

# Ch. 1. Basic concepts of Fluid Flow

## CFD - Computational Fluid Dynamics

### ⑥ Conservation principles

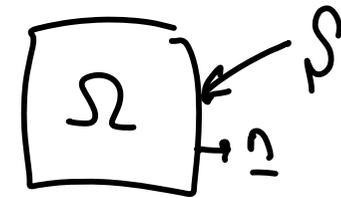
Control volume approach

Reynolds transport theorem



### ⑦ mass conservation

$$\frac{\partial}{\partial t} \int_{\Omega} \rho d\Omega + \int_S \rho \underline{u} \cdot \underline{n} dS = 0$$



$\rho$ : density     $\underline{u}$ : velocity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0$$

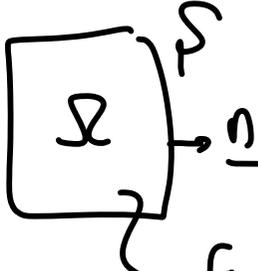
$$\sum_{i=1}^3 \frac{\partial}{\partial x_i} (\rho u_i)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1} (\rho u_1) + \frac{\partial}{\partial x_2} (\rho u_2) + \frac{\partial}{\partial x_3} (\rho u_3) = 0$$

② momentum conservation

$$\frac{\partial}{\partial t} \int_{\Omega} \rho \underline{u} d\Omega + \int_S \rho \underline{u} (\underline{u} \cdot \underline{n}) dS = \int_{\Omega} \underline{f} d\Omega + \int_S \underline{f} dS$$

$\int_{\Omega} \underline{f} d\Omega$  : body force       $\int_S \underline{f} dS$  : surface force



$$\underline{T} = - \left( p + \frac{2}{3} \mu \nabla \cdot \underline{u} \right) \underline{I} + 2\mu \underline{D}$$

stress tensor

$p$ : pressure,  $\mu$ : viscosity

$\underline{D} = \frac{1}{2} [\nabla \underline{u} + (\nabla \underline{u})^T]$  : strain-rate tensor (deformation-rate tensor)

$\underline{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  : identity tensor

$$T_{ij} = - \left( p + \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \right) \delta_{ij} + 2\mu D_{ij}$$

$\delta_{ij}$  : Kronecker delta  $\begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$$\frac{\partial u_k}{\partial x_k} = \sum_{k=1}^3 \frac{\partial u_k}{\partial x_k} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, \quad \left( \frac{\partial u_\alpha}{\partial x_\alpha} : \text{no summation on } \alpha \right)$$

$$D_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\tau_{ij} = 2\mu D_{ij} - \frac{2}{3}\mu \frac{\partial u_k}{\partial x_k} \delta_{ij} : \text{viscous part of stress tensor}$$

$$\Rightarrow T_{ij} = -p \delta_{ij} + \tau_{ij}$$

$$\Rightarrow \frac{\partial}{\partial t} \int_{\Omega} \rho \underline{u} \, d\Omega + \int_S \rho \underline{u} (\underline{u} \cdot \underline{n}) \, dS = \int_S \underline{T} \cdot \underline{n} \, dS + \int_{\Omega} \rho \underline{b} \, d\Omega$$

body force per unit mass

$$\frac{\partial}{\partial t} \int_{\Omega} \rho \underline{u} \, d\Omega + \int_S \rho \underline{u} (\underline{u} \cdot \underline{n}) \, ds = \int \underline{f}$$

$$\left( \begin{array}{l} \underline{T} = - \left( p + \frac{2}{3} \mu \nabla \cdot \underline{u} \right) \underline{I} + 2\mu \underline{D} \\ T_{ij} = - \left( p + \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \right) \delta_{ij} + 2\mu D_{ij}, \quad D_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\ T_{ij} = 2\mu D_{ij} - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij} \\ T_{ij} = -p \delta_{ij} + \tau_{ij} \end{array} \right)$$

$$\Rightarrow \frac{\partial}{\partial t} \int_{\Omega} \rho \underline{u} \, d\Omega + \int_S \rho \underline{u} (\underline{u} \cdot \underline{n}) \, ds = \int_S \underline{T} \cdot \underline{n} \, ds + \int_{\Omega} \rho \underset{\uparrow}{\underline{b}} \, d\Omega$$

body force per unit mass

$$\frac{\partial}{\partial t} (\rho \underline{u}) + \nabla \cdot (\rho \underline{u} \underline{u}) = \nabla \cdot \underline{\tau} + \rho \underline{b}$$

$$\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) = \frac{\partial \tau_{ij}}{\partial x_j} + \rho b_i$$

$$\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho b_i$$

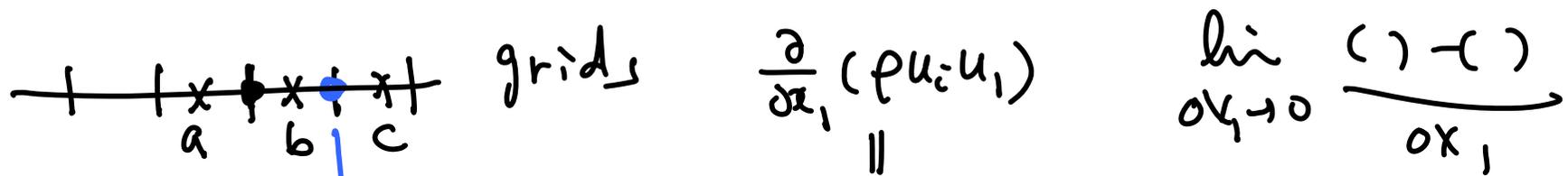
strong conservative form

$$\rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} + u_i \frac{\partial}{\partial x_j} (\rho u_j) + \rho u_j \frac{\partial u_i}{\partial x_j}$$

$$u_i \left( \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) \right) = 0 \text{ because of continuity}$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho b_i$$

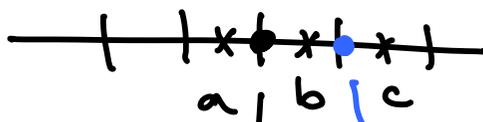
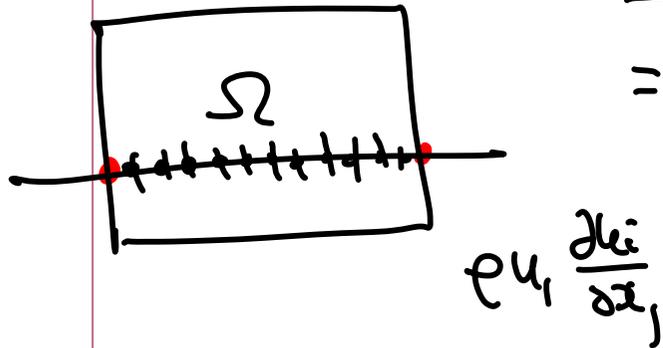
non-conservative form



$$\left[ (\rho u_i u_i)_b - (\rho u_i u_i)_a \right] / \Delta x_1$$

$$\left[ (\rho u_i u_i)_c - (\rho u_i u_i)_b \right] / \Delta x_1 \quad +$$

$$= \left[ (\rho u_i u_i)_c - (\rho u_i u_i)_a \right] / \Delta x_1$$



$$\rho u_i \cdot \frac{u_b - u_a}{\Delta x_1}$$

$$+ \rho u_i \cdot \frac{u_{ic} - u_{ib}}{\Delta x_1} = \dots$$

⇒ we prefer conservative form of governing eqs.!



③ conservation of scalar quantities

$$\frac{\partial}{\partial t} \int_{\Omega} \rho \phi d\Omega + \int_S \rho \phi (\underline{u} \cdot \underline{n}) dS = \Sigma f_{\phi}$$

$$f_{\phi} = \int_S \Gamma \nabla \phi \cdot \underline{n} dS \quad : \text{diffusion}$$

Fourier law for heat eq.

Fick's law for mass diffusion

$$\rightarrow \frac{\partial}{\partial t} \int_{\Omega} \rho \phi d\Omega + \int_S \rho \phi (\underline{u} \cdot \underline{n}) dS = \int_S \Gamma \nabla \phi \cdot \underline{n} dS$$

convection
diffusion

$$+ \int_{\Omega} \delta \phi d\Omega$$

source/sink

$$\frac{\partial}{\partial t} (\rho \phi) + \nabla \cdot (\rho \phi \underline{u}) = \nabla \cdot (\Gamma \nabla \phi) + \delta \phi$$

$$\frac{\partial}{\partial t} (\rho \phi) + \frac{\partial}{\partial x_j} (\rho \phi u_j) = \frac{\partial}{\partial x_j} \left( \Gamma \frac{\partial \phi}{\partial x_j} \right) + \delta \phi$$



$$\left\{ \begin{array}{l} \frac{\partial u_i^*}{\partial x_i^*} = 0 \quad \text{continuity eq} \\ \frac{\partial u_i^*}{\partial t^*} + \frac{\partial}{\partial x_j^*} (u_i^* u_j^*) = -\frac{\partial p^*}{\partial x_i^*} + \frac{1}{Re} \nabla^{2*} u_i^* + \frac{1}{Fr} \gamma_i^* \quad \text{Navier-Stokes eq.} \\ \frac{\partial T^*}{\partial t^*} + \frac{\partial}{\partial x_j^*} (u_j^* T^*) = \frac{1}{RePr} \nabla^{2*} T^* \quad \text{normalized gravitational acceleration vector} \end{array} \right.$$

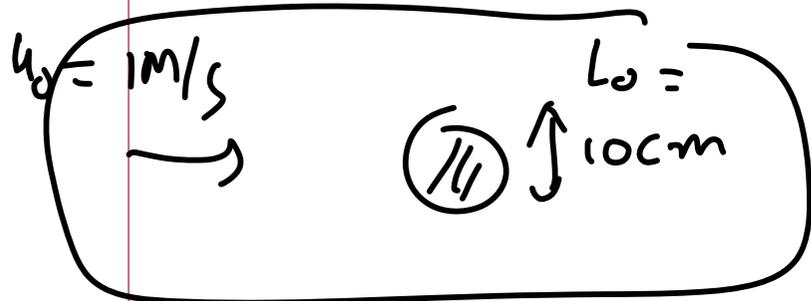
⇒ very difficult to solve  $\nabla^{2*} = \frac{\partial^2}{\partial x_i^* \partial x_i^*}$

∴ nonlinearity → turbulence

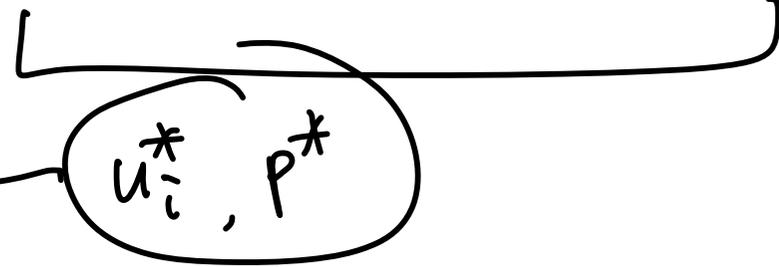
5 unknowns  $(u_i^*, p^*, T^*)$  — coupled  
no gov. eq for  $p^*$

existence? uniqueness? → We don't know yet.

"Navier-Stokes equations" by Roger Temam.



$$Re = \frac{u d}{\nu} = \frac{1 \times 0.1}{1.5 \times 10^{-5}} = \frac{2}{3} \times 10^4$$



① Simplified mathematical models

① Incompressible flow :  $\rho \equiv \text{const}$      $Ma < 0.3$

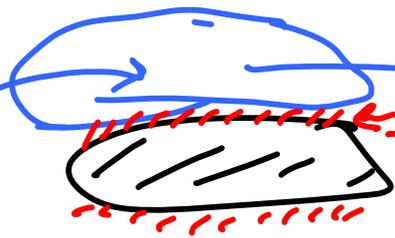
$$\begin{cases} \frac{\partial u_i}{\partial x_i} = 0 \\ \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} (u_i u_j) = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left( \mu \frac{\partial u_i}{\partial x_j} \right) + b_i \end{cases} \quad \begin{array}{l} \text{still difficult} \\ \text{to solve} \end{array}$$

② Inviscid flow :  $\mu \equiv 0 \rightarrow \underline{\tau} = -p \underline{I}$

$$\left( \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0 \right.$$

$$\left. \frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \rho b_i \right) \text{ Euler eqs.}$$

ⓐ high Re, viscous effect or turbulence effect exists only near the wall.



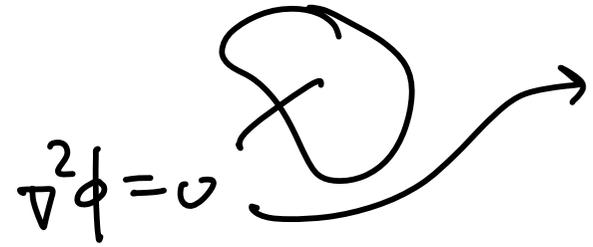
no need to put dense grids.  
very thin @ high Re.

③ potential flow :  $\mu = 0$  &  $\nabla \times \underline{u} = 0$   
inviscid irrotational flow

$\nabla \times \underline{u} = 0 \rightarrow \underline{u} = \nabla \phi$ ,  $\phi$ : velocity potential

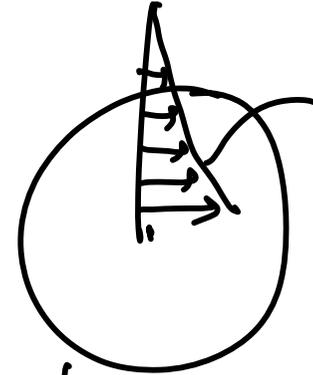
incomp. flow  $\nabla \cdot \underline{u} = 0 \rightarrow \nabla^2 \phi = 0$

$\phi \rightarrow \underline{u} = \nabla \phi \rightarrow p$  from Bernoulli eq



$\nabla \times \underline{u}$   
||

vortex vs. vorticity



$u_\theta \sim \frac{1}{r}$   
vortex



$\nabla \times \underline{u} \neq 0$  no vortex

Q criterion,  $\lambda_2$  method (Jeong & Hussain)

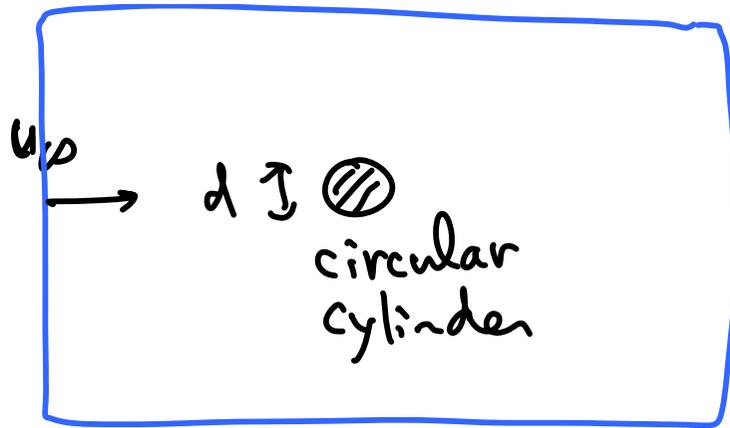
④ Creeping flow : Stokes flow  
very low velocity  $\rightarrow Re \ll 1$

viscous  
pressure terms  $\gg$  convection term  
body force (nonlinear)

$\rightarrow$  Stokes eq. : linear eq.

$$\left( \begin{array}{l} \frac{\partial u_i}{\partial x_i} = 0 \\ \frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \mu \frac{\partial u_i}{\partial x_j} \right) + \rho b_i \end{array} \right.$$

porous media  
coating  
micro device



requires huge domain size  
for Stokes flow.  
(external flow)

- ⑤ Boussinesq approx. ... for low Ma number flow  
 If density variation is not large,  
 treat  $\rho$  as constant in unsteady & convection terms  
 and treat  $\rho$  as variable only in the gravitational term

$$(\rho - \rho_0) g_i = -\rho_0 g_i = \beta (T - T_0)$$

$\beta$  ← coefficient of volumetric expansion

- ⑥ Boundary layer approximation



no reverse flow  
 no recirculation

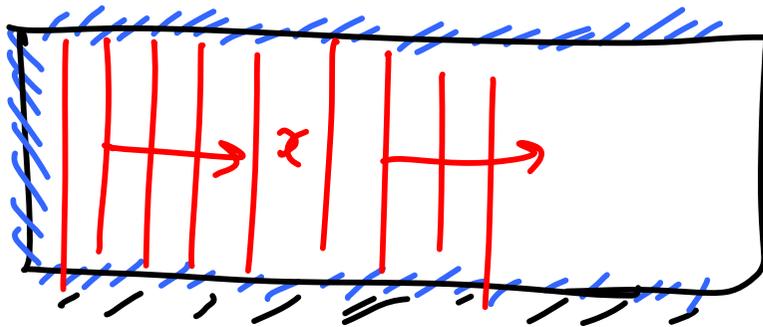
thin shear layer above the wall

$$\left( \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y} \right) \leftarrow \frac{\partial}{\partial x_1} \ll \frac{\partial}{\partial x_2}, \quad v \ll u$$

$$\cancel{\frac{\partial}{\partial t}(\rho u)} + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = -\frac{\partial p}{\partial x} + \mu \left( \cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\cancel{\frac{\partial}{\partial t}(\rho v)} + \cancel{\frac{\partial}{\partial x}(\rho uv)} + \cancel{\frac{\partial}{\partial y}(\rho v^2)} = -\frac{\partial p}{\partial y} + \mu \left( \cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} \right)$$

$$\hookrightarrow \frac{\partial p}{\partial x} = 0 \rightarrow p = p(y)$$

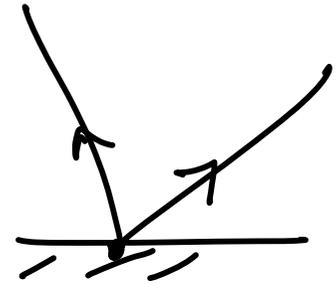


... b.c's

① Mathematical classification of flows

$$au_{xx} - 2bu_{xy} + cu_{yy} = f(x, y, u, u_x, u_y)$$

}	$b^2 - ac > 0$	hyperbolic	→
	$= 0$	parabolic	
	$< 0$	elliptic	



numerical method should respect the properties of the eq.

① hyperbolic flows - unsteady inviscid comp. flow  
steady supersonic flow

② parabolic flows - boundary layer approx.

③ elliptic flow - recirculating flow  
unsteady incomp flow

④ mixed flow type - steady transonic flow

- supersonic: hyperbolic
- subsonic: elliptic

CFD — ① fluid mechanics  
L ② numerical analysis .... J. Comp. Phys.