Controllability and Observability

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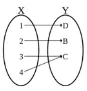
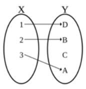
Some Linear Algebra

- Consider a matrix $L \in \Re^{m \times p}$, which may then be thought of as a **linear mapping** $L : \mathcal{X} \to \mathcal{Y}$ from the domain $\mathcal{X} \approx \Re^p$ to the range $\mathcal{Y} \approx \Re^m$.

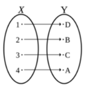


  – **Range-space** $\mathcal{R}(L) := \{y \in \Re^m \mid y = Lx, \text{ for some } x \in \Re^p\} \in \mathcal{Y}$.

  – **Null-space** $\mathcal{N}(L) := \{x \in \Re^p \mid Lx = 0\} \in \mathcal{X}$.



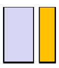
  – L is called **surjective (or onto)** if $\mathcal{R}(L) = \mathcal{Y}$ (i.e., column rank of $L = m$ with $\mathcal{Y} = \Re^m$).



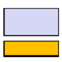
  – L is called **injective (or one-to-one)** if $\mathcal{N}(L) = \emptyset$ (i.e., if $x_1 \neq x_2$, $Lx_1 \neq Lx_2$; not many-to-one).

  – L is called **bijective** if surjective and injective.

- **Fundamental theorem of linear algebra** (w.r.t. Euclidean metric $||x||^2 := x^T I x$):



  – $\Re^m = \mathcal{R}(L) \oplus \mathcal{N}(L^T)$, i.e., $\forall y \in \Re^m$, $\exists x \in \Re^p$ and $y_n \in \Re^m$ s.t., $\boxed{y = Lx + y_n}$ with $L^T y_n = 0$ and $||y||^2 = ||Lx||^2 + ||y_n||^2$.



  – $\Re^p = \mathcal{R}(L^T) \oplus \mathcal{N}(L)$, i.e., $\forall x \in \Re^p$, $\exists y \in \Re^m$ and $x_n \in \Re^p$ s.t., $\boxed{x = L^T y + u_n}$ with $Lu_n = 0$ and $||x||^2 = ||L^T y||^2 + ||u_n||^2$.

  – $\mathcal{R}(LL^T) = \mathcal{R}(L)$ and $\mathcal{N}(L) = \mathcal{N}(L^T L)$.

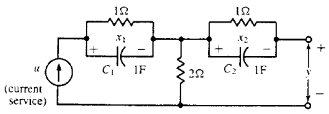
Controllability

- Consider a dynamical system with state transition map:

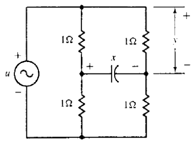
$$x(t) = s(t, t_o, x_o, u([t_o; t_1])$$

We say it is **controllable on** $[t_o, t_1]$, $t_1 > t_o$ if $\forall x_o, x_f$, $\exists u([t_o; t_1])$ s.t., $x_f = s(t_1, t_o, x_o, t([t_o; t_1])$. We also say it is **controllable at** t_o, if $\forall x_o, x_f$, $\exists t_1 > t_o$ and $u([t_o; t_1])$ s.t., $x_f = s(t_1, t_o, x_o, t([t_o; t_1])$.

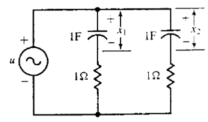
  - CTRB at t_o implies the system can be transferred from any initial state x_o at t_o to any finial state t_f in finite time.
  - For LTV system, two stat'ts equivalent and CTRB may change w/ time (i.e., t_o important). For LTI system, if CTRB at t_o, CTRB $\forall t$.
  - CTRB is a fundamental property to check before any control design.
  - If some states not CTRB, yet, still AS $\Rightarrow$ stabilizable; if not CTRB and unstable $\Rightarrow$ unstable pz-cancelation.



x_2: uncontrollable from u
x_1: unobservable from y



x: not CTRB, not OBSV due to symmetry



x_1+x_2: CTRB, yet, not x_1-x_2

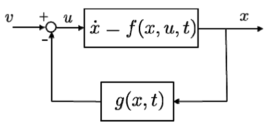
Effect of State-FB on CTRB

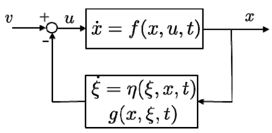
- Consider a dynamical system with state transition map:

$$x(t) = s(t, t_o, x_o, u([t_o; t_1])$$

We say it is **controllable on** $[t_o, t_1]$, $t_1 > t_o$ if $\forall x_o, x_f$, $\exists u([t_o; t_1])$ s.t., $x_f = s(t_1, t_o, x_o, t([t_o; t_1])$. We also say it is **controllable at** t_o, if $\forall x_o, x_f$, $\exists t_1 > t_o$ and $u([t_o; t_1])$ s.t., $x_f = s(t_1, t_o, x_o, t([t_o; t_1])$.

- A system $\dot{x} = f(x, u, t)$ is CTRB iff CTRB under static state feedback.
  - CTRB of $\dot{x} = f'(x, v, t)$ & $\dot{x} = f(x, u, t)$ equivalent w/ $u = v - g(x, t)$.
  - Static state-FB doesn't change original CTRB.
- If $\dot{x} = f(x, u, t)$ with dynamic state-FB is CTRB, oiginal system is CTRB. Even if original CTRB, dynamic state-FB may not CTRB (un-OBSV).
  - $v = u + g(x, \xi, t) : (x_o, \xi_o) \to (x_1, \xi_1) \Rightarrow u = v - g(x, \xi, t) : x_o \to x_1$.
  - Even dynamic state-FB can't make un-CTRB system CTRB.

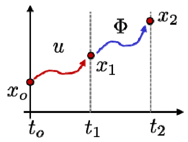




Reachability Map and CT-LTV CTRB

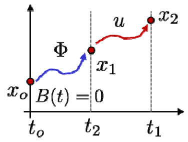
- Consider CT-LTV system $\dot{x} = A(t)x + B(t)u$, $x(t_o) = x_o$ with the state response given by:

$$x(t_1) = \Phi(t_1, t_o)x_o + \int_{t_o}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$



- Define **reachability map:**

$$\boxed{L_r(t_o, t_1)(u(\cdot)) := \int_{t_o}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau}$$



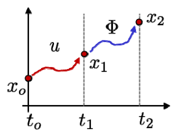
which is a linear map w.r.t. the control input $u([t_o; t_1])$.

- **Th. 6-11R1:** CT-LTV system is CTRB on $[t_o, t_1]$ iff $L_r(t_o, t_1)$ is surjective (i.e., can produce any vector in $\Re^n$).
- **Th. 6-11R2:** If CT-LTV system is CTRB on $[t_o, t_1]$, it is also CTRB on $[t_o, t_2]$ $\forall t_2 \geq t_1$.
  - (Pf) Since CTRB on $[t_o, t_1]$, can obtain any x_1 at t_1 $\Rightarrow$ given x_2, choose x_1 s.t., $x_2 = \Phi(t_2, t_1)x_1$ $\Rightarrow$ can steer to x_2 using $u([t_o, t_1])$: $x_0 \to x_1$ and $u([t_1, t_2]) = 0$ w/ $x_2 = \Phi(t_2, t_1)x_1$.
  - For CT-LTV system, CTRB on $[t_o, t_1]$ doesn't imply CTRB on $[t_o, t_2]$, $t_2 < t_1$. For CT-LTI, CTRB on $[t_o, t_1]$ implies CTRB for any interval.

Reachability Map and DT-LTV CTRB

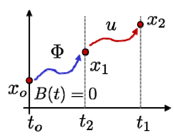
- Consider DT-LTV system $x_{k+1} = A(k)x_k + B(k)u_k$ $x(k_o) = x_o$ with the state response given by: with $\Phi(k_1, k_o) = \Pi_{k=k_o}^{k-1} A(k)$,

$$x(k_1) = \Phi(k_1, k_o)x_o + \sum_{k=k_o}^{k_1-1} \Phi(k_1, k+1)B(k)u(k)$$



- **Reachability map:**

$$\boxed{\begin{aligned} L_r(k_o, k_1)(u_{k_o}, ..., u_{k_1-1}) : &= \sum_{k=k_o}^{k_1-1} \Phi(k_1, k+1)B(k)u_k \\ &= L_r(k_o, k_1)U_{k_o;k_1-1} \end{aligned}}$$



where $L_r(k_o, k_1) := [\Phi_{k_1,k_o+1}B_{k_o}, \Phi_{k_1,k_o+2}B_{k_o+1}, \ldots, \Phi_{k_1,k_1-1}B_{k_1-1}] \in \Re^{n \times p(k_1-k_o)}$ and $U = [u_{k_o}, ..., u_{k_1-1}]^T \in \Re^{p(k_1-k_o)}$.

- **Th. 6-11RD1:** DT-LTV system is CTRB on $[k_o, k_1]$ iff $L_r(k_o, k_1)$ is surjective (i.e., column rank of $L_r(k_o, k_1) = n$).
- **Th. 6-11RD2:** If DT-LTV system is CTRB on $[k_o, k_1]$ and $A(k)$ is non-singular $k_1 \leq k \leq k_2$, it is also CTRB on $[k_o, k_2]$.
  - Non-singular $A(k)$ necessary to pull x_2 to x_1 via invertible Φ_{k_2,k_1}.
  - DT-LTV CTRB on $[k_o, k_1]$ doesn't imply CTRB on $[k_o, k_2]$, $k_2 < k_1$. DT-LTI CTRB on $[k_o, k_1]$ implies CTRB for any interval.

Reachability Grammian

- Consider CT-LTV system $\dot{x} = A(t)x + B(t)u$, with **reachability map:**

$$L_r(0,T)(u(\cdot)) := \int_0^T \Phi(T,\tau)B(\tau)u(\tau)d\tau$$

- If we split $[0,T]$ into N sub-interval, we have:

$$L_r(0,T)(u(\cdot)) \approx \sum_{k=1}^{N} \Phi(T, \tfrac{k}{N}T)B(\tfrac{k}{N}T)u(\tfrac{k}{N}T)\tfrac{T}{N} =: \bar{L}_r(0,T)U$$

where $\bar{L}_r(0,T) := [\Phi(T, \frac{1}{N}T)B(\frac{1}{N}T), ..., \Phi(\frac{N}{N}T, T)B(\frac{N}{N}T)] \times \frac{T}{N}$ and $U := [u(\frac{1}{N}T), ..., u(\frac{N-1}{N}T), u(T)]^T$.

- Since $\mathcal{R}(L_r L_r^T) = \mathcal{R}(L_r)$ and $T > 0$, CTRB on $[0,T]$ iff $\text{rank}(\bar{L}_r \bar{L}_r^T) = n$.
- With $\frac{T}{N}$ removed and $N \to \infty$, we then achieve **reachabillty grammian:**

$$\boxed{W_r(t_o,t_1) := \int_{t_o}^{t_1} \Phi(t_1,\tau)B(\tau)B^T(\tau)\Phi^T(t_1,\tau)d\tau}$$

which is sometimes also called **controllability grammian** (Chen).

Reachability Grammian and CTRB

- For CT-LTV system $\dot{x} = A(t)x+B(t)u$, **reachabillty grammian** is given by:

$$\boxed{W_r(t_o,t_1) := \int_{t_o}^{t_1} \Phi(t_1,\tau)B(\tau)B^T(\tau)\Phi^T(t_1,\tau)d\tau}$$

  - $W_r(t_o,t_1)$ is symmetric and PSD (if singular) or PD (if non-singular).
  - $W_r(t_o,t_1)$ captures not only how input u can directly affect x via $B(t)$, but also through the dynamics Φ over $[t_o,t_1]$.
  - **Controllability (to zero) grammian (Rugh):**

$$W_c(t_o,t_1) := \int_{t_o}^{t_1} \Phi(t_o,\tau)B(\tau)B^T(\tau)\Phi^T(t_o,\tau)d\tau$$

    is related to $W_r(t_o,t_1)$ by $W_r(t_o,t_1) = \Phi(t_1,t_o)W_c(t_o,t_1)\Phi^T(t_1,t_0)$.
  - For CT-LTI, $W_r(t_o,t_1) = \int_0^{t_1-t_o} e^{A\tau}BB^Te^{A^T\tau}d\tau$ and $W_c(t_o,t_1) = \int_0^{t_1-t_o} e^{-A\tau}BB^Te^{-A\tau}d\tau$.
  - DT-LTV system: $W_r(k_o,k_1) := \sum_{k=k_o}^{k_1-1} \Phi(k_1,k+1)B(k)B^T(k)\Phi^T(k_1,k_1) = L_r(k_o,k_1)L_r^T(k_o,k_1) \in \Re^{n\times n}$

- **Th. 6-11R3:** CT-LTV system is CTRB on $[t_o,t_1]$ iff its reachability grammian $W_r(t_o,t_1)$ is non-singular.

CTRB of CT-LTV System - I

Th. 6-11: CT-LTV system is CTRB at t_o iff $\exists t_1 > t_o$ s.t., the controllability grammian $W_r(t_o, t_1)$ is non-singular.

- ($\Leftarrow$): Recall $x(t_1) = \Phi(t_1, t_o)x_o + \int_{t_o}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$. Then, with

  $$u(t) := -B^T(t)\Phi^T(t_1, t)W_r^{-1}(t_o, t_1)[\Phi(t_1, t_o)x_o - x_1]$$

  we can steer from any $x_o = x(t_o)$ to any $x_1 = x(t_1) \Rightarrow$ CTRB on $[t_o, t_1]$ $\Rightarrow$ CTRB at t_o ($u(t)$ is the min. norm control w.r.t. Euclidean metric).

- ($\Rightarrow$) Suppose $W_r(t_o, t_1)$ singular for all $t_1 \to$ should't be CTRB?

  If $W_r(t_o, t_1)$ singular, $W_r(t_o, t_1)$ only PSD $\Rightarrow \exists\, v \in \Re^n$ s.t., $\forall t_1 \geq t_o$,

  $$0 = v^T W_r(t_o, t_1)v = \int_{t_o}^{t_1} ||B^T(\tau)\Phi^T(t_1, \tau)v||d\tau$$

  v in un-CTRB subspace

  i.e., $v^T\Phi(t_1, \tau)B(\tau) \equiv 0$, $\forall \tau \in [t_o, t_1]$, $\forall t_1 \geq t_o$ (from continuity of B, Φ).

  Now, suppose CTRB at $t_o \Rightarrow \exists\, u(t)$ to drive x from $x_o := \Phi(t_o, t_1)v$ to $x_1(t_1) = 0$ for some $t_1 \geq t_o$, i.e., $0 = v + \int_{t_o}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$. Yet,

  $$0 = v^T v + v^T \int_{t_o}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau = v^T v \neq 0$$

CTRB of CT-LTV System - II

- Th. 6.11 requires $\Phi(t_1, \tau) \Rightarrow$ better if can check CTRB w/o computing Φ.

- Define $M_o(t) := B(t)$ and

  $$M_{m+1}(t) := \underbrace{-A(t)M_m(t)}_{\text{effect thru A}} + \underbrace{\tfrac{d}{dt}M_m(t)}_{\text{effect of dB/dt}}$$

- We then have recursive relation: $\frac{\partial}{\partial t}[\Phi(t_2, t)M_m(t)] = \Phi(t_2, t)M_{m+1}(t)$, e.g., from $\Phi(t_2, t)B(t) = \Phi(t_2, t)M_o(t)$,

  $$\frac{\partial}{\partial t}[\Phi(t_2, t)M_o(t)] = -\Phi(t_2, t)A(t)M_o(t) + \Phi(t_2, t)\frac{d}{dt}M_o(t) = \Phi(t_2, t)M_1(t)$$

Th. 6-12: CT-LTV system with $A(t), B(t) \in C^{n-1}$ is CTRB on $[t_o, t_1]$ if, $\exists t_1 > t_o$ s.t.,

$$\text{rank}[M_o(t_1), M_1(t_1), ..., M_{n-1}(t_1)] = n$$

- Suppose not CTRB $\Rightarrow W_r(t_o, t_1)$ singular $\forall t_1 > t_o \Rightarrow \exists v \in \Re^n$ s.t.,

  $$0 = v^T W_r(t_o, t_1)v = \int_{t_o}^{t_1} ||B^T(\tau)\Phi^T(t_1, \tau)v||^2 d\tau$$

  i.e., $v^T\Phi(t_1, \tau)M_o(\tau) \equiv 0 \Rightarrow v^T\Phi(t_1, \tau)M_1(\tau) \equiv 0 ... \Rightarrow v^T\Phi(t_1, \tau)M_m(\tau) \equiv 0 \Rightarrow v^T\Phi(t_1, \tau)[M_o(\tau), M_1(\tau), ..., M_m(\tau)] = 0$ $\forall \tau \geq t_o$ implying a contradiction:

  $$[M_o(\tau), M_1(\tau), ..., M_m(\tau)] < n, \quad \forall \tau \geq t_o$$

CT-LTV CTRB: Example 6.13

Th. 6-12: For CT-LTV system with $A(t), B(t) \in \mathcal{C}^{n-1}$, define $M_o(t) := B(t)$ and $M_{m+1}(t) := -A(t)M_m(t) + \frac{d}{dt}M_m(t)$. Then, it is CTRB on $[t_o, t_1]$ if, $\exists t_1 > t_o$ s.t.,

$$\text{rank}[M_o(t_1), M_1(t_1), ..., M_{n-1}(t_1)] = n$$

- This is the CT-LTV version of the well-known CT-LTI CTRB condition:

$$\text{rank}(\mathcal{C}) := \text{rank}[B, AB, A^2B, ..., A^{n-1}B] = n$$

- $A(t)M_m(t)$ and $\frac{d}{dt}M_m(t)$ respectively represents how the m-th propagated input can affect the state thru the dynamics $A(t)$ and thru the time-varying component of $B(t)$.
- (Ex 6.13:) Consider $\dot{x} = \begin{bmatrix} t & -1 & 0 \\ 0 & -t & t \\ 0 & 0 & t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u.$
  - $M_o(t) = B(t) = [0; 1; 1]$.
  - $M_1(t) = -A(t)M_o + \frac{d}{dt}M_o = [1; 0; -t]$.
  - $M_2(t) = -A(t)M_1 + \frac{d}{dt}M_1 = [-t; t^2; t^2 - 1]$.
  - rank $[M_oM_1M_2] = 3$ with determinant $t^2 + 1$ CTRB at every t.

Minimum Norm Control

- Recall $u(t) = -B^T(t)\Phi^T(t_1, t)W_r^{-1}(t_o, t_1)[\Phi(t_1, t_o)x_o - x_1]$, which is **minimum-norm control**, i.e., $\min_u \int_{t_o}^{t_1} u^T(t)u(t)dt$ with (x_o, x_1).
- Consider DT-LTV system with

$$x(t_1) = \Phi(k_1, k_o)x_o + L_r(k_o, k_1)U_{k_o:k_1-1}$$

- Minimum-norm control is then given by $\min_{U \in \Re^{p(k_1-k_o)}} \frac{1}{2}U^TU$, subject to $L_r(k_o, k_1)U = x_1 - \Phi(k_1, k_o)x_o =: x_d$, or, using **Lagrange-multiplier** $\lambda \in \Re^n$,

$$\boxed{\min_{U \in \Re^{p(k_1-k_o)}} L(U, \lambda) := \frac{1}{2}U^TU + \lambda^T[L_r(k_1, k_o)U - x_d]}$$

- Necessary conditions for optimality:

$$\frac{\partial L}{\partial U}\Big|_{U^*,\lambda^*} = 0 \quad \Rightarrow \quad U^* + L_r^T(k_o, k_1)\lambda^* = 0$$
$$\frac{\partial L}{\partial \lambda}\Big|_{U^*,\lambda^*} = 0 \quad \Rightarrow \quad L_r(k_o, k_1)U^* - x_d = 0$$

i.e., $\lambda^* = -W_r^{-1}(k_o, k_1)L_r(k_o, k_1)U^* = -W_r^{-1}(k_o, k_1)x_d$ and

$$\boxed{U^* = -L_r^T(k_o, k_1)W_r^{-1}(t_o, t_1)[\Phi(k_1, k_o)x_o - x_1]}$$

with the optimal cost $L(U^*, \lambda^*) = \frac{1}{2}x_d^TW_r^{-1}(k_o, k_1)x_d$.

Minimum Norm Control

- For DT-LTV system, the minimum-norm control $\min_{U\in\Re^{p(k_1-k_o)}} \frac{1}{2}U^TU$, subject to $L_r(k_o,k_1)U = x_1 - \Phi(k_1,k_o)x_o$ is

$$\boxed{U^* = -L_r^T(k_o,k_1)W_r^{-1}(t_o,t_1)[\Phi(k_1,k_o)x_o - x_1]}$$

  with the optimal cost $L^* = \frac{1}{2}x_d^TW_r^{-1}(k_o,k_1)x_d$.
- For CT-LTV system, the minimum-norm control $\min_u \int_{t_o}^{t_1} \frac{1}{2}u^T(t)u(t)dt$ subject to $\int_{t_o}^{t_1}\Phi(t_1,\tau)B(\tau)u(\tau)d\tau = x_1 - \Phi(t_1,t_o)x_o$ is

$$\boxed{u(t) = -B^T(t)\Phi^T(t_1,t)W_r^{-1}(t_o,t_1)[\Phi(t_1,t_o)x_o - x_1]}$$

  with the optimal cost $L^* = \frac{1}{2}x_d^TW_r^{-1}(t_o,t_1)x_d$.
- The larger λ_i of W_c is, the easier to control x along the direction of its eigenvector; if not CTRB, the cost of control becomes ∞.
- Longer the final time $t_1 \Rightarrow$ larger $W_r(t_o,t_1) \Rightarrow$ easier to attain $x_1 = x(t_1)$.
- From $W_r(t_o,t_1) = \int_{t_o}^{t_1}\Phi(t_1,\tau)B(\tau)B^T(\tau)\Phi^T(t_1,\tau)d\tau$,

$$\frac{\partial 2L^*}{\partial t_o} = x_d^T\frac{\partial W_r^{-1}}{\partial t_o}x_d = -x_d^TW_r^{-1}\frac{\partial W_r}{\partial t_o}W_r^{-1}x_d = x_d^TW_r^{-1}\Phi BB^T\Phi W_r^{-1}x_d \geq 0$$

  i.e., the later t_o is, the more difficult to control to attain $x_1 = x(t_1)$.

ENGINEERING

Controllability of CT-LTI System

Th. 6-1: For CT-LTI system, the following statements are equivalent:

1. (A,B) is CTRB.
2. CTRB gramian $W_r(t) = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau$ is non-singular $\forall t > 0$.
3. $\text{rank}(\mathcal{C}) := \text{rank}\begin{bmatrix} B & AB & A^2B & ... & A^{n-1}B \end{bmatrix} = n$, where $\mathcal{C}\in\Re^{n\times np}$.
4. $\begin{bmatrix} A-\lambda I & B \end{bmatrix} \in \Re^{n\times(n+p)}$ has full-row rank $\forall\lambda(A)$.
5. If A is Hurwitz, $AP + PA^T = -BB^T$ has unique PD solution $P = W_r(\infty)$.

- $(1 \leftrightarrow 2)$: Already shown for CT-LTV system in Th. 6-11.
- $(3 \rightarrow 2)$: Already shown in Th. 6-12.
- $(2 \rightarrow 3)$: Suppose $\text{rank}(\mathcal{C}) < n \Rightarrow \exists v\in\Re^n$ s.t.,

$$v^T\mathcal{C} = 0 \quad\Rightarrow\quad v^TA^kB = 0,\ \forall k = 0,1,...,n-1$$

  Further, using Cayley-Hamilton theorem (i.e., given CE of A, $\det(\lambda I - A) = \lambda^n + \alpha_1\lambda^{n-1} + ...\alpha_n$, $\Delta(A) = A^n\alpha_1A^{n-1} + ... + \alpha_nI = 0$),

$$v^Te^{At}B = v^T[I + At + \frac{A^tt^2}{2!} + ...]B = v^T[I + \beta_1(t)A + ... + \beta_{n-1}(t)A^{n-1}]B = 0$$

  implying $v^TW_r(t)v = \int_0^t ||v^Te^{A\tau}B||^2d\tau = 0$, i.e., $W_r(t)$ is singular $\forall t \geq 0$.

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CT-LTI System CTRB (cont'd)

Th. 6-1: For CT-LTI system, the following statements are equivalent:

1. (A, B) is CTRB.
2. CTRB gramian $W_r(t) = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau$ is non-singular $\forall t > 0$.
3. $\text{rank}(\mathcal{C}) := \text{rank}\begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} = n$, where $\mathcal{C} \in \Re^{n \times np}$.
4. $\begin{bmatrix} A - \lambda I & B \end{bmatrix} \in \Re^{n \times (n+p)}$ has full-row rank $\forall \lambda(A)$.
5. If A is Hurwitz, $AP + PA^T = -BB^T$ has unique PD solution $P = W_r(\infty)$.

- $(2 \leftrightarrow 5)$: For this, we need an extened version of Lyapunov theorem

  Th. 5-6: if A is Hurwitz, $A^TP + PA = -N$ has a unique solution $P := \int_0^\infty e^{A^Tt}Ne^{At}dt$ for **any** N.

  (Pf: Th. 5-6) Suppose not $\Rightarrow \exists$ another solution P' s.t., $A^T(P - P') + (P - P')A = 0 \Rightarrow e^{A^Tt}[A^T(P-P') + (P-P')A]e^{-A^Tt} = 0 \Rightarrow \frac{d}{dt}[e^{A^Tt}(P - P')e^{At}] = 0 \Rightarrow e^{A^Tt}(P - P')e^{At}\Big|_0^\infty = -(P - P') = 0$.
- $(2 \rightarrow 5)$: From Th. 5-6, a unique P exists w/ $P = W_r(\infty)$, which is well-defined (from A Hurwitz) and PD (from 2).
- $(5 \rightarrow 2)$: Solution $P = W_r(\infty)$ is PD $\Rightarrow W_r(t)$ is also PD $\forall t > 0$ from continuity of e^{At}.

ENGINEERING

CT-LTI System CTRB (cont'd)

Th. 6-1: For CT-LTI system, the following statements are equivalent:

1. (A, B) is CTRB.
2. CTRB gramian $W_r(t) = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau$ is non-singular $\forall t > 0$.
3. $\text{rank}(\mathcal{C}) := \text{rank}\begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} = n$, where $\mathcal{C} \in \Re^{n \times np}$.
4. $\begin{bmatrix} A - \lambda I & B \end{bmatrix} \in \Re^{n \times (n+p)}$ has full-row rank $\forall \lambda(A)$.
5. If A is Hurwitz, $AP + PA^T = -BB^T$ has unique PD solution $P = W_r(\infty)$.

- $(3 \rightarrow 4)$: Suppose not, i.e., $\exists q \in \Re^n$ s.t., for some $\lambda_i(A)$, q: left eigen-vector

  $$q^T[A - \lambda_i I \; B] = 0 \quad \Rightarrow \quad q^TA = \lambda_i q^T, \; q^TB = 0$$
  $$q^TA^2 = \lambda_i^2 q^T, \ldots, \; q^TA^k = \lambda_i^k q^T, q^TB = 0$$

  This then implies that

  $$q^T[B \; AB \; \ldots \; A^{n-1}B] = [q^TB \; \lambda_i q^TB \; \ldots \; \lambda_i^{n-1}q^TB] = 0$$

  that is, $\text{rank}(\mathcal{C}) < n$.
- $(4 \rightarrow 3)$: For this, we need the following theorems.

ENGINEERING

CTRB Invariance and Decomposition

Th. 6-2, 6.3: CT-LTI system CTRB is invariant under any similarity transform $x = P^{-1}\bar{x}$ with

$$\text{rank}(\mathcal{C}) = \text{rank}[B\ AB\ \dots\ A^{n-1}B] = \text{rank}(\bar{\mathcal{C}}) = \text{rank}[\bar{B}\ \bar{A}\bar{B}\ \dots\ \bar{A}^{n-1}\bar{B}]$$

(Pf) This is a direct consequence of: from $\bar{A} = PAP^{-1}$ and $\bar{B} = PB$ with full-rank $P \in \Re^{n\times n}$,

$$\text{rank}[\bar{B}\ \bar{A}\bar{B}\ \dots\ \bar{A}^{n-1}\bar{B}] = \text{rank}(P[B\ AB\ \dots A^{n-1}B]) = \text{rank}[B\ AB\ \dots A^{n-1}B]$$

Th. 6-6: Consider CT-LTI system $\dot{x} = Ax + Bu$, $y = Cx + Du$, with $\text{rank}(\mathcal{C}) = n_1 \le n$. Then, $\exists$ similarity-TF $x = P^{-1}\bar{x}$ s.t., the transformed system is given by

$$\begin{pmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{pmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + Du$$

where $\bar{x}_c \in \Re^{n_1}$ and $\bar{x}_{\bar{c}} \in \Re^{n-n_1}$. Further, **the CTRB reduced system**

$$\dot{\bar{x}}_c = \bar{A}_c\bar{x}_c + \bar{B}_c u, \quad \bar{y} = \bar{C}_c\bar{x}_c + Du$$

is CTRB and has the same TF as the original system.

Canonical Decomposition - I

Th. 6-6: For CT-LTI system with $\text{rank}(\mathcal{C}) = n_1 \le n$, $\exists\ x = P^{-1}\bar{x}$ s.t.,

$$\begin{pmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{pmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + Du$$

where $\dot{\bar{x}}_c = \bar{A}_c\bar{x}_c + \bar{B}_c u$, $\bar{y} = \bar{C}_c\bar{x}_c + Du$ is CTRB has the same original TF.

- $\text{rank}(\mathcal{C}) = n_1 \Rightarrow$ define q_i s.t., $\text{span}[B\ AB\ \dots\ A^{n-1}B] \approx \text{span}[q_1\ q_2\ ..\ q_{n_1}]$.
- Construct $P^{-1} = [q_1\ q_2\ \dots q_{n_1} \mid q_{n_1+1}\ \dots q_n] =: [Q_c \mid Q_{\bar{c}}]$, where $Q_c \in \Re^{n\times n_1}$ and $Q_{\bar{c}} \in \Re^{n\times(n-n_1)}$ s.t., $\text{rank}[Q_c \mid Q_{\bar{c}}] = n$.
- Q_c is then A-invariant, i.e., $\text{span}\{AQ_c\} \in \text{span}\{Q_c\}$ (from CH-theorem).
- Define $P = \begin{bmatrix} P_c \\ P_{\bar{c}} \end{bmatrix}$ s.t., $P_cQ_c = I, P_CQ_{\bar{c}} = 0$, $P_{\bar{c}}Q_c = 0$, $P_{\bar{c}}Q_{\bar{c}} = I$.
- Then, the transformed matrices are given by

$$PAP^{-1} = \begin{bmatrix} P_cAQ_c & P_cAQ_{\bar{c}} \\ P_{\bar{c}}AQ_c & P_{\bar{c}}AQ_{\bar{c}} \end{bmatrix}, \quad PB = \begin{bmatrix} P_cB \\ P_{\bar{c}}B \end{bmatrix}, \quad CP^{-1} = [CQ_c \mid CQ_{\bar{c}}]$$

  where $P_{\bar{c}}AQ_c = 0$ (from A-invariance of Q_c) and $P_{\bar{c}}B = 0$ (from $P_{\bar{c}}Q_c = 0$) $\Rightarrow$ transformed dynamics structure proved.

Canonical Decomposition - II

Th. 6-6: For CT-LTI system with rank$(\mathcal{C}) = n_1 \le n$, $\exists\, x = P^{-1}\bar{x}$ s.t.,

$$\begin{pmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{pmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + Du$$

where $\dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u$, $\bar{y} = \bar{C}_c \bar{x}_c + Du$ is CTRB has the same original TF.

- $\bar{A} = PAP^{-1}$, $\bar{B} = PB$, with $\bar{A}_c = P_c A Q_c$, $\bar{B}_c = P_c B$ and $\bar{C}_c = CQ_c$.
- CTRB of $(\bar{A}, \bar{B})$: using the structures of $\bar{A}$, $\bar{B}$,

$$\bar{\mathcal{C}} = \begin{bmatrix} \boxed{\begin{matrix} \bar{B}_c & \bar{A}_c\bar{B}_c & \dots & \bar{A}^{n_1-1}\bar{B}_c \end{matrix}} & \bar{A}^{n_1}\bar{B}_c & \dots & \bar{A}^{n-1}\bar{B}_c \\ 0 \quad\quad 0 \quad\quad \dots \quad\quad 0 & 0 & \dots & 0 \end{bmatrix}$$

  with rank$(\bar{\mathcal{C}}) = $ rank$(\bar{\mathcal{C}}_c) = n_1$ (due to CH-therorem).
- From $H(s) = C(sI - A)^{-1}B + D = CP^{-1}(sI - \bar{A})^{-1}PB = D$ and the structure of the transformed dynamics:

$$H(s) = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} (sI - \bar{A}_c)^{-1} & \star \\ 0 & (sI - \bar{A}_{\bar{c}})^{-1} \end{bmatrix} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + D$$
$$= \bar{C}_c(sI - \bar{A}_c)^{-1}\bar{B}_c + D$$

CT-LTI System CTRB (cont'd)

Th. 6-1: For CT-LTI system, the following statements are equivalent:

1. (A, B) is CTRB.
2. CTRB gramian $W_r(t) = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau$ is non-singular $\forall t > 0$.
3. rank$(\mathcal{C}) :=$ rank$\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n$, where $\mathcal{C} \in \Re^{n \times np}$.
4. $\begin{bmatrix} A - \lambda I & B \end{bmatrix} \in \Re^{n \times (n+p)}$ has full-row rank $\forall \lambda(A)$.
5. If A is Hurwitz, $AP + PA^T = -BB^T$ has unique PD solution $P = W_r(\infty)$.

- $(4 \to 3)$: Suppose rank$(\mathcal{C}) < n \Rightarrow$ from Th. 6-6, can transform with

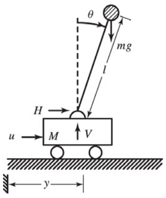
$$\bar{A} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

  Denote an eigenvalue of $\bar{A}_{\bar{c}}$ by λ_i w/ $q_i^T\bar{A}_{\bar{c}} = \lambda_i q_i^T \Rightarrow q_i^T(\bar{A}_{\bar{c}} - \lambda_i I) = 0$.
  Define $q := [0; q_i]$ (i.e., (λ_i, q_i) is an **un-CTRB mode**). Then,

$$q^T\left[\bar{A} - \lambda_i I \ \bar{B}\right] = \begin{pmatrix} 0 \\ q_i \end{pmatrix}^T \begin{bmatrix} \bar{A}_c - \lambda_i I & \bar{A}_{12} & \bar{B}_c \\ 0 & \bar{A}_{\bar{c}} - \lambda_i I & 0 \end{bmatrix} = 0$$

  i.e., item 4 violated for (A, B) w/ CTRB invariance btw (A, B) and $(\bar{A}, \bar{B})$.

Example: Cart-Pendulum



Linearized dynamics:

$$M\ddot{y} = u - mg\theta, \quad ML\ddot{\theta} = (m+M)g\theta - u$$

with 4 states $x = [y, \dot{y}, \theta, \dot{\theta}]$ and or 2-DOF with 1 control
1 input $u \Rightarrow$ under-actuated unstable system
$\Rightarrow$ can control all 4 states by one control?

- State-space representation: $\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{m+N}{ML}g & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{ML} \end{bmatrix} u.$
- CTRB space: $\mathcal{C} = \text{span} \begin{bmatrix} 0 & L & 0 & -mLg \\ L & 0 & -mLg & 0 \\ 0 & -1 & 0 & (m+M)g \\ -1 & 0 & (m+M)g & 0 \end{bmatrix}$

pendubot

- CTRB thanks to the gravity g if $M = 0$ (i.e., singular)
  - Gravity is necessary for CTRB by breaking symmetry
  - If not gravity $\Rightarrow$ not CTRB (pendubot, acrobot)

Example: CTRB Decomposition

- CT-LTI system: $\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y = [1\ 1\ 1]x.$
- Define CTRB transformation: $x = P^{-1}\bar{x} = [Q_c\ Q_{\bar{c}}]\bar{x}$.
- CTRB space: $Q_c = \texttt{orth}(\texttt{ctrb}(\texttt{A},\texttt{B}))$ (2-dim)
- unCTRB space: $Q_{nc} = \texttt{null}(\texttt{Q}_c')$ (1-dim)
- $\bar{A} = PAP^{-1}$, $\bar{B} = PB$, $\bar{C} = CP^{-1}$, $\bar{D} = D$.
- Not CTRB and not stabilizable

```
Qc =

   -0.6778    0.2016
   -0.2851   -0.9585
   -0.6778    0.2016

Qnc =

   -0.7071
    0.0000
    0.7071

Abar =

    1.3865    1.2992   -0.0000
   -0.1150    0.6135    0.0000
   -0.0000    0.0000    1.0000

Bbar =

   -0.2851   -1.3555
   -0.9585    0.4032
    0.0000         0

Cbar =

   -1.6406   -0.5553   -0.0000
```

- $H(s) = \begin{bmatrix} \frac{s^2-1}{s^3-3s^2+3s-1} & \frac{2s-2}{s^2-2s-1} \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s^2-2s+1} & \frac{2}{s-1} \end{bmatrix} = H_c(s)$
- Canceled unstable pole $s = 1$ constitutes the unCTRB/unstable state, whereas other two states can be controlled by u.
- Minimal realization with 2 states and OBSV $H_c = (A_c, B_c, C_c, D)$

Observability

Def 6.O1: A state-space dynamical system is said to be **observable** on $[t_o, t_1]$, if, for any initial condition $x(t_o)$, given the output $y(t)$ and the input $u(t)$ $\forall t \in [t_o, t_1]$, we can uniquely determine $x(t_o)$.

* Equivalent to say estimate $x(t)$ from $u(t), y(t)$, even if $x(t_o)$ unknown.

- Consider CT-LTV system:

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u$$

with the output response given by:

$$\overset{\text{known}}{y(t)} = C(t)\Phi(t,t_o)x_o + C(t)\int_{t_o}^{t} \Phi(t,\tau)B(\tau)\underset{\text{known}}{u(\tau)}d\tau + D(t)\underset{\text{known}}{u(t)}$$

- Since $y(t), u(t)$ known $\forall t \in [t_o, t_1]$, OBSV condition boils simply down to: given $\bar{y}(t) = y(t) - C(t)\int_{t_o}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$, $t \in [t_o, t_1]$, can we estimate x_o by observing $\bar{y}(t) = C(t)\Phi(t,t_o)x_o$?

- That is, equivalently, OBSV of the following simple system:

$$\dot{\bar{x}} = A(t)\bar{x}, \quad \bar{y}(t) = C(t)\Phi(t,t_o)\bar{x}(t_o)$$

Observability Map and Grammian

- Now, for CT-LTV system OBSV on $[t_o, t_1]$, we can consider OBSV of

$$\dot{x}(t) = A(t)x(t), \quad y(t) = C(t)x(t) = C(t)\Phi(t,t_o)x(t_o)$$

- **Observability map:**

$$L_o(t_o,t)(\cdot) : x_o \mapsto y(t) = C(t)\Phi(t,t_o)x_o, \quad t \in [t_o, t_1]$$

which is a linear map w.r.t. the argument x_o.

- For OBSV, this obserability map $L_o(t_o, t)$ should not possess an invariant **null-space**, since, if not, $\exists x_{\bar{o}} \in \Re^n$ s.t.,

$$y(t) = L_o(t_o,t)(x_o + \alpha x_{\bar{o}}) = C(t)\Phi(t,t_o)(x_o + \alpha x_{\bar{o}}), \quad \forall t \in [t_o, t_1]$$

where $x_{\bar{o}}$ at a certain t' could be ok with time-varying information gathering (e.g., rotating $C(t)\Phi(t,t_o)$: cf. persistency of excitation).

- Recall $\mathcal{N}(L_o) = \mathcal{N}(L_o^T L_o)$ and, similarly for $W_r(t_o, t_1)$, we define **observability grammian:**

$$\boxed{W_o(t_o,t_1) := \int_{t_o}^{t_1} \Phi^T(\tau,t_o)C^T(\tau)C(\tau)\Phi(\tau,t_o)d\tau}$$

Observability of CT-LTV Systems

- **Th. 6-O11:** CT-LTV system $\dot{x} = A(t)x$, $y = C(t)x$ is **observable** at t_o, iff $\exists t_1 > t_o$ s.t., the following OBSV grammian $W_o(t_o, t_1)$ is non-singular:

$$W_o(t_o, t_1) = \int_{t_o}^{t_1} \Phi(\tau, t_o) C^T(\tau) C(\tau) \Phi(\tau, t_o) d\tau$$

- (Proof) From $y = C(t)x = C(t)\Phi(t, t_o)x_o$, we have

$$\int_{t_o}^{t_1} \Phi^T(\tau, t_o) C^T(\tau) y(\tau) d\tau = \int_{t_o}^{t_1} \Phi^T(\tau, t_o) C^T(\tau) C(t) \Phi(\tau, t_o) d\tau \cdot x_o$$

thus, if the above condition holds, x_o is uniquely determined by

$$x_o = W_o^{-1}(t_o, t_1) \int_{t_o}^{t_1} \Phi^T(\tau, t_o) C^T(\tau) y(\tau) d\tau$$

- **Th. 6-O12:** Define $N_{m+1}(t) := N_m(t)A(t) + \frac{d}{dt}N_m(t)$ with $N_o(t) := C(t)$. Then, CT-LTV system with $A(t), C(t) \in \mathcal{C}^{n-1}$ is OBSV at t_o if, $\exists t_1 > t_o$ s.t.,

$$\text{rank}[N_o(t_1); N_1(t_1); ..., N_{n-1}(t_1)] = n$$

Observability of CT-LTI System

Th. 6-1: For CT-LTI system, the following statements are equivalent:

1. (A, C) is OBSV.
2. OBSV grammian $W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$ is non-singular $\forall t > 0$.
3. $\text{rank}(\mathcal{O}) := \text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$, where $\mathcal{O} \in \Re^{nm \times n}$.
4. $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \in \Re^{(n+m)\times n}$ has full-column rank $\forall \lambda(A)$.
5. If A is Hurwitz, $A^T P + PA = -C^T C$ has unique PD solution $P = W_o(\infty)$.

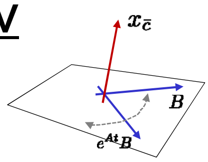
Th. 6-5 (Duality of CTRB & OBSV): For the CT-LTI system,

(A, B) CTRB $\Leftrightarrow$ (A^T, B^T) OBSV; (A, C) OBSV $\Leftrightarrow$ (A^T, C^T) CTRB

- A direct consequence of

$$W_r(0, t) = \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau, \quad W_o(0, t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

Geometry of CTRB and OBSV



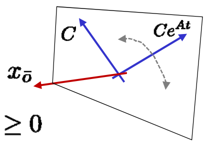
- For CT-LTI CTRB: $W_r(t)$ is non-singular iff $\text{rank}(\mathcal{C}) = n$.
- For CT-LTI OBSV: $W_o(t)$ is non-singular iff $\text{rank}(\mathcal{O}) = n$.
- If not CTRB, $\exists\, x_{\bar c} \in \Re^n$ s.t.,

$$x_{\bar c}^T \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = 0 \;\Rightarrow\; x_{\bar c}^T A^k B = 0 \;\Rightarrow\; x_{\bar c}^T e^{At} B = 0,\ \forall \tau \geq 0$$

$$x_{\bar c}^T W_r(t) x_{\bar c} = \int_0^t x_{\bar c}^T e^{A\tau} B B^T e^{A^T\tau} x_{\bar c} d\tau = \int_0^t ||x_{\bar c}^T e^{A\tau} B||^2 d\tau = 0$$

i.e., control successively generated by $e^{At}B$ **always orthogonal to** $x_{\bar c}$.

- If not OBSV, $\exists\, x_{\bar o} \in \Re^n$ s.t.,



$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_{\bar o} = 0 \;\Rightarrow\; CA^k x_{\bar o} = 0 \;\Rightarrow Ce^{At} x_{\bar o} = 0,\ \forall \tau \geq 0$$

$$x_{\bar o}^T W_o(t) x_{\bar o} = \int_0^t x_{\bar o}^T e^{A^T\tau} C^T C e^{A\tau} x_{\bar o} d\tau = \int_0^t ||Ce^{A\tau} x_{\bar o}||^2 d\tau = 0$$

i.e., null-space successively eliminated by Ce^{At} **always contains** $x_{\bar o}$.

OBSV Decomposition - I

Th. 6-O6: For CT-LTI system with $\text{rank}(\mathcal{O}) = n_2 \leq n$, $\exists\, x = T^{-1}\bar x$ s.t.,

$$\begin{pmatrix} \dot{\bar x}_o \\ \dot{\bar x}_{\bar o} \end{pmatrix} = \begin{bmatrix} \bar A_o & 0 \\ \bar A_{21} & \bar A_{\bar o} \end{bmatrix} \begin{pmatrix} \bar x_o \\ \bar x_{\bar o} \end{pmatrix} + \begin{bmatrix} \bar B_o \\ \bar B_{\bar o} \end{bmatrix} u, \quad y = \begin{bmatrix} \bar C_o & 0 \end{bmatrix} \begin{pmatrix} \bar x_o \\ \bar x_{\bar o} \end{pmatrix} + Du$$

where $\dot{\bar x}_o = \bar A_o \bar x_o + \bar B_o u$, $\bar y = \bar C_o \bar x_o + Du$ is OBSV, has the same original TF.

- $\text{rank}(\mathcal{O}) = \text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n_2 \Rightarrow$ construct $Q_{\bar o} = \text{null}(\mathcal{O}) \in \Re^{n \times (n-n_2)}$.
- Construct $T^{-1} = [Q_{\bar o} \mid Q_o]$, where $Q_o \in \Re^{n \times n_2}$ s.t., $\text{rank}[Q_{\bar o} \mid Q_o] = n$.
- Null-space of $\mathcal{O}$ is A-invariant, e.g., $Q_{\bar o} \in \text{null}(\mathcal{O}) \Rightarrow AQ_{\bar o} \in \text{null}(\mathcal{O})$.
- Define $T = \begin{bmatrix} P_{\bar o} \\ P_o \end{bmatrix}$ s.t., $P_{\bar o} Q_{\bar o} = I, P_{\bar o} Q_o = 0,\ P_o Q_{\bar o} = 0,\ P_o Q_o = I$.
- Then, the transformed matrices are given by

$$TAT^{-1} = \begin{bmatrix} P_{\bar o} A Q_{\bar o} & P_{\bar o} A Q_o \\ P_o A Q_{\bar o} & P_o A Q_o \end{bmatrix}, \quad TB = \begin{bmatrix} P_{\bar o} B \\ P_o B \end{bmatrix}, \quad CT^{-1} = [CQ_{\bar o} \mid CQ_o]$$

OBSV Decomposition - II

Th. 6-O6: For CT-LTI system with $\text{rank}(\mathcal{O}) = n_2 \le n$, $\exists\, x = T^{-1}\bar{x}$ s.t.,

$$\begin{pmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{pmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u, \quad y = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} + Du$$

where $\dot{\bar{x}}_o = \bar{A}_o \bar{x}_o + \bar{B}_o u$, $\bar{y} = \bar{C}_o \bar{x}_o + Du$ is OBSV, has the same original TF.

- OBSV of $(\bar{A}, \bar{C})$: using the structures of $\bar{A}$, $\bar{C}$,

$$\bar{\mathcal{O}} = \begin{bmatrix} \boxed{\begin{matrix} \bar{C}_o \\ \vdots \\ \bar{C}_o \bar{A}_o^{n_2-1} \end{matrix}} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \vdots & \vdots \\ \bar{C}_o \bar{A}_o^{n-1} & 0 \end{bmatrix} \quad \Rightarrow \quad \text{rank}(\bar{\mathcal{O}}) = \text{rank}(\bar{\mathcal{O}}_c) = \text{rank}(\mathcal{O}) = n_2$$

- From $H(s) = C(sI - A)^{-1}B + D = CT^{-1}(sI - \bar{A})^{-1}TB + D$ and the structure of the transformed dynamics:

$$H(s) = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} (sI - \bar{A}_o)^{-1} & 0 \\ \star & (sI - \bar{A}_{\bar{o}})^{-1} \end{bmatrix} \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} + D$$
$$= \bar{C}_o (sI - \bar{A}_o)^{-1} \bar{B}_o + D$$

Example: OBSV Decomposition

- CT-LTI system: $\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y = [1\ 1\ 1]x.$
- Check OBSV: $\texttt{rank(obsv((A,C))} = 2$.
- OBSV sub-space: $Q_o = \texttt{orth(obsv((A,C))'} = \text{span}(\mathcal{O}^T)$.
- non-OBSV sub-space: $Q_{no} = \texttt{null(Q}_o'\texttt{)}$ (1-dim).
- $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$.
- Not OBSVB and not detectable
- $H(s) = \begin{bmatrix} \frac{s^2-1}{s^3-3s^2+3s-1} \\ \frac{2s-2}{s^2-2s-1} \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s^2-2s+1} \\ \frac{2}{s-1} \end{bmatrix} = H_c(s)$
- Canceled unstable pole $s = 1$ constitutes the unstable/unOBSV mode, whereas other two states can be observed from y, u.
- Minimal realization with 2 states and CTRB $H_o = (A_o, B_o, C_o, D)$
- Note that $Q_o = Q_c$ and $Q_{\bar{o}} = Q_{\bar{c}}$, i.e., CTRB/OBSV subspaces concident.

```
Qo =

  -0.2492    0.6617
  -0.9358   -0.3525
  -0.2492    0.6617

Qno =

  -0.7071
   0.0000
   0.7071

Qc =

  -0.6778    0.2016
  -0.2851   -0.9585
  -0.6778    0.2016

Qnc =

  -0.7071
   0.0000
   0.7071

Abar_o =

   1.4665    0.1757   -0.0000
  -1.2385    0.5335    0.0000
  -0.0000   -0.0000    1.0000

Bbar_o =

  -0.9358   -0.4985
  -0.3525    1.3235
  -0.0000    0.0000

Cbar_o =

  -1.4343    0.9710    0.0000
```

Kalman Decomposition

Th. 6-7: For CT-LTI system $\dot{x} = Ax + Bu$, $y = Cx + Du$, $\exists P \in \Re^{n\times n}$ s.t., with $x = P^{-1}\bar{x}$, it can be transformed to:

$$\begin{pmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{pmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{pmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{pmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{bmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} + Du$$

- Given $\mathcal{R}(\mathcal{C})$ and $\mathcal{N}(\mathcal{O})$,

$$P^{-1} = \begin{bmatrix} Q_{co} & \boxed{Q_{c\bar{o}}} & Q_{\bar{c}o} & \boxed{Q_{\bar{c}\bar{o}}} \end{bmatrix}$$

  - $Q_{c\bar{o}} = \mathcal{R}(\mathcal{C}) \cap \mathcal{N}(\mathcal{O}) \rightarrow$ CTRB/unOBSV mode
  - $Q_{co} = \mathcal{R}(\mathcal{C}) \setminus Q_{c\bar{o}} \rightarrow$ CTRB/OBSV mode
  - $Q_{\bar{c}\bar{o}} = \mathcal{N}(\mathcal{O}) \setminus Q_{c\bar{o}} \rightarrow$ unOBSV/CTRB mode
  - $Q_{\bar{c}o} = \Re^n \setminus (Q_{co} \cup Q_{c\bar{o}} \cup Q_{\bar{c}\bar{o}}) \rightarrow$ unCTRB/OBSV mode.

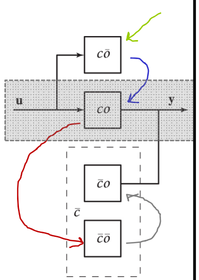
- $\bar{x}_{co}, \bar{x}_{c\bar{o}}, \bar{x}_{\bar{c}o}, \bar{x}_{\bar{c}\bar{o}}$ respectively specify dynamics of their corresponding modes.

Kalman Decomposition

Th. 6-7: For CT-LTI system $\dot{x} = Ax + Bu$, $y = Cx + Du$, $\exists P \in \Re^{n\times n}$ s.t., with $x = P^{-1}\bar{x}$, it can be transformed to:

$$\begin{pmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{pmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{pmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{pmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{bmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} + Du$$



- Given $\mathcal{R}(\mathcal{C})$ and $\mathcal{N}(\mathcal{O})$, $P^{-1} = \begin{bmatrix} Q_{co} & Q_{c\bar{o}} & Q_{\bar{c}o} & Q_{\bar{c}\bar{o}} \end{bmatrix}$
- $\bar{x}_{co}, \bar{x}_{c\bar{o}}, \bar{x}_{\bar{c}o}, \bar{x}_{\bar{c}\bar{o}}$ respectively specify dynamics of these modes.
- CTRB/OBSV $(\bar{A}_{co}, \bar{B}_{co}, \bar{C}_{co}, D)$ produces the same TF matrix, i.e.,

$$H(s) = C(sI - A)^{-1}B + D = \bar{C}_{co}(sI - \bar{A}_{co})^{-1}\bar{B}_{co} + D$$

  i.e., minimal realization of $H(s)$.

- If unCTRB or unOBSV modes are unstable, need to change system stucture itself (e.g., change or add actuators and sensors).

Kalman Decomposition: Example

$$\dot{x} = \begin{bmatrix} 0 & -1 & -1 & -3 \\ -9 & 2 & 6 & 9 \\ 4 & -2 & -5 & -4 \\ 3 & -1 & -2 & -4 \end{bmatrix} \begin{pmatrix} -1 \\ 6 \\ -3 \\ -2 \end{pmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix}$$

RC =

-0.2291	-0.7286
0.8630	0.0718
-0.3463	0.6807
-0.2877	-0.0239

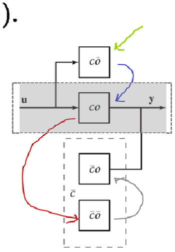
NO =

0.0038	-0.4082
-1.0000	-0.0093
0.0076	-0.8165
0.0038	-0.4082

- $\lambda(A) = \{-1, -2, -2 \pm j\}$.
- $H(s) = \frac{s^3+5s^2+8s+4}{s^4+7s^3+19s^2+23s+10}$, $p = \{-1, -2, -2 \pm j\}$, $z = \{-1, -2, -2\}$.
- $\texttt{rank(ctrb(A,B))} = 2$ and $\texttt{rank(obsv(A,C))} = 2$.
- $\mathcal{R}(\mathcal{C}) = \texttt{orth(ctrb(A,B)}$ and $\mathcal{N}(\mathcal{O}) = \texttt{orth(null(obsv(A,C)'))}$.
- $P^{-1} = \begin{bmatrix} Q_{co} & Q_{c\bar{o}} & Q_{\bar{c}o} & Q_{\bar{c}\bar{o}} \end{bmatrix}$
- $Q_{c\bar{o}} = \mathcal{R}(\mathcal{C}) \cap \mathcal{N}(\mathcal{O}) \Rightarrow Q_{c\bar{o}} = \mathcal{R}(\mathcal{C})[a;b] = \mathcal{N}(\mathcal{O})[c;d]$, i.e.,

$$[\mathcal{R}(\mathcal{C}) \; \mathcal{N}(\mathcal{O})][a;b;c;d] = 0$$

yet, $\text{null}([\mathcal{R}(\mathcal{C}) \; \mathcal{N}(\mathcal{O})]) = \emptyset \Rightarrow Q_{c\bar{o}} = \emptyset$.

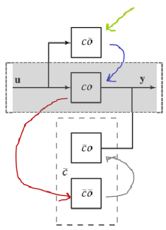


- $Q_{c\bar{o}} \cup Q_{co} = \mathcal{R}(\mathcal{C}) \Rightarrow Q_{co} = \mathcal{R}(\mathcal{C})$, i.e., all the CTRB modes are also OBSV (some OBSV modes may still be unCTRB).

Kalman Decomposition: Example

$$\dot{x} = \begin{bmatrix} 0 & -1 & -1 & -3 \\ -9 & 2 & 6 & 9 \\ 4 & -2 & -5 & -4 \\ 3 & -1 & -2 & -4 \end{bmatrix} \begin{pmatrix} -1 \\ 6 \\ -3 \\ -2 \end{pmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix}$$

- $P^{-1} = \begin{bmatrix} Q_{co} & Q_{c\bar{o}} & Q_{\bar{c}o} & Q_{\bar{c}\bar{o}} \end{bmatrix}$



- $Q_{c\bar{o}} = \emptyset \Rightarrow Q_{co} = \mathcal{R}(\mathcal{C})$.
- $\mathcal{N}(\mathcal{O}) = Q_{c\bar{o}} \cup Q_{\bar{c}\bar{o}} \Rightarrow Q_{\bar{c}\bar{o}} = \mathcal{N}(\mathcal{O})$.
- $\text{null}[\mathcal{R}(\mathcal{C}) \; \mathcal{N}(\mathcal{O})] = \emptyset \Rightarrow \text{span}[\mathcal{R}(\mathcal{C}) \; \mathcal{N}(\mathcal{O})] = \Re^4 \Rightarrow Q_{\bar{c}o} = \emptyset$.
- $P^{-1} = \begin{bmatrix} Q_{co} & Q_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \mathcal{R}(\mathcal{C}) & \mathcal{N}(\mathcal{O}) \end{bmatrix}$
- $H(s) = \bar{C}_{co}[sI - \bar{A}_{co}]^{-1}\bar{B}_{co} + D = \frac{-17.5s - 35.11}{s^2+4s+5}$.

Abar =

-1.0053	12.4965	-0.0000	-0.0000
-0.1592	-2.9947	0.0000	0.0000
0.0000	0.0000	1.9767	4.9635
-0.0000	-0.0000	-2.3850	-4.9767

Bbar =

7.0216
-0.8352
0.0000
-0.0000

Cbar =

0.0586	-0.7047	0.0000	0.0000

- Recall
  - $\lambda(A) = \{-1, -2, -2 \pm j\}$.
  - $H(s) = \frac{s^3+5s^2+8s+4}{s^4+7s^3+19s^2+23s+10}$, $p = \{-1, -2, -2 \pm j\}$, $z = \{-1, -2, -2\}$.

CTRB/OBSV Canonical Forms

- So far, we have witnessed that coprimness of $H(s) = \frac{N(s)}{D(s)}$ should have something with its minimal CTRB/OBSV state-space realization (A, B, C, D).
- Consider $H(s) = \frac{s^2-1}{s^3-3s^2+3s-1} (= \frac{s+1}{s^2-2s+1})$.
- CTRB canonical form of $H(s)$:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} x$$

which is CTRB, yet, not OBSV. However, CTRB canonical form of co-prime form is both CTRB and OBSV with $H(s) = \frac{s+1}{s^2-2s+1}$.

- OBSV canonical form of $H(s)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

which is OBSV, yet, not CTRB. However, CTRB canonical form of co-prime form is both CTRB and OBSV with $H(s) = \frac{s+1}{s^2-2s+1}$.

Coprime Fraction and Minimal Realization

- **Th. 7-1:** Consider $H(s) = \frac{N(s)}{D(s)}$. Then, its CTRB (or OBSV) canonical form is also OBSV (or CTRB) iff $N(s)$ and $D(s)$ are coprime polynomials.

  - Now, define $\deg(H(s))$ to be the order of the denomenator of $H(s)$ after removing common factor among $N(s)$ and $D(s)$ (i.e., order of coprime $H(s)$).

- **Th. 7-2:** The following statements are equivalent:
  - (A, B, C, D) is a minimal realization of $H(s)$.
  - (A, B, C, D) is CTRB and OBSV.
  - $\dim(A) = \deg(H(s))$.

  - Consider $H(s) = \frac{s^2-1}{4(s^3-1)} = \frac{s+1}{4(s^2+s+1)}$ with $\deg(H(s)) = 2$, Then, **any** 2-dimensional realization (A, B, C, D) is a minimal realization, which will also be CTRB and OBSV.

  - If (A, B, C, D) is CTRB and OBSV, poles of $H(s)$ = eigenvalues of A, implying that

$$\text{internal stability} \quad \Leftrightarrow \quad \text{BIBO stability}$$

Recall that, in general, internal stability $\Rightarrow$ BIBO stability.

Equivalence of Minimal Realizations

Th. 7-3M: Any minimal realizations of $H(s)$ are equivalent.

- Recall that, with $\bar{A} = PAP^{-1}$, $\bar{B} = PB$ and $\bar{C} = CP^{-1}$, $\bar{D} = D$,

$$\bar{\mathcal{C}} = P\mathcal{C}, \quad \bar{\mathcal{O}} = \mathcal{O}P^{-1}$$

- Note also that $\mathcal{O}A\mathcal{C} = \bar{\mathcal{O}}PAP^{-1}\bar{\mathcal{C}} = \bar{\mathcal{O}}\bar{A}\bar{\mathcal{C}}$.
- We then have

$$\bar{A} = \bar{\mathcal{O}}^{-1}\mathcal{O} \cdot A \cdot \mathcal{C}\bar{\mathcal{C}}^{-1}$$

where, with all the matrices full-rank,

$$\mathcal{C}\bar{\mathcal{C}}^{-1} \cdot \bar{\mathcal{O}}^{-1}\mathcal{O} = \mathcal{C}\mathcal{C}^{-1}P^{-1} \cdot P\mathcal{O}^{-1}\mathcal{O} = I$$

implying that we can choose the similarity transform

$$P = \bar{\mathcal{O}}^{-1}\mathcal{O}, \quad \text{with} \quad P^{-1} = \mathcal{C}\bar{\mathcal{C}}^{-1}$$

(Ex) $H(s) = \frac{s+1}{s^2-2s+1}$. CTRB form: $\left(\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1\ 1], 0\right)$; OBSV form: $\left(\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, [0\ 1], 0\right) \Rightarrow P = \mathcal{O}_o^{-1}\mathcal{O}_c = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$ w/ $PA_cP^{-1} = A_o$.

Minimal and Characteristic Polynomials

- Consider two matrices with Jordan form:

$$A_1 = T_1 \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} T_1^{-1}, \quad A_2 = T_2 \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} T_2^{-1}$$

- **Characteristic polynomial:** with the multiplicity of λ_i,

$$\boxed{\Delta_1(\lambda) = \Delta_2(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)^2(\lambda - \lambda_3)}$$

which specifies the dimension of state vector, yet, can't tell stability of A.
- **Minimal polynomial:** with the index of λ_i,

$$\boxed{\psi_1(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)^2(\lambda - \lambda_3) \neq \psi_2(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)}$$

which specifies stability of A, but not dimension of A.

- **Th. 7-M1:** Suppose (A, B, C, D) is minimal realization of $H(s)$. Then,
  - Monic least common denominator of all entries of $H(s) = \psi_A(s)$.
  - Monic least common denominator of all minors of $H(s) = \Delta_A(s)$

MIMO Minimal Realization

- **Th. 7-M1:** Suppose (A,B,C,D) is minimal realization of $H(s)$. Then,
  - Monic least common denominator of all entries of $H(s) = \psi_A(s)$.
  - Monic least common denominator of all minors of $H(s) = \Delta_A(s)$
- (Ex 7.5) $H(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \\ \frac{-1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s} \end{bmatrix} \Rightarrow \psi_A(s) = s(s+1)(s+2)(s+3)$. Also, $m_1(s) = \frac{1}{(s+1)(s+2)}$, $m_2(s) = \frac{s+4}{(s+1)(s+3)}$, $m_3(s) = \frac{3}{s(s+1)(s+2)(s+3)}$ $\Rightarrow \Delta(s) = s(s+1)(s+2)(s+3) \Rightarrow$ four states necessary with $A \in \Re^{4\times 4}$ having all distinct eigenvalues $\{0,-1,-2,-3\}$.
- (Ex 7.4) $H_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \Rightarrow \psi(s) = s+1 = \Delta(s) \Rightarrow$ only one state is necessary with $A=-1$ ($y = y_1 = y_2 = \frac{1}{s+1}(u_1+u_2)$). $H_2(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \Rightarrow \psi(s) = s+1$, $\Delta(s) = (s+1)^2 \Rightarrow A \in \Re^{2\times 2}$ with repeated yet non-deficient $\lambda_1 = \lambda_2 = -1$.
- **Th. 7-M2:** (A,B,C,D) is a minimal realization of proper rational $H(s) \in \mathcal{C}^{m\times p}$ iff
$$\dim A = \deg(H(s))$$
where $\deg(H(s))$ = order of LCD of all minors of $H(s)$.

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Minimal Realization: Example

- **Th. 7-M2:** (A,B,C,D) is a minimal realization of proper rational $H(s) \in \mathcal{C}^{m\times p}$ iff
$$\dim A = \deg(H(s))$$
where $\deg(H(s))$ = order of LCD of all minors of $H(s)$.
- (Ex 7.6):
$$H(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{1}{(s+2)^2} \end{bmatrix}$$
- Using $(A,B,C,D) = \texttt{ssdata}(H)$:

```
A =

   -0.5000         0         0         0         0         0
         0   -2.5000   -1.0000         0         0         0
         0    1.0000         0         0         0         0
         0         0         0   -2.0000         0         0
         0         0         0         0   -4.0000   -2.0000
         0         0         0         0    2.0000         0

B =

    2.0000         0
    0.5000         0
         0         0
         0    2.0000
         0    0.5000
         0         0

C =

   -3.0000         0         0    1.5000         0         0
         0         0    1.0000         0         0    1.0000

D_r =

     2     0
     0     0
```

- Using $(A_r, B_r, C_r, D_r) = \texttt{minreal}(sys_{org})$:

```
A_r =

   -0.7192    0.4546   -0.2440
    0.5904   -1.7353    0.5292
   -0.0508   -0.0228   -2.0455

B_r =

    1.9704    0.1037
    0.4723   -0.1052
   -0.0406    2.0476

C_r =

   -2.7962   -0.9041    1.5603
   -0.1971    0.8267    0.0525

D_r =

     2     0
     0     0
```


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Balanced Model Reduction

- Now, suppose that all modes are stablizable and detectable. Then, it would be possible to use only the CTRB/OBSV part of Kalman decomposition to describe the system IO-behavior.
- Even among these CTRB/OBSV modes, some may be easy to control/observe (i.e., large $x^T W_c x$ and $x^T W_o x$), yet, others may be very difficult to do so (i.e., non-zero, yet, very small $x^T W_c x$ and $x^T W_o x$).
- If some stabilizable/detectable modes are very difficult to control/observe (i.e., almost unCTRB/unOBSV), their omission will not affect the system's IO-behavior that much $\Rightarrow$ **model reduction**.
- Model reduction is particularly useful/necessary to reduce the model obtained by spatial discretization of PDE systems (e.g., deformable object).
- It would then nice if we can "realign" the state vectors so that some modes are easy to control/observe while other modes are difficult to control/observe, i.e., define similarity-TF s.t.,

$$\boxed{\bar{W}_c = \bar{W}_o = \Sigma = \text{diag}[\sigma_1, \sigma_2, ...\sigma_n]}$$

and retain only modes with large enough σ_i. In this case, realization has "balanced" $\bar{W}_c, \bar{W}_o \Rightarrow$ **balanced model reduction**.

Equivalence of WcWo

Th. 7-5: (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ are minimal realization (i.e., also equivalent w/ $x = P^{-1}\bar{x}$). Then, $W_c \cdot W_o$ and $\bar{W}_c \cdot \bar{W}_o$ have the same eigenvalues and futher all of them are real and non-negative.

- For $(\bar{A}, \bar{C})$, we can compute $\bar{W}_c$ from $\bar{A}\bar{W}_c + \bar{W}_c\bar{A}^T = -\bar{B}\bar{B}^T$.
- Using $\bar{A} = PAP^{-1}$ and $\bar{B} = PB$, we can then obtain:

$$AP^{-1}\bar{W}_c P^{-T} + P^{-1}\bar{W}_c P^{-T} A^T = -BB^T$$

  implying that $W_c = P^{-1}\bar{W}_c P^{-T}$. Similarly, $W_o = P^T \bar{W}_o P$.
- Thus, $W_c W_o = P^{-1}\bar{W}_c \bar{W}_o P \Rightarrow W_c W_o$ and $\bar{W}_c \bar{W}_o$ share same eigenvalues (rather not suprising...).
- $W_c \succ 0 \Rightarrow W_c = Q^T \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}} Q = R^T R$ with $\Lambda = \text{diag}[\lambda_1, ..., \lambda_n]$, $\lambda_i > 0$.
- Further, using $\det(AB) = \det(A)\det(B)$ with $\det(A^{-1}) = 1/\det(A)$,

$$\begin{aligned}\det(\lambda I - W_c W_o) &= \det(\lambda R^T R^{-T} - R^T R W_o) \\ &= \det(R^T)(\lambda I - R W_o R^T)\det(R^{-T}) = \det(\lambda I - R W_o R^T)\end{aligned}$$

  implying that $\lambda_i(W_c W_o) > 0$ (rather suprising!).

Balanced Realization

Th. 7-6: For any minimal realization (A, B, C), there exists an equivalent realization $(\bar{A}, \bar{B}, \bar{C})$ s.t.,

$$\bar{W}_c = \bar{W}_o = \Sigma = \text{diag}[\sigma_1, \sigma_2, ...\sigma_n]$$

- From Th. 7-5, since $\{\lambda_i(RW_oR^T)\} = \{\lambda_i(\bar{W}_c\bar{W}_o)\}$, if $\bar{W}_c = \bar{W}_o = \Sigma$,
$$RW_oR^T = U\Sigma^2U^T, \quad U^TU = I$$
- Further, using $W_c = R^TR$ and $\bar{W}_c\bar{W}_o = PW_cW_oP^{-1}$,
$$\Sigma^2 = \bar{W}_c\bar{W}_o = PW_cW_oP^{-1} = PR^TU \cdot \Sigma^2 \cdot U^TR^{-T}P^{-1}$$
- Choose $P^{-1} = R^TU\Sigma^{-\frac{1}{2}}$ with $P = \Sigma^{\frac{1}{2}}U^TR^{-T}$ $\Rightarrow$ $\bar{W}_c = PW_cP^{-1} = \Sigma$ and $\bar{W}_c = PW_cP^{-1} = \Sigma$.
- We can also decompose the system s.t.,
$$\begin{pmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + Du$$
- Reduced system: $\dot{\bar{x}}_1 = A_{11}\bar{x}_1 + B_1u$, $y = C_1\bar{x}_1 + Du$, whose steady-state behavior may be altered (e.g., different dc-gain).
- Inject original steady-state behavior: $\bar{x}_2 = -A_{22}^{-1}(A_{21}\bar{x}_1 + B_2u) \Rightarrow$

Balanced Reduction

Th. 7-6: For any minimal realization (A, B, C), there exists an equivalent realization $(\bar{A}, \bar{B}, \bar{C})$ s.t.,

$$\bar{W}_c = \bar{W}_o = \Sigma = \text{diag}[\sigma_1, \sigma_2, ...\sigma_n]$$

- $P^{-1} = R^TU\Sigma^{-\frac{1}{2}}$ with $\bar{W}_c = \bar{W}_o = \Sigma$ and
$$\begin{pmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + Du$$
- Reduced system with steady-state behavior possibly altered:
$$\dot{\bar{x}}_1 = A_{11}\bar{x}_1 + B_1u, \quad y = C_1\bar{x}_1 + Du$$
- To retain original steady-state behavior, using $\bar{x}_2 = -A_{22}^{-1}(A_{21}\bar{x}_1 + B_2u)$,
$$\dot{\bar{x}}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\bar{x}_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u$$
$$y = (C_1 - C_2A_{22}^{-1}A_{21})\bar{x}_1 + (D - C_2A_{22}^{-1}B_2)u$$

Balanced Reduction: Example

```
eig(A)

sys = ss(A,B,C,D);

W_c = gram(sys, 'c');
W_o = gram(sys, 'o');

[U_c,S_c,V_c] = svd(W_c);
R = sqrtm(S_c)*U_c';
[U,S2,V] = svd(R*W_o*R');
mSigma = sqrtm(S2);
```

A =

-1	-1	-1	-3	-2	4
9	-3	6	-9	3	0
4	-1	-5	-4	-6	0
3	1	-2	-4	-1	1
-3	-1	2	4	-1	1
-6	-2	4	5	2	-1

B =

-2	4
3	0
-6	0
-1	1
-1	1
2	-1

C =

1	-2	-3	0	-5	-1
3	1	-2	-4	-1	1

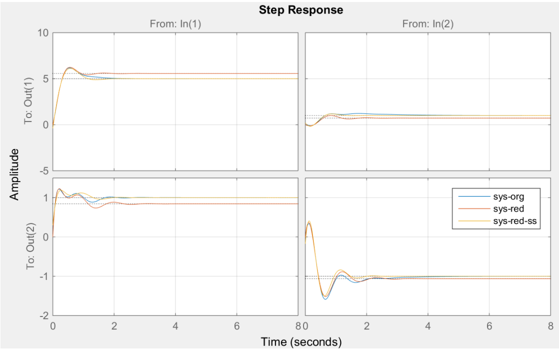
D =

0	0
0	0

```
>> eig(A)

ans =

  -2.0097 + 6.2403i
  -2.0097 - 6.2403i
  -4.2100 + 4.1231i
  -4.2100 - 4.1231i
  -1.0824 + 0.0000i
  -1.4782 + 0.0000i
```



mSigma =

3.9784	0	0	0	0	0
0	1.9285	0	0	0	0
0	0	1.3993	0	0	0
0	0	0	0.9521	0	0
0	0	0	0	0.4028	0
0	0	0	0	0	0.1857