457.643 Structural Random Vibrations
In-Class Material: Class 26 (Final)

VI. Nonlinear Random Vibration Analysis (contd.)

◎ "Standard" ELM: procedure to find equivalent linear coefficients

1) In general, the equivalent linear coefficient (minimizing the mean-square error) matrix is derived as (Kozin 1987)

$$\boxed{\mathbf{A} = \frac{\mathrm{E}[\boldsymbol{g}(\boldsymbol{y})\boldsymbol{y}^{\mathrm{T}}]}{\mathrm{E}[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}]}}$$

But, this formula is impractical because (1) the distribution of $\boldsymbol{y}$ is unknown, and (2) it is not straightforward to compute the expectation $\mathrm{E}[\cdot]$ that involves the nonlinear responses.

2) "Restricted" ELM: $\boldsymbol{y}$ is assumed to be *nearly* Gaussian (e.g. the input stochastic process is Gaussian, and the nonlinearity is not strong)

When $\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{f}$ is alternatively formulated as $\boldsymbol{q}(\boldsymbol{y}, \dot{\boldsymbol{y}}, \boldsymbol{f}) = \mathbf{0}$, the equivalent linear coefficient matrix is derived as (Atalik & Utku 1976)

$$\boxed{A_{ij} = \mathrm{E}\left[\frac{\partial q_i(\boldsymbol{y})}{\partial y_j}\right]}$$

Example: Application of this approach to standard MDOF system

$\boldsymbol{q}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \boldsymbol{f}$ can be linearized to $\mathbf{M}^{e}\ddot{\boldsymbol{x}} + \mathbf{C}^{e}\dot{\boldsymbol{x}} + \mathbf{K}^{e}\boldsymbol{x} = \boldsymbol{f}$ where

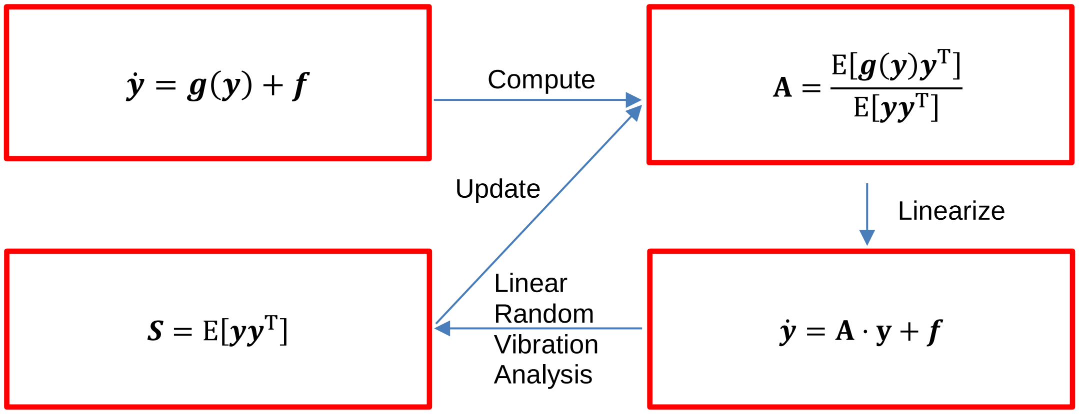
$$\boxed{M_{ij}^{e} = \mathrm{E}\left[\frac{\partial q_i}{\partial \ddot{x}_j}\right], C_{ij}^{e} = \mathrm{E}\left[\frac{\partial q_i}{\partial \dot{x}_j}\right], K_{ij}^{e} = \mathrm{E}\left[\frac{\partial q_i}{\partial x_j}\right]}$$

For the given type of a nonlinear system, one needs to derive the closed-form expressions of these expectations in terms of $\mathrm{E}[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}]$ so that one can obtain the moments by solving (equivalent) linear random vibration problem iteratively (Details shown below for the Bouc-Wen class model).

3) "Unrestricted" ELM (Pradlwarter & Schuëller 1991)

- Not limited to "Gaussian response" assumption
- Need to identify joint distribution model for the given class of nonlinear problem (and how to obtain the moments as well)

◎ Nonlinear random vibration analysis by standard ELM



◎ (Standard restricted) ELM for Bouc-Wen model (Wen 1980)

Suppose that a system with Bouc-Wen element(s) is subjected to a zero-mean Gaussian (filtered) white noise.

1) Derivation of analytical (closed-form) expression for equivalent linear coefficients:

   The nonlinear differential equation about the evolution of the auxiliary variable, i.e.

   $\dot{z} = \dot{x} \cdot [A - |z|^n(\gamma + \beta \mathrm{sgn}(\dot{x}z))]$

   This can be alternatively described as

   $q(\dot{x}, z, \dot{z}) = \dot{z} - \dot{x} \cdot \left[A - |z|^n\left(\gamma + \beta \mathrm{sgn}(\dot{x}z)\right)\right] = 0$

   This nonlinear differential equation is linearized to

   $a_0\dot{z} + a_1\dot{x} + a_2 z = 0$

   From Atalik & Utku (1976), i.e. $A_{ij} = \mathrm{E}\left[\frac{\partial q_i}{\partial ()_j}\right]$

   a) $a_0 = \mathrm{E}\left[\frac{\partial q}{\partial \dot{z}}\right] = \mathrm{E}[1] = 1$

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b) when $n=1,\ a_1 = \mathrm{E}\left[\frac{\partial q}{\partial \dot{x}}\right] = \mathrm{E}[-A + \gamma|z| + |z|\beta\mathrm{sgn}(\dot{x}z) + \dot{x}|z|\beta\cdot 2\delta(\dot{x})\mathrm{sgn}(z)]$

One can show that $\mathrm{E}[|z|] = \sqrt{\frac{2}{\pi}}\sigma_z$ and

$$\mathrm{E}[|z|\mathrm{sgn}(\dot{x}z)] = \mathrm{E}[|z|\mathrm{sgn}(\dot{x})\mathrm{sgn}(z)] = \mathrm{E}[z\cdot\mathrm{sgn}(\dot{x})] = \sqrt{\frac{2}{\pi}}\frac{\mathrm{E}[\dot{x}z]}{\sigma_{\dot{X}}}$$

Here a useful formula for zero-mean Gaussian, introduced in Atalik & Utku (1976), $\mathrm{E}[\mathbf{y}h(\mathbf{y})] = \mathrm{E}[\mathbf{y}\mathbf{y}^{\mathrm{T}}]\cdot\mathrm{E}[\nabla h(\boldsymbol{y})]$ is used for the derivation.

Finally,

$$a_1 = \sqrt{\frac{2}{\pi}}\left[\beta\frac{\mathrm{E}[z\dot{x}]}{\sigma_{\dot{X}}} + \gamma\sigma_z\right] - A$$

c) when $n=1,\ a_2 = \mathrm{E}\left[\frac{\partial q}{\partial z}\right] = \mathrm{E}[\dot{x}\mathrm{sgn}(z)\gamma + \dot{x}\mathrm{sgn}(z)\beta\mathrm{sgn}(\dot{x}z) + \dot{x}|z|\beta\mathrm{sgn}(\dot{x})2\delta(z)]$

$$a_2 = \sqrt{\frac{2}{\pi}}\left[\gamma\frac{\mathrm{E}[\dot{x}z]}{\sigma_z} + \beta\sigma_{\dot{X}}\right]$$

2) Construct an equivalent linear system

$\mathbf{y} = \begin{Bmatrix} x \\ z \\ \dot{x} \end{Bmatrix}$, and $\dot{\mathbf{y}} = \begin{Bmatrix} \dot{x} \\ \dot{z} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -a_2/a_0 & -a_1/a_0 \\ -\alpha\omega_0^2 & -(1-\alpha)\omega_0^2 & -2\xi\omega_0 \end{bmatrix}\begin{Bmatrix} x \\ z \\ \dot{x} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ f(t)/m \end{Bmatrix}$

$$\dot{\boldsymbol{y}} = \boldsymbol{G}\boldsymbol{y} + \boldsymbol{f}$$

3) Perform linear random vibration analysis

e.g. if $f(t)$ is a white noise, the 2nd moment ($\mathbf{S}$) follows the Lyapunov equation (Lin 1967)

$$\mathbf{GS} + \mathbf{SG}^{\mathrm{T}} + \mathbf{B} = \mathbf{0}$$

where $B_{ij} = 0$ except $B_{33} = 2\pi\Phi_0$ (→ Φ_0 is the PSD of the white noise $f(t)$), and

$$\mathbf{S} = \mathrm{E}[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}] = \mathrm{E}\begin{bmatrix} x^2 & xz & x\dot{x} \\ xz & z^2 & z\dot{x} \\ x\dot{x} & z\dot{x} & \dot{x}^2 \end{bmatrix}$$

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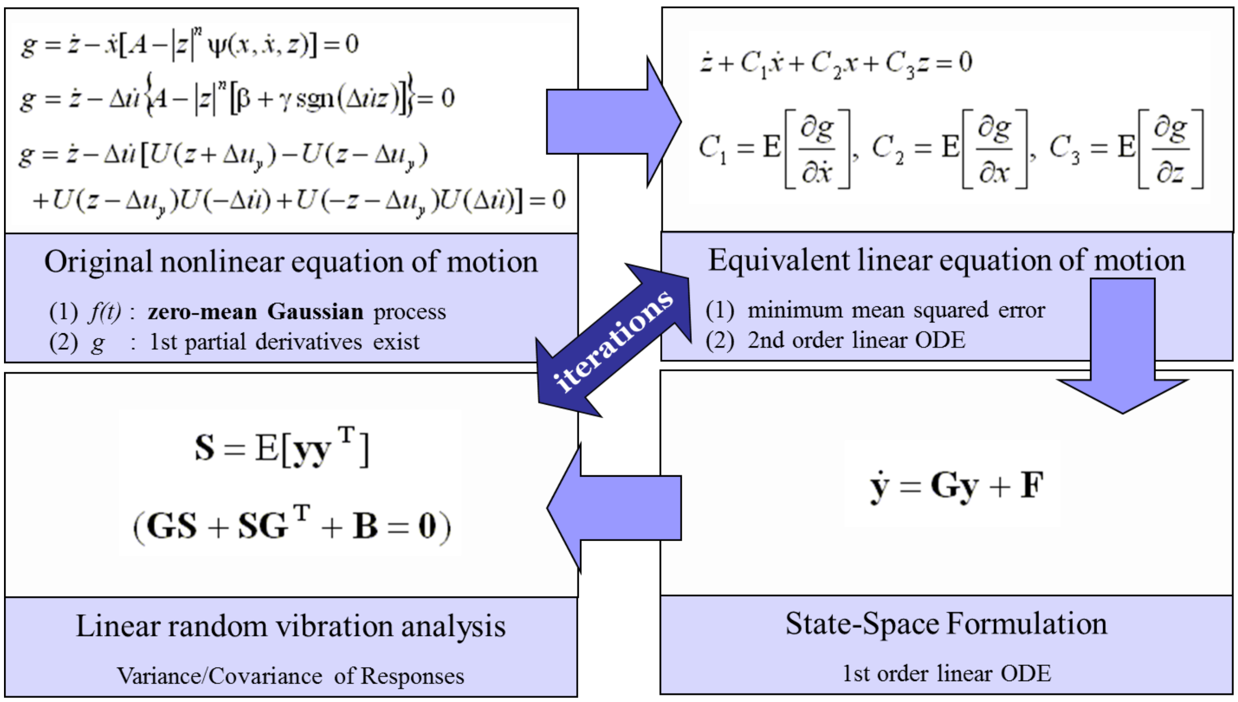
This random vibration analysis approach can be used for filtered white noise case as well by introducing the ground displacement x_g to the state-space vector, i.e. adding another DOF representing the filter.

4) Re-compute a_1 and a_2 based on newly calculated $\mathbf{S} = \mathrm{E}[\boldsymbol{y}\boldsymbol{y}^{\mathrm{T}}]$

5) Repeat 1)-4) until the solution converges.



References:

Song, J., A. Der Kiureghian, and J.L. Sackman (2007). Seismic interaction in electrical substation equipment connected by nonlinear rigid bus conductors. *Earthquake Engineering and Structural Dynamics*, Vol. 36, 167-190.

Ok, S.-Y., J. Song, and K.-S. Park (2008). Optimal design of hysteretic dampers connecting adjacent structures using multi-objective genetic algorithm and stochastic linearization method. *Engineering Structures*, Vol. 30, 1240-1249.

Q: Random vibration analysis by <u>structural reliability analysis methods</u>?

Der Kiureghian, A. (2000). The geometry of random vibrations and solutions by FORM and SORM. *Probabilistic Engineering Mechanics*, 15(1), 81-90.

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◎ Discrete representation of a random process

1) Discrete representation (in time domain)

$$\boxed{f(t) = \mu(t) + \sum_{i=1}^{n} u_i s_i(t) = \mu(t) + \mathbf{u}^{\mathrm{T}}\mathbf{s}(\mathrm{t})}$$

where $u_i, i = 1, \dots, n$ are uncorrelated standard normal random variables

$\mathbf{s}(t)$ is a vector of deterministic time-varying basis function which is identified based on the correlation structure of the process, e.g. Karhunen-Loève expansion

For example, EQ input can be modeled as a filtered white noise

$$f(t) = \int_0^{\mathrm{t}} u(\tau)s(t-\tau)d\tau \cong \sum u_i s_i(t)$$

where $s(\cdot)$ denotes the unit impulse response function of the filter.

2) Discrete representation (in *frequency* domain)

For example (Broccardo and Der Kiureghian 2015), white noise can be discretized as

$$\ddot{x}_g(t) = \sigma \sum_{j=1}^{n/2} \left[u_j \cos(\omega_j t) + \hat{u}_j \sin(\omega_j t)\right]$$

◎ Response of linear structure to Gaussian excitation

$$x(t) = \int_0^{\mathrm{t}} f(\tau)h(t-\tau)d\tau = \int_0^{\mathrm{t}} \sum_{i=1}^{n} u_i s_i(\tau)\, h(t-\tau)d\tau$$

where $h(\cdot)$ is the unit impulse response function of the structure, and thus the response is

$$\boxed{x(t) = \sum_{i=1}^{n} u_i a_i(t) = \boldsymbol{a}^{\mathrm{T}}(t)\boldsymbol{u}}$$

where $a_i(t) = \int_0^t s_i(\tau)h(t-\tau)d\tau$

In summary, the response of a linear structure to a Gaussian input can be described as a linear function of uncorrelated standard normal random variables (owing to the discrete representation).

◎ Failure probability by discrete representation

The instantaneous failure probability of the linear response is

$$P(x(t_0) \geq x_0) = P\left(x_0 - \boldsymbol{a}^{\mathrm{T}}(t_0)\boldsymbol{u} \leq 0\right)$$

This is a structural reliability problem with a linear limit state function $g(\mathbf{u}) = x_0 - \boldsymbol{a}^{\mathrm{T}}(t_0)\boldsymbol{u}$

From structural reliability theories, the failure probability is obtained by a closed-form solution

$$P(x(t_0) \geq x_0) = \Phi[-\beta(x_0, t_0)] = \Phi\left[-\frac{x_0}{\|\boldsymbol{a}(t_0)\|}\right]$$

One can also compute crossing rate, first-passage failure probability, etc. by structural reliability analysis in the standard normal space (Der Kiureghian, 2000).

This idea was utilized for efficient topology optimization with constraints on instantaneous failure probability, first-passage probability and system reliability.

Chun, J., J. Song, and G.H. Paulino (2016). Structural topology optimization under constraints on instantaneous failure probability. *Structural and Multidisciplinary Optimization*, 53(4): 773-799.

Chun, J., J. Song, and G.H. Paulino (2019). System-reliability-based design and topology optimization of structures under constraints on first-passage probability. *Structural Safety*. 76, 81-94.

◎ ELM by discrete representation: Tail ELM (TELM)

For nonlinear system and/or non-Gaussian process, first-order reliability method (FORM) or second-order reliability method (SORM) can be used to compute the probabilities approximately. This idea was further developed to propose the tail equivalent linearization method (TELM; Fujimura and ADK 2007).

Fujimura, K., and A. Der Kiureghian (2007). Tail-equivalent linearization method for nonlinear random vibration. *Probabilistic Engineering Mechanics*, 22:63-76.

Broccardo, M., and A. Der Kiureghian (2015). Multicomponent nonlinear stochastic dynamic analysis by tail-equivalent linearization. *Journal of Engineering Mechanics*, 142(3):04015100.

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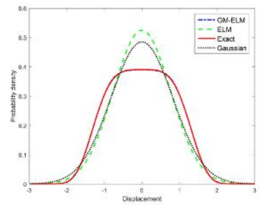
◎ Gaussian Mixture Based ELM (GM-ELM; Wang and Song 2017)

Wang, Z., and J. Song (2017). Equivalent linearization method using Gaussian mixture (GM-ELM) for nonlinear random vibration analysis. *Structural Safety*. Vol. 64, 9-19.

Yi, S., Z. Wang, and J. Song (2018). Bivariate Gaussian mixture based equivalent linearization method (GM-ELM) for stochastic seismic analysis of nonlinear structures. *Earthquake Engineering and Structural Dynamics*. Vol. 47(3), 678-696.

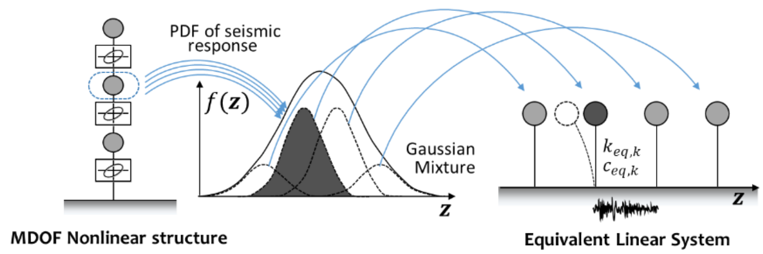
Idea: represent response of a nonlinear system (non-Gaussian distribution) by the superposition of linear responses (Gaussian distributions).



1) Equivalent linear system (ELS)

Gaussian Mixture approximation of instantaneous response PDF ($\mathbf{z} = \{x\}$ or $\mathbf{z} = \{x, \dot{x}\}$)

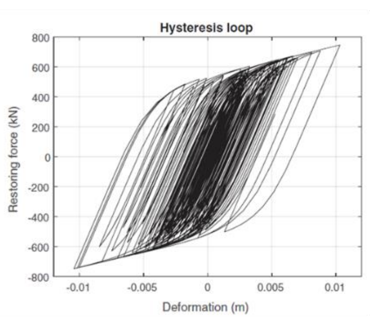
$$f(\mathbf{z}) \simeq \sum_{k=1}^{K} \alpha_k f(\mathbf{z}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

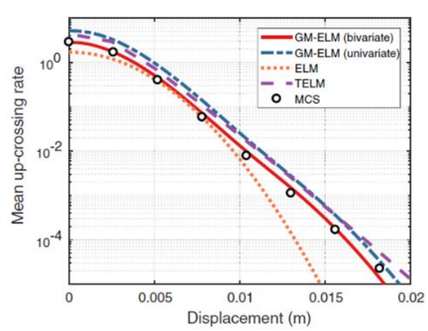


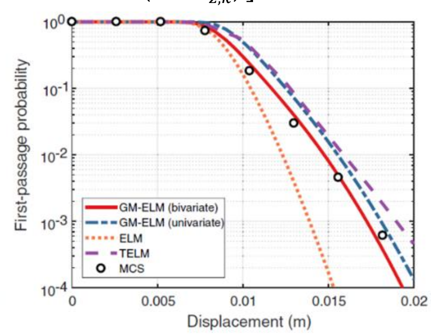
- $\boldsymbol{\mu}_k$: shifted base (and velocity) of the *k*-th linear oscillator
- $\boldsymbol{\Sigma}_k$: variances of *k*-th linear oscillator: $\lambda_{0,k}$ (and $\lambda_{2,k}$) → system parameters $c_{eq,k}$, $k_{eq,k}$
- α_k: rate of occurrence

2) (Linear) random vibration analysis using ELS

- Instantaneous failure probability: $P(Z \geq a) \simeq \sum_{k=1}^{K} \alpha_k \left[1 - \Phi\left(\frac{a-\mu_{z,k}}{\sigma_{z,k}}\right)\right]$
- Mean upcrossing rate: $\nu^+(a) \simeq \sum_{k=1}^{K} \alpha_k \nu_k^+(a - \mu_k)$
- First-passage failure probability: $\mathrm{P}\left(\max_{0<t<T} Z(t) > a\right) \simeq 1 - A\exp(-\nu^+(a)T)$
- Response spectrum analysis: $\mathrm{E}\left[\max_{0<t<T} |Z(t)|\right] = \left[\sum_{k=1}^{K} \alpha_k \Gamma^2 S_d^2(\omega_k, \xi_k)\left(1 + \frac{\mu_{z,k}^2}{\sigma_{z,k}^2}\right)\right]^{1/2}$, etc.







➔ Recently improved for nonstationary responses and scale-independent "universal" ELS to facilitate seismic fragility analysis (Yi et al., under review)

// End of Course. Thank you! -- J.S. //