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457.643 Structural Random Vibrations
In-Class Material: Class 24

V. Crossings & Failure Analysis (contd.)

◎ (Upper) bound on first-passage probability using crossing rate

$$P(\text{at least one failure in }(0,t]) = \sum_{i=1}^{\infty} P(i\text{ crossing(s) in }(0,t])$$

Note

$$\int_0^{t} \nu(a;t)dt = \mathrm{E}[N(a;t)]$$
$$= \text{mean no. of crossings in }(0,t]$$
$$= \sum_{i=1}^{\infty} i \cdot P(i\text{ crossing(s) in }(0,t])$$

$$\therefore P(\text{at least one failure}) \le \int_0^{t} \nu(a;t)dt$$

This approximation works well when crossing events are rare, but may not work if it is a narrow-band process (because if there is crossing, multiple crossings can occur).

◎ Probability distribution of "global" peak and first-passage probability

$X_\tau = \max_{0\le t\le\tau} X(t)$ (cf. $|X(t)|$ ~ two-sided)

Relationship between first-passage probability and CDF of the global peak:

$p_X(a;\tau) = \quad - F_{X_\tau}(a)$ where

$$F_{X_\tau}(a) = P(X(0) \le a \cap \qquad \text{upcrossings above level }a\text{ in }(0,t])$$
$$\cong F_X(a;0)\cdot P(\qquad \text{upcrossings above level }a\text{ in }(0,\mathrm{t}])$$

Two methods to obtain the probability of __________ upcrossings:

- Poisson assumption
- Vanmarcke's formula (Prof. Erik Vanmarcke)

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◎ First-passage probability by Poisson assumption

In this approach, it is assumed that upcrossing events form a Poisson process.

This approach works relatively well if the threshold value a is ________ or the process is a _____-band process (because correlation between crossing events is _____ in these cases).

$$P(x \text{ crossing(s) in } (0,\tau]) = \frac{m(\tau)^x}{x!}\exp[-m(\tau)]$$

$$\therefore P(0 \text{ crossings in } (0,\tau]) = \exp[-m(\tau)] = \exp\left[-\int_0^\tau \qquad dt\right]$$

Therefore, the first-passage probability by Poisson assumption is

$$\boxed{p_X(a;\tau) = 1 - F_X(a;0)\cdot\exp\left[-\int_0^\tau \nu^+(a;t)dt\right]}$$

Note: the first-passage probability takes the form $1 - A\cdot L_X(a;\tau) = 1 - A\cdot\exp\left(-\int_0^\tau \alpha(a;t)dt\right)$. The approach by Vanmarcke aims to improve the accuracy of A and $\alpha(a;t)$.

Example: Stationary Gaussian process with zero-mean

$$\nu^+(a) = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left(-\frac{a^2}{2\lambda_0}\right)$$

$$F_X(a;0) = P(X \le a) = \Phi\left(\frac{a}{\sigma_X}\right) = \Phi(r)$$

$$\boxed{p_X(a;\tau) = 1 - \Phi\left(\frac{a}{\sigma_X}\right)\cdot\exp\left[-\frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left(-\frac{a^2}{2\lambda_0}\right)\cdot\tau\right]}$$

Note: For two-sided crossing, $\mathrm{F}_{|X|}(a;0) = 1 - 2\Phi(-r)$ and $2\nu_X^+(a)$ are used instead.

Furthermore, from the CDF of the global peak, $F_{X_\tau}(a) = \exp\left[-\frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left(-\frac{a^2}{2\lambda_0}\right)\cdot\tau\right]$, Davenport (1964) derived the relationship between the statistics of the global peak (μ_{X_τ} and σ_{X_τ}) and the standard deviation of the process $X(t)$ as follows:

$\mu_{X_\tau} = p\sigma_X$ and $\sigma_{X_\tau} = q\sigma_X$

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The so-called "peak factors" were derived from the CDF as

$$\boxed{\begin{aligned} p &= \sqrt{2\ln[\nu_X^+(0)\tau]} + \frac{0.5772}{\sqrt{2\ln[\nu_X^+(0)\tau]}} \\ q &= \frac{\pi}{\sqrt{6}}\frac{1}{\sqrt{2\ln[\nu_X^+(0)\tau]}} \end{aligned}}$$

Note:

- For the two-sided peak, replace $\nu_X^+(0)$ by $\nu_X(0) = \nu_X^+(0) + \nu_X^-(0)$
- These peak factors work relatively well for wide-band processes and high thresholds because the CDF was derived based on ____________ assumption.
- Der Kiureghian (1980) proposed improved versions that work for general cases based on Vanmarcke's formula (discussed later)

◎ First-passage probability by Vanmarcke (1975)

Recall, the first-passage probability was derived in the form

$$p_X(a;\tau) = 1 - A \cdot \exp\left(-\int_0^\tau \alpha(a;t)dt\right) = 1 - A \cdot L_X(a;t)$$

where A denotes the probability of the "safe start" and $L_X(a;t) = \exp\left(-\int_0^\tau \alpha(a;t)dt\right)$ represents the conditional probability of the first-passage failure given "safe start"

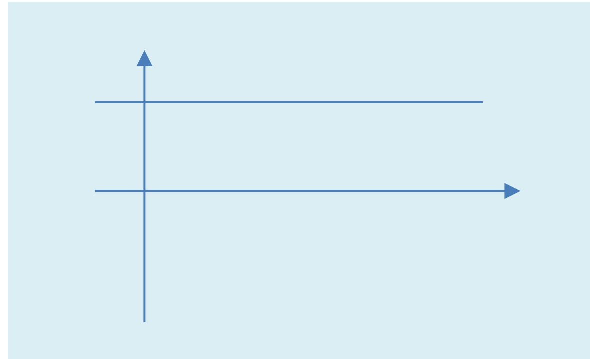
When the first-passage probability is described as above, one can show that $\alpha(a;t)$ is interpreted as (See L&S, pp.497-499)

$$\alpha(a;t) = \lim_{\Delta t \to 0} \frac{\mathrm{E}[\text{No. of crossings in } (t, t+\Delta t) \mid \text{no prior crossings up to } t]}{\Delta t}$$

In words, $\alpha(a;t)$ in the formulation above should be "   " mean crossing rate given _______________

※ In the Poisson assumption based approach, $\alpha(a;t)$ is approximated by __________, which is "   " mean crossing rate. This means the Poisson approach neglects __________ ___________ between crossing events. This is why the approach works well when the threshold is high and the process is ________-band.

Vanmarcke (1975) took into account the statistical dependence between the crossing events by introducing the envelope process and the "clump" size, i.e. the average number of crossings of the original process per a crossing of the envelope process.



For example, the clump size of a stationary Gausssian process with zero-mean is

$$\mathrm{E}[\mathrm{CS}] = \frac{1}{1 - \exp\left(-\sqrt{2\pi}\delta^{1.2} r\right)}$$

where δ is the bandwidth parameter and $r = a/\sigma_X$ is the normalized threshold.

- $\delta \cong 0$ (narrow band): $\mathrm{E}[\mathrm{CS}]$ large (envelope crossing → many process crossings)
- $\delta \cong 1$ (wide band): $\mathrm{E}[\mathrm{CS}] \cong 1$ (one crossing per one envelope crossing)

Based on this, the first-passage probability is estimated by

$$p_X(a;\tau) = 1 - B \cdot \exp\left(-\int_0^\tau \eta^+(a;t)dt\right)$$

$$\boxed{\begin{aligned} B &= P(E(0) < a) = \int_0^a f_E(e;0)de \\ \eta^+(a;t) &= \frac{P(E(t) \geq a) \cdot \nu_X^+(0;t)}{P(E(t) < a)}\left[1 - \exp\left(\frac{-\nu_E^+(a;t)}{P(E(t) \geq a) \cdot \nu_X^+(0;t)}\right)\right] \end{aligned}}$$

For a stationary Gaussian process with zero-mean, using the envelope process by Cramer and Leadbetter (1967), the first-passage probability is expressed using

$$\boxed{\begin{aligned} B &= 1 - \exp(-r^2/2) \\ \eta_X^+(a;t) &= \nu_X^+(a;t)\frac{1 - \exp\left(-\sqrt{2\pi}\delta^{1.2} r\right)}{1 - \exp(-r^2/2)} \end{aligned}}$$

Note: For two-sided crossings, use $\nu_X(a;t)$ instead of $\nu_X^+(a;t)$, and $\sqrt{\pi/2}$ instead of $\sqrt{2\pi}$

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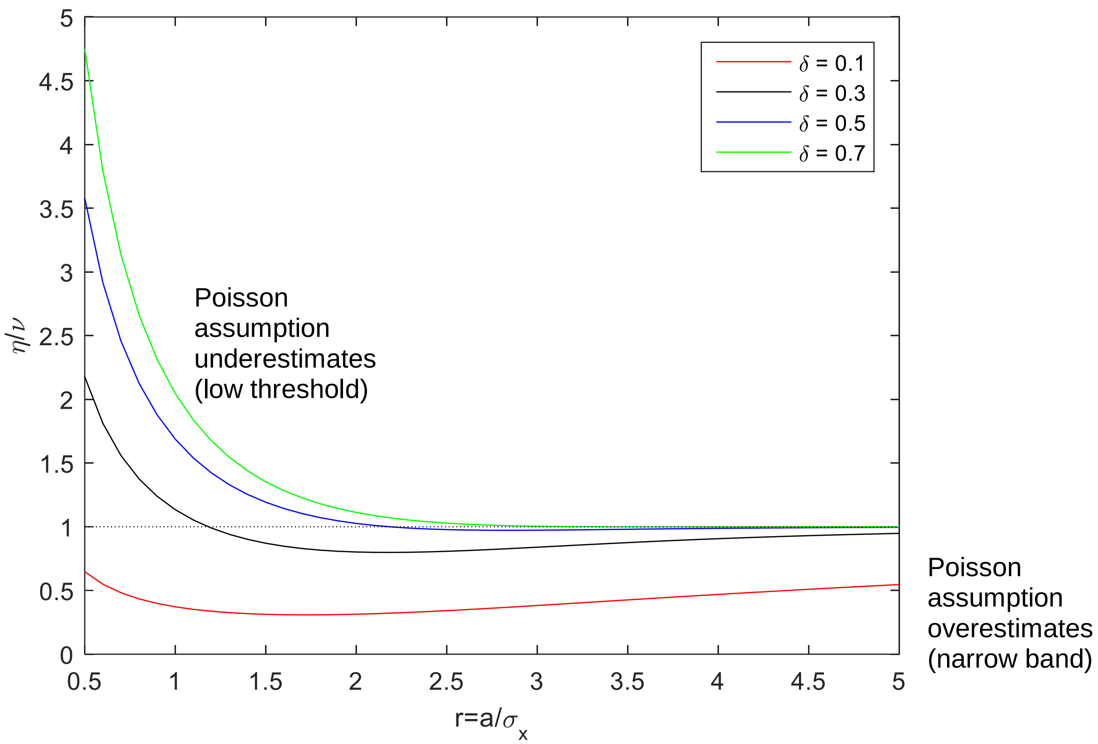
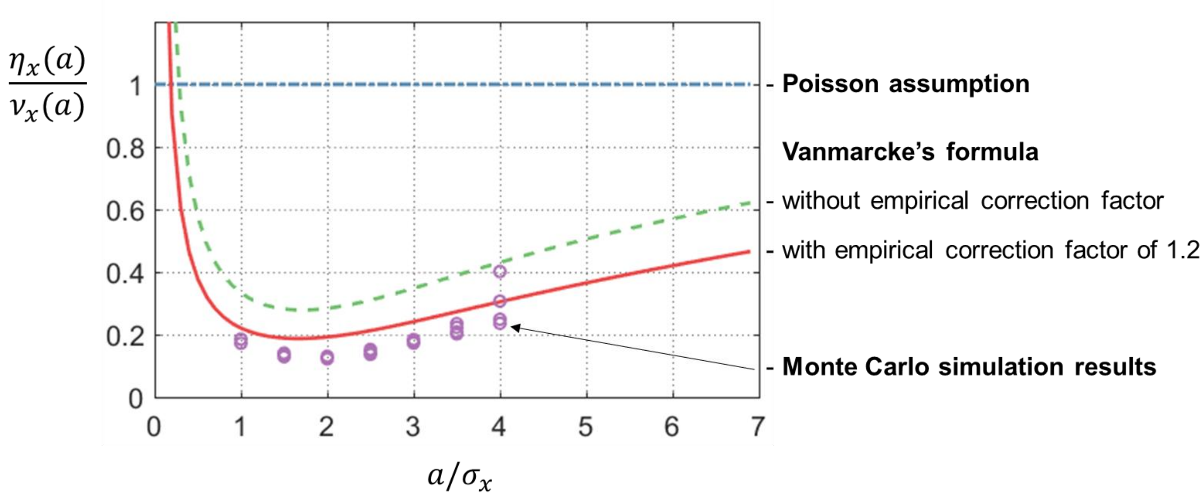
※ $\eta_X^+(a)/\nu_X^+(a)$ for a stationary Gaussian process with zero-mean:



※ Also see Figure 4(a) in Song and Der Kiureghian (2006) ($\delta = 0.26$)

※ (From L&S) 1% of critical damping ($\delta \simeq 0.11$) – (Plot created by Ms. Sang-ri Yi)



Vanmarcke's formula and simulation data shows largest discrepancy near $a = 2\sigma_x$ (about 60-70%). For $a = 2\sigma_x$ and the large values of time t, the Vanmarcke's formula will significantly overpredict the first-passage probability.

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◎ Peak factors (improved for narrow-band process)

To account for the effect of the statistical dependence between crossing events, Der Kiureghian (1980) derived peak factors based on Vanmarcke's formula (for two-sided peak):

$$p = 1.253 + 0.209\nu_e\tau \qquad 0 < \nu_e\tau \le 2.1$$
$$= \sqrt{2\ln(\nu_e\tau)} + \frac{0.5772}{\sqrt{2\ln(\nu_e\tau)}} \qquad 2.1 < \nu_e\tau$$

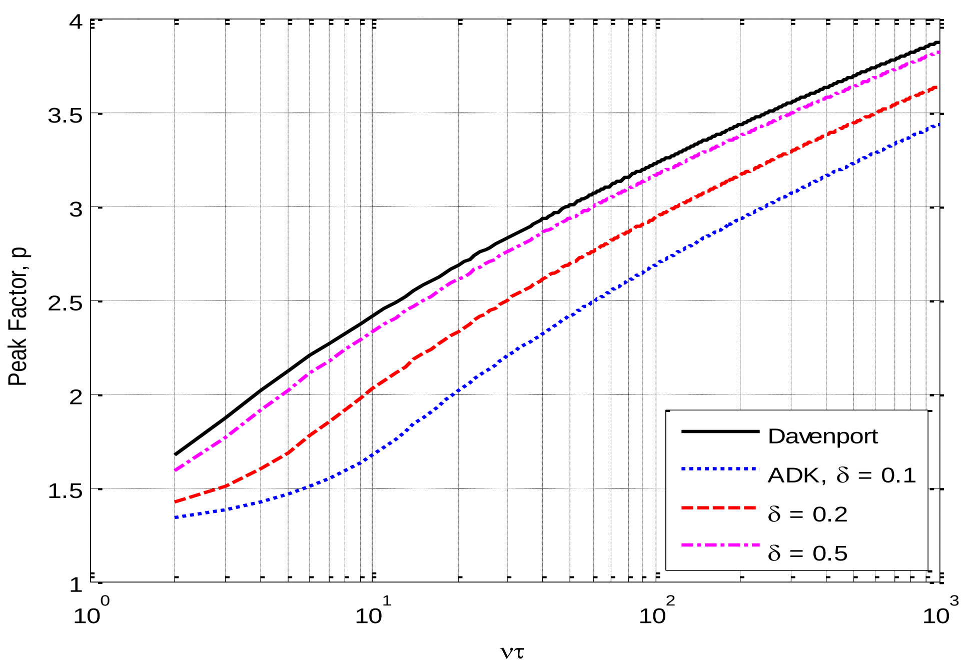
$$q = 0.658 \quad 0 < \nu_e\tau \le 2.1$$
$$= \frac{1.20}{\sqrt{2\ln(\nu_e\tau)}} - \frac{5.40}{13 + [2\ln(\nu_e\tau)]^{3.2}} \qquad 2.1 < \nu_e\tau$$

where $\nu_e = 2\delta\nu_X(0) \quad 0 < \delta \le 0.1$

$$= (1.63\delta^{0.45} - 0.38)\nu_X(0) \quad 0.1 < \delta \le 0.69$$
$$= \nu_X(0) \quad 0.69 < \delta < 1$$

For the one-sided peak, replace $\nu_X(0)$ by $\nu_X^+(0)$, and δ by 2δ.

Example: two-sided peak factors for stationary Gaussian with zero-mean



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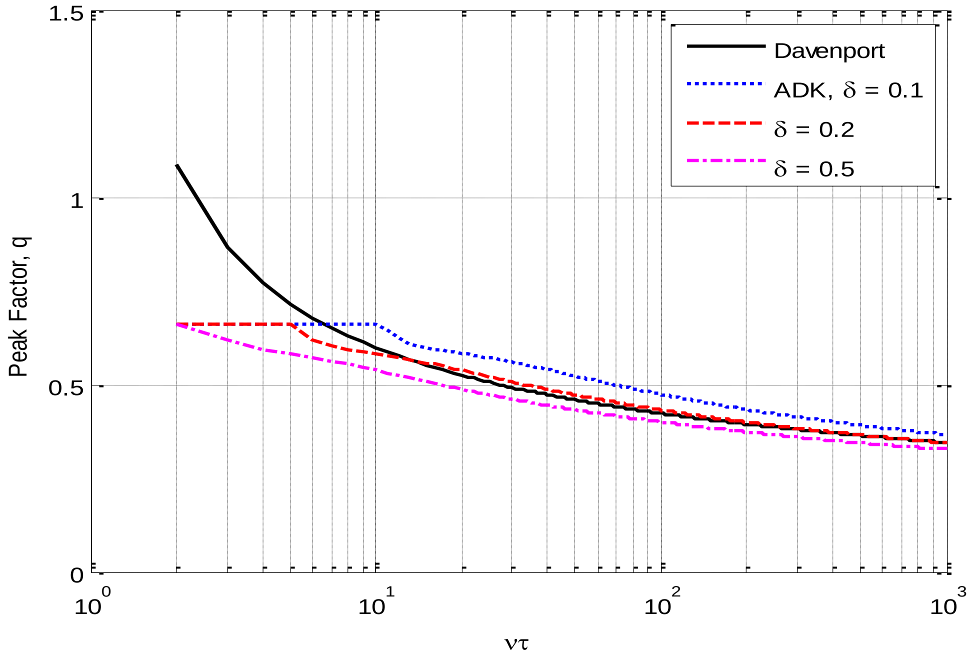

Note: When $\nu\tau = 10 \times 20 = 200$ (a rough upperbound for typical earthquake responses), $p = 2.93\sim3.43$ and $q = 0.37\sim0.43$

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◎ Extension of first-passage probability concept to multiple stochastic processes

$$P\left(\max_{0\le t<\tau} X_1(t) > a_1 \cap \max_{0\le t<\tau} X_2(t) > a_2\right)?$$

Song, J., and A. Der Kiureghian (2006). Joint first-passage probability and reliability of systems under stochastic excitation. *Journal of Engineering Mechanics. ASCE*, 132(1), 65-77.

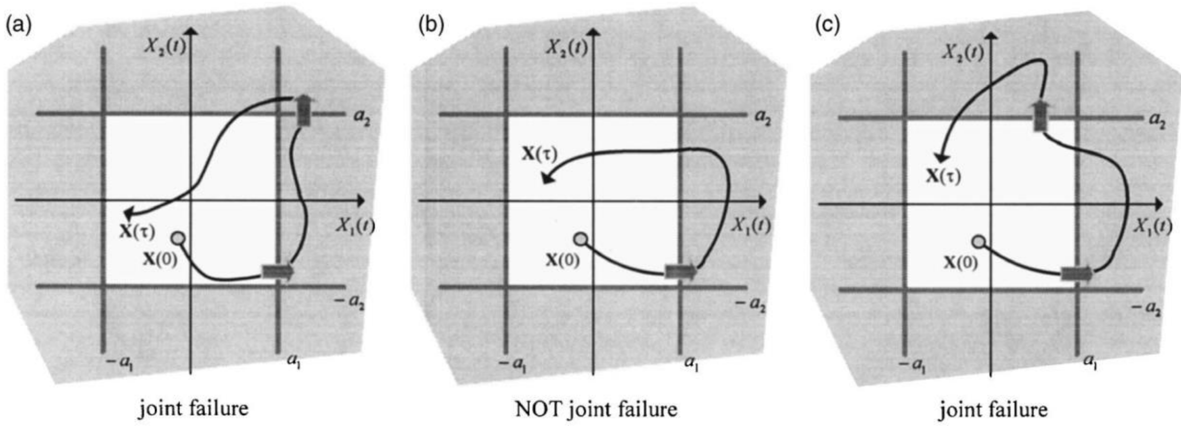

Fig. 1. Trajectories of a vector process and relation to the joint failure event

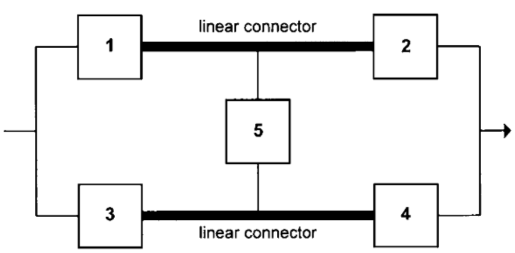

Fig. 8. System with five interconnected equipment items

LP Bounds method (Song and ADK, 2003) + Joint First-Passage (Song and ADK, 2006) → System fragility curve

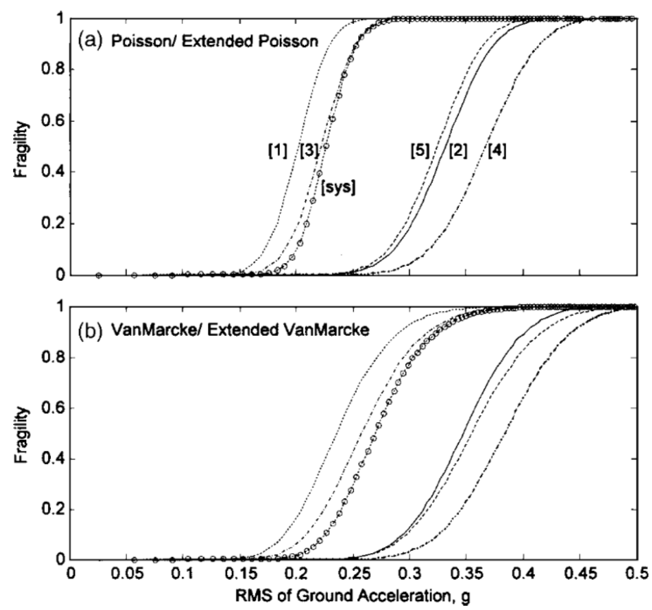

Fig. 9. Equipment and system fragility estimates by (a) extended Poisson approximation and (b) extended VanMarcke approximation