:

```
x1 = rnorm(10000,2,4) # normal dist. mu=2, sigma=4
x2 = rnorm(10000,2,4) # normal dist. mu=2, sigma=4
x3 = rnorm(10000,2,1) # normal dist. mu=2, sigma=2
x4 = rgamma(10000,shape=1,rate=1/2)
# gamma dist. mu=1*2=2, sigma = 1*2^2=4

qqplot(x1, x2) # same dist, same mu & sigma
qqplot(x1, x3) # same dist, different mu & sigma
qqplot(x1, x4) # different dist, same mu & sigma
```

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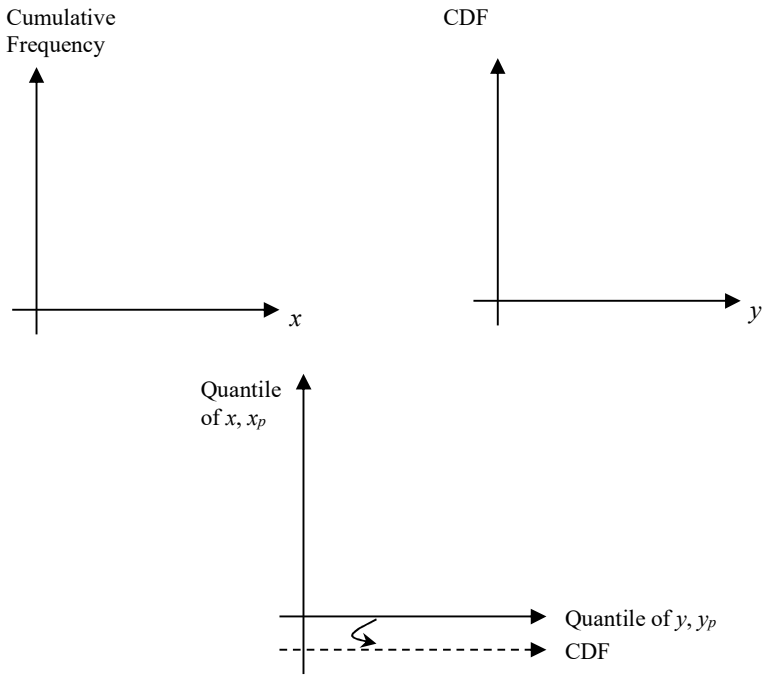
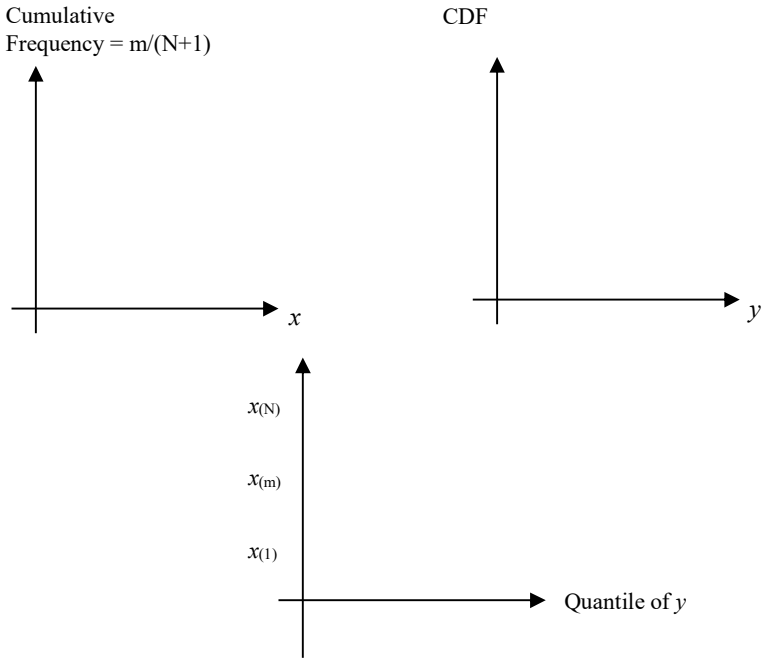
2. Probability Paper (Plot)

A sample data set $\{x_1, x_2, ..., x_n\}$ and a probabilistic distribution $F_Y(y)$



Compute quantile $y_p = F_Y^{-1}(p)$ → "Probability Plot"
Use a special graph paper (that has pre-calculated CDF) → "Probability Paper"

(a) Probability paper for Normal distribution: Use **<u>standard</u>** Normal random variable



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(b) Probability paper for Lognormal distribution

Take the ( ) of the same values and use the normal probability plot

Why?

(c) Probability paper for a general distribution

Find the standardized distribution for the distribution to test through a linear transformation

(d) **[R practice]** Probability plots against correct and wrong assumed distribution types:

```
# Probability Paper
install.packages("e1071")
library(e1071)

probplot(x1, qdist=qnorm) # correct dist.
probplot(x1, qdist=qlnorm) # wrong dist.
probplot(x4, qdist=qgamma, shape=1) # correct dist.
```

Example 1: Shifted exponential distribution

$$f_Y(y) = \begin{cases} \lambda \exp[-\lambda(x-a)] & x \geq a \\ 0 & \text{otherwise} \end{cases}$$

Example 2: Gumbel distribution

$$F_X(x) = \exp\left[-e^{-\alpha(x-u)}\right]$$

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Example 3: Construct a Rayleigh distribution probability paper for the sample, {1.413, 5.451, 1.805, 6.606, 3.464, 3.589, 4.793, 2.107, 1.023, 2.551}. The PDF of the Rayleigh distribution is

$$f_X(x) = \frac{x}{b^2}\exp\left(-\frac{x^2}{2b^2}\right), \quad 0 < x$$

(1) Find the standard variate and its CDF, quantile for the m-th data.

(2) Order the sample in the increasing order and compute the cumulative frequency of each data and its corresponding quantile.

m	$X_{(m)}$	$m/(n+1)$	$s_p = s_{m/(n+1)}$
1	1.023	1/11 = 0.09	0.437
2	1.413	2/11 = 0.18	0.634
3	1.805	0.27	0.798
4	2.107	0.36	0.951
5	2.551	0.45	1.101
6	3.464	0.55	1.256
7	3.589	0.64	1.422
8	4.793		
9	5.451		
10	6.606		

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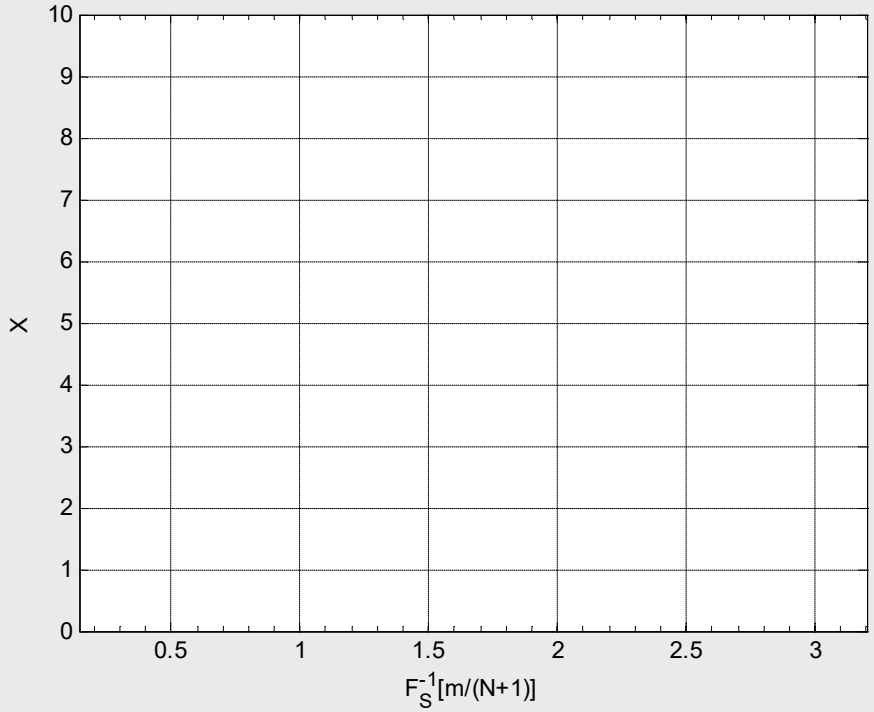
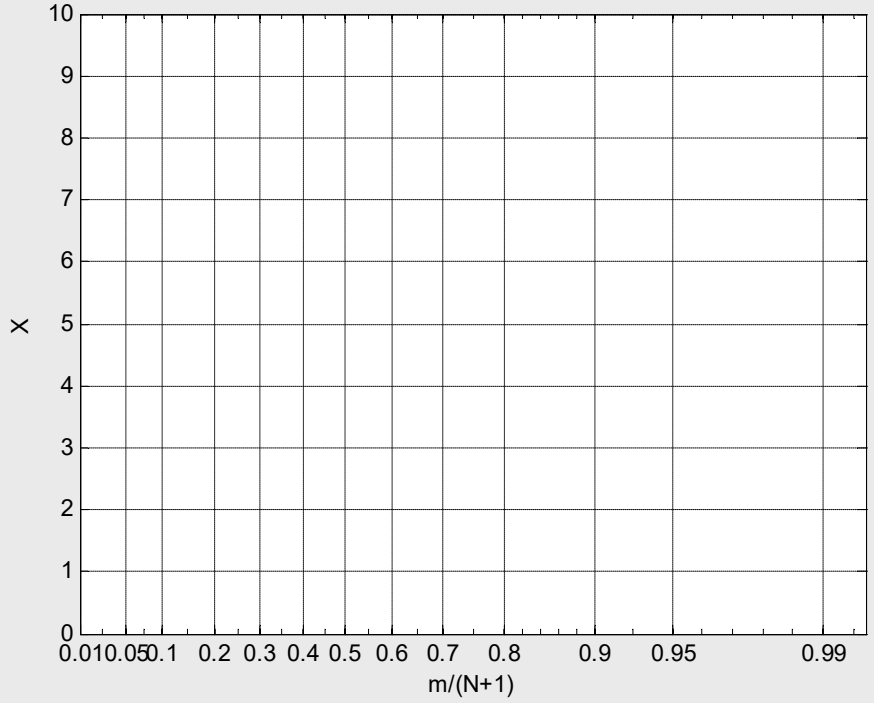
(3) Plot the pair of the standard variate and the corresponding sample value with arithmetic scales on both axes → "Probability Plot"



(4) Plot the pair of the standard variate and the corresponding sample value on the given probability paper of the Rayleigh distribution → "Probability Paper"



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```
# Example 3 - Construct a Probability Plot by R
x = c(1.413,5.451,1.805,6.606,3.464,3.589,4.793,2.107,1.023,2.551)
m = seq(1,length(x))
xm = sort(x)
x_CDF = m/(length(x)+1)
sp = sqrt(-2*log(1-x_CDF))
plot(sp,xm,pch=16,xlab="Quantile for Rayleigh, sp",ylab="Sample
Data, X")
grid(nx=NULL, ny=NULL, col="blue")
```