Ch. 10 Vector Integral Calculus. Integral Theorems

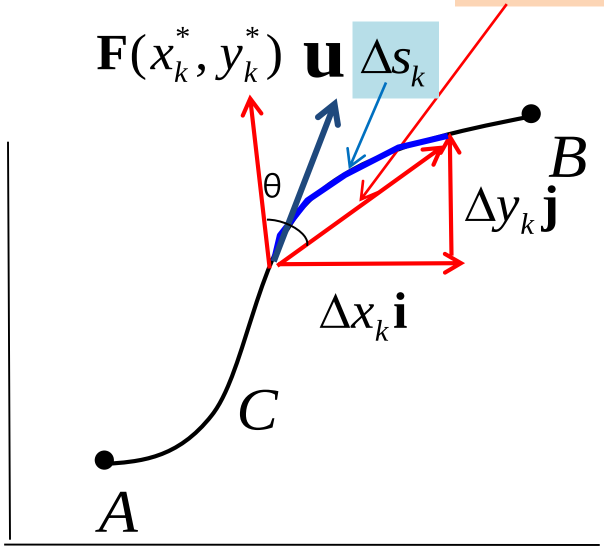
서울대학교
조선해양공학과
서유택
2018.10

※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

Introduction

☑ Work Done Equals the Gain in Kinetic Energy (운동에너지)

$$\Delta r_k = \Delta x_k \mathbf{i} + \Delta y_k \mathbf{j}$$



If Δs_k is small, $\mathbf{F}(x_k^*, y_k^*)$ is constant force, and $\Delta s_k \approx \Delta r_k$

Approximate work done by F over the subarc is

$$(\| \mathbf{F}(x_k^*, y_k^*) \| \cos\theta) \| \Delta r_k \| = \mathbf{F}(x_k^*, y_k^*) \cdot \Delta r_k$$
$$= F_1(x_k^*, y_k^*)\Delta x_k + F_2(x_k^*, y_k^*)\Delta y_k$$

By summing the elements of work and passing to limit,

$$W = \int_C F_1(x, y)dx + F_2(x, y)dy \quad or \quad W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

The work done by a force *F* along a curve *C* is due entirely to the tangential component of *F*

10.1 Line Integrals

☑ Concept of a line integral (선적분)

: A simple and natural generalization of a definite integral known from calculus

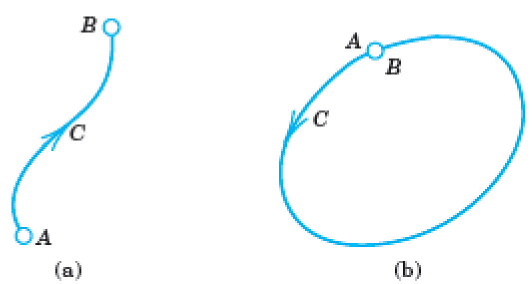
- Line Integral (선적분) or Curve Integral (곡선적분): We integrate a given function (Integrand, 피적분함수) along a curve C in space (or in the plane).

- Path of Integration

  C : $\mathbf{r}(t) = [\ x(t)\ ,\ y(t)\ ,\ z(t)\ ] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\ (\ a<t<b\ )$

- General Assumption

: Every path of integration of a line integral is assumed to be piecewise smooth.



Oriented Curve

10.1 Line Integrals

☑ Definition and Evaluation of Line Integrals

- A line integral of **F**(r) over a curve *C* (= work integral)

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt \qquad \mathbf{r}' = \frac{d\mathbf{r}}{dt}$$

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z')\,dt$$

- The interval a≤t≤b on t-axis: the positive direction, the increasing t

☑ Example 1 Evaluation of a Line Integral in the Plane

- Find the value of the line integral when F(r) = [– y , – xy] = – y**i** – xy**j** along C. C : r(t) = [cos t , sin t] = cos t **i** + sin t **j**, where $0 \le t \le \pi/2$.

- x(t) = cos t, y(t) = sin t ,

Sol) F(r(t)) = – y(t)**i** – x(t)y(t)**j** = – sin t **i** – cos t sin t **j**

- By differentiation

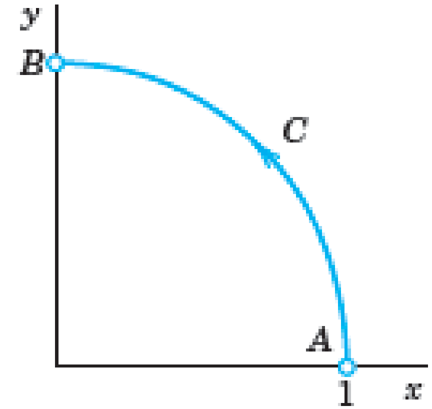
$$\Rightarrow \quad \int_C \mathbf{F}(\mathbf{r})\bullet d\mathbf{r} = \int_0^{\pi/2}[-\sin t, -\cos t \sin t]\bullet[-\sin t, \cos t]dt$$

$$\mathbf{r}'(t) = [-\sin t, \cos t] = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$= \int_0^{\pi/2}\left(\sin^2 t - \cos^2 t \sin t\right)dt$$

$$\boxed{\sin^2 t = \frac{1}{2}(1-\cos 2t)} \qquad \boxed{\text{set } \cos t = u, \quad -\sin t = du}$$

$$= \int_0^{\pi/2}\frac{1}{2}(1-\cos 2t)dt - \int_1^0 u^2(-du) = \frac{\pi}{4} - 0 - \frac{1}{3} \approx 0.4521$$



Example 1

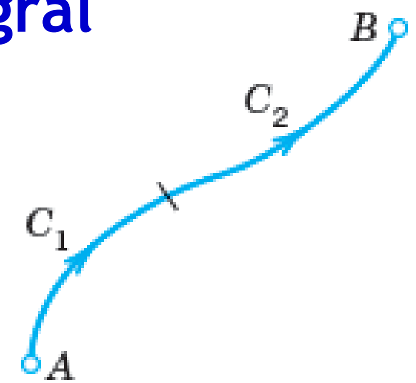
10.1 Line Integrals

☑ Simple general properties of the line integral

(5a) $\displaystyle\int_C k\mathbf{F}\cdot d\mathbf{r} = k\int_C \mathbf{F}\cdot d\mathbf{r}$ $\quad (k \text{ constant})$

(5b) $\displaystyle\int_C (\mathbf{F}+\mathbf{G})\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{G}\cdot d\mathbf{r}$

(5c) $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$



Formula (5c)

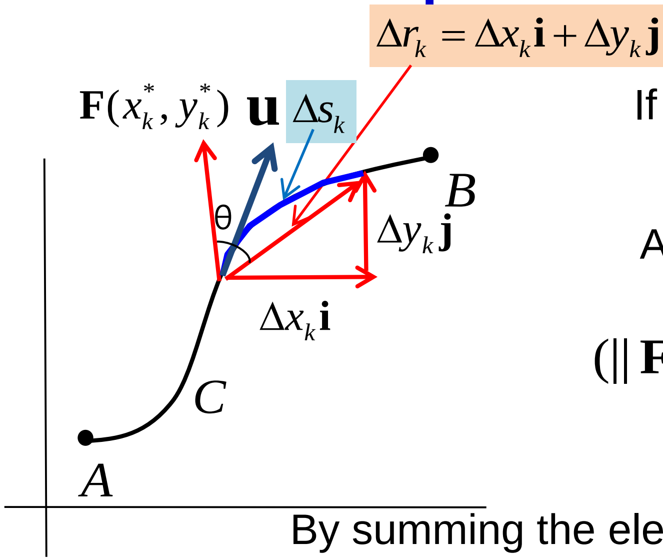
☑ Theorem 1 Direction-Preserving Parametric Transformations (방향을 유지하는 매개변수 변환)

- Any representations of C that give the same positive direction on C also yield the same value of the line integral

ex) $\mathbf{r}(t)=[2t^2, t^4],\ 1\le t\le 3 \quad \Rightarrow t^* = t^2, \mathbf{r}(t^*)=[2t^*, t^{*2}],\ 1\le t^*\le 9$

$$\Rightarrow \int_C \mathbf{F}(\mathbf{r}^*)\bullet d\mathbf{r}^* = \int_C \mathbf{F}(\mathbf{r})\bullet d\mathbf{r}$$

☑ Work Done Equals the Gain in Kinetic Energy



If Δs_k is small, $\mathbf{F}(x_k^*, y_k^*)$ is constant force, and $\Delta s_k \approx \Delta r_k$

Approximate work done by F over the subarc is

$$(\| \mathbf{F}(x_k^*, y_k^*) \| \cos\theta) \| \Delta r_k \| = \mathbf{F}(x_k^*, y_k^*) \cdot \Delta r_k$$
$$= F_1(x_k^*, y_k^*)\Delta x_k + F_2(x_k^*, y_k^*)\Delta y_k$$

By summing the elements of work and passing to limit,

$$W = \int_C F_1(x, y)dx + F_2(x, y)dy \quad or \quad W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{u}(s)\frac{ds}{dt} \qquad d\mathbf{r} = \mathbf{u}ds$$

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'dt = \int_C \mathbf{F}(\mathbf{r}) \cdot \mathbf{u}ds \qquad \mathbf{u} = \frac{d\mathbf{r}}{ds}, \ \mathbf{r}' = \frac{d\mathbf{r}}{dt}$$

The work done by a force *F* along a curve *C* is due entirely to the tangential component of *F*

10.1 Line Integrals

☑ Ex.4 Work Done Equals the Gain in Kinetic Energy

Let F be a force and t be time, then $d\mathbf{r}/dt = \mathbf{v}$, velocity.

By Newton's second law,

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{v}(t)\, dt$$

$$\mathbf{F} = m\mathbf{r}''(t) = m\mathbf{v}'(t)$$

$$\Rightarrow \quad W = \int_a^b \mathbf{F} \cdot \mathbf{r}' dt = \int_a^b m\mathbf{v}'(t) \cdot \mathbf{v}(t)\, dt$$

$$= \int_a^b m \left(\frac{\mathbf{v} \cdot \mathbf{v}}{2} \right)' dt = \frac{m}{2} |\mathbf{v}|^2 \Big|_{t=a}^{t=b}$$

$$\boxed{(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'}$$

10.1 Line Integrals

☑ Example

Find the work done by

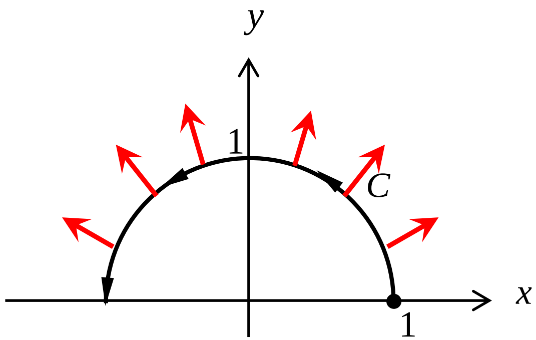
(a) F=x**i** +y**j** and

(b) F=¾**i**+½**j** along the curve C traced by r(t)=cos t **i**+sin t **j** from t=0 to t=π.

Solution)

$$(a)\ \mathbf{F} = x\mathbf{i} + y\mathbf{j}$$

Q : solve it

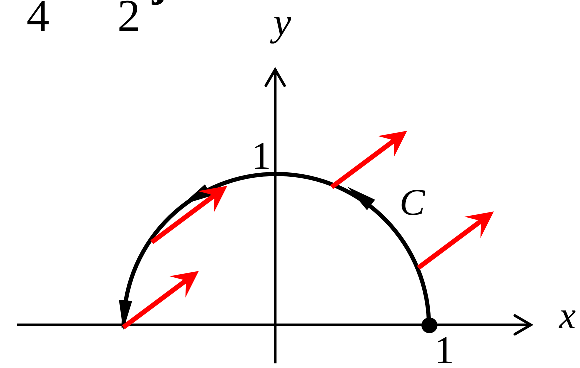


$$W = \int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (x\mathbf{i} + y\mathbf{j}) \bullet d\mathbf{r}$$

$$= \int_0^{\pi} (\cos t\mathbf{i} + \sin t\mathbf{j}) \bullet (-\sin t\mathbf{i} + \cos t\mathbf{j}) dt$$

$$= \int_0^{\pi} (-\cos t \sin t + \sin t \cos t) dt = 0$$

$$(b)\ \mathbf{F} = \frac{3}{4}\mathbf{i} + \frac{1}{2}\mathbf{j}$$



$$W = \int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \left(\frac{3}{4}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) \bullet d\mathbf{r}$$

$$= \int_0^{\pi} \left(\frac{3}{4}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) \bullet (-\sin t\mathbf{i} + \cos t\mathbf{j}) dt$$

$$= \int_0^{\pi} \left(-\frac{3}{4}\sin t + \frac{1}{2}\cos t\right) dt$$

$$= \left[\frac{3}{4}\cos t + \frac{1}{2}\sin t\right]_0^{\pi} = -\frac{3}{2}$$

☑ Other Forms of Line Integrals:

- When $\mathbf{F} = F_1\mathbf{i}$ or $F_2\mathbf{j}$ or $F_3\mathbf{k}$

$$\int_C F_1\,dx, \quad \int_C F_2\,dy, \quad \int_C F_3\,dz$$

- When without taking a dot product, we can obtain a line integral whose value is a vector rather than a scalar

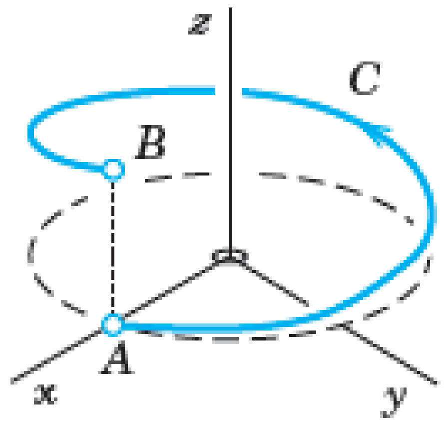
$$\int_C \mathbf{F}(\mathbf{r})\,dt = \int_a^b \mathbf{F}(\mathbf{r}(t))\,dt = \int_a^b \left[F_1(\mathbf{r}(t)),\ \ F_2(\mathbf{r}(t)),\ \ F_3(\mathbf{r}(t))\right]dt$$

☑ Ex.5 Integrate $\mathbf{F}(\mathbf{r}) = [\ xy\ ,\ yz\ ,\ z\ ]$ along the helix.

$$\mathbf{r}(t) = [\cos t, \sin t, 3t] = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 3t\,\mathbf{k}$$

$$\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\,dt = \left[-\frac{1}{2}\cos^2 t,\ \ 3\sin t - 3t\cos t,\ \ \frac{3}{2}t^2\right]\Bigg|_0^{2\pi}$$

$$= \left[0,\ \ -6\pi,\ \ 6\pi^2\right]$$



10.1 Line Integrals

☑ Theorem 2 Path Dependence (경로 관련성)

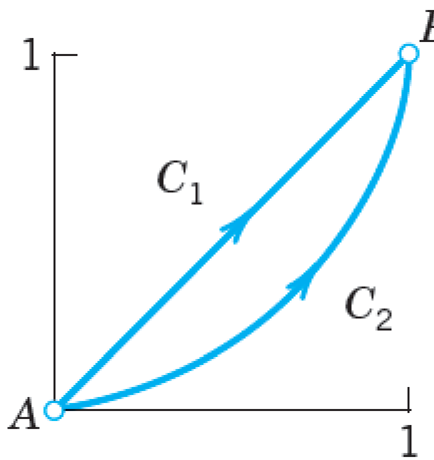
- The line integral generally depends not only on **F** and on the endpoints *A* and *B* of the path, but also on the path itself which the integral is taken.

☑ **Ex.** Integrate $\mathbf{F} = [\,0\,,\,xy\,,\,0\,]$ on the straight segment $C_1 : \mathbf{r}_1(t)=[t, t, 0]$ and the parabola $C_2 : \mathbf{r}_2(t)=[t, t^2, 0]$ with $0 \le t \le 1$, respectively.

Sol)

$$\mathbf{F}(\mathbf{r}_1(t))\cdot\mathbf{r}_1'(t) = t^2 \quad \Rightarrow \quad \int_{C_1} \mathbf{F}(\mathbf{r}_1)\cdot d\mathbf{r}_1 = \frac{1}{3}$$

$$\mathbf{F}(\mathbf{r}_2(t))\cdot\mathbf{r}_2'(t) = 2t^4 \quad \Rightarrow \quad \int_{C_2} \mathbf{F}(\mathbf{r}_2)\cdot d\mathbf{r}_2 = \frac{2}{5}$$



Proof of Theorem 2

10.2 Path Independence of Line Integrals

☑ Theorem 1 Path Independence (경로 독립성, 경로 무관성)

- A line integral with continuous F_1, F_2, F_3 in a domain *D* in space is path independent in *D* if and only if $\mathbf{F} = [F_1, F_2, F_3]$ is the gradient of some function f in *D* (**F** is gradient field),

$$\mathbf{F} = \text{grad } f \quad \left(F_1 = \frac{\partial f}{\partial x},\ F_2 = \frac{\partial f}{\partial y},\ F_3 = \frac{\partial f}{\partial z}\right) \Rightarrow \int_A^B \left(F_1 dx + F_2 dy + F_3 dz\right) = f(B) - f(A)$$

✓ Proof

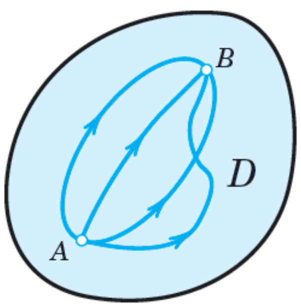
$$\int_C \left(F_1 dx + F_2 dy + F_3 dz\right) = \int_C \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz\right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt$$

$$= \int_a^b \frac{df}{dt} dt = f\left[x(t), y(t), z(t)\right]_{t=a}^{t=b}$$

$$= f\left(x(b), y(b), z(b)\right) - f\left(x(a), y(a), z(a)\right)$$

$$= f(B) - f(A)$$



$$\therefore \int_C \left(F_1 dx + F_2 dy + F_3 dz\right) = f(B) - f(A)$$

A line integral is independent of path.

10.2 Path Independence of Line Integrals

☑ Ex.1 Path Independence

Show that the integral $\int_C (2xdx + 2ydy + 4zdz)$ is path independent in any domain in space and find its value in the integration from A : (0, 0, 0) to B : (2, 2, 2).

Sol) $\mathbf{F} = [2x, 2y, 4z] = \text{grad } f$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 2x = F_1, \quad \frac{\partial f}{\partial y} = 2y = F_2, \quad \frac{\partial f}{\partial z} = 4z = F_3$$

$$\Rightarrow \quad f = x^2 + y^2 + 2z^2$$

Hence the integral is independent of path according to Theorem 1.

$$\int_C (2xdx + 2ydy + 4zdz) = f(B) - f(A) = f(2,2,2) - f(0,0,0) = 4 + 4 + 8 = 16$$

10.2 Path Independence of Line Integrals

☑ Theorem 2 Path Independence

- The integral is path independent in a domain *D* if and only if its value around every closed path in *D* is zero.

Proof 1) path independence → integral is zero

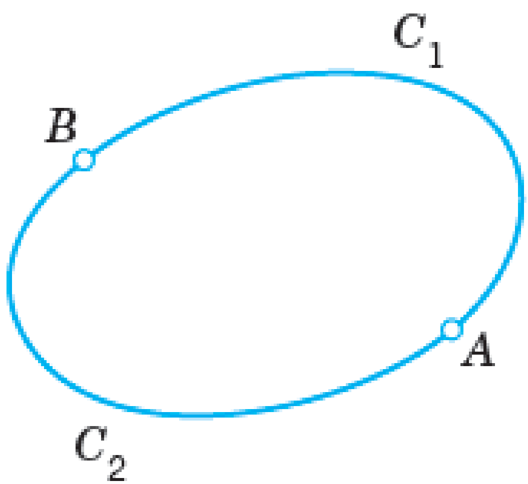
$$\int_{C_1} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{C_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \qquad C_1 : A \to B,\ C_2 : B \to A$$

$$\int_A^B \mathbf{F}(\mathbf{r}_1) \cdot d\mathbf{r} = \int_A^B \mathbf{F}(\mathbf{r}_2) \cdot d\mathbf{r} \;\Rightarrow\; \int_A^B \mathbf{F}(\mathbf{r}_1) \cdot d\mathbf{r} - \int_A^B \mathbf{F}(\mathbf{r}_2) \cdot d\mathbf{r} = 0$$

$$\Rightarrow \int_A^B \mathbf{F}(\mathbf{r}_1) \cdot d\mathbf{r} + \int_B^A \mathbf{F}(\mathbf{r}_2) \cdot d\mathbf{r} = 0$$

$$\Rightarrow \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$$

The integral around closed path is zero.



10.2 Path Independence of Line Integrals

☑ Theorem 2 Path Independence

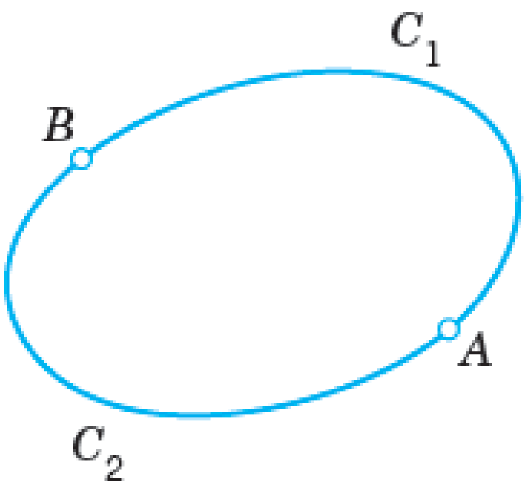
- The integral is path independent in a domain *D* if and only if its value around every closed path in *D* is zero.

Proof 2) path independence ← integral is zero

for given any points *A* and *B* and any two curves C_1 and C_2 from *A* to *B*

$$\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}(\mathbf{r}_1)\cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F}(\mathbf{r}_2)\cdot d\mathbf{r}_2 = 0$$

$$C_1 : A \to B,\ C_2 : B \to A$$



$$\int_A^B \mathbf{F}(\mathbf{r}_1)\cdot d\mathbf{r}_1 + \int_B^A \mathbf{F}(\mathbf{r}_2)\cdot d\mathbf{r}_2 = 0$$

Move 2nd term to the right.

$$\underline{\int_A^B \mathbf{F}(\mathbf{r}_1)\cdot d\mathbf{r}_1} = -\int_B^A \mathbf{F}(\mathbf{r}_2)\cdot d\mathbf{r}_2 = \underline{\int_A^B \mathbf{F}(\mathbf{r}_2)\cdot d\mathbf{r}_2}$$

In conclusion, a line integral is path independent .

10.2 Path Independence of Line Integrals

☑ Work. Conservative and Nonconservative (Dissipative, 소산하는) Physical Systems

- **Theorem 2**: work is path independent in *D* if and only if its value is zero for displacement around every closed path in *D*.
- **Theorem 1**: this happens if and only if F is the gradient of a potential in *D*.

⇒ F and the vector field defined by F are called conservative in *D* because mechanical energy is conserved

⇒ no work is done in the displacement from a point *A* and back to *A*.

- For instance, the gravitational force is conservative;

✓ if we throw a ball vertically up, it will return to our hand with the same kinetic energy it had when it left our hand.

✓ If this does not hold, nonconservative or dissipative physical system.

10.2 Path Independence of Line Integrals

☑ Theorem 3* Path Independence

- The integral is path independent in a domain *D* in space if and only if the differential form has continuous coefficient functions F_1, F_2, F_3 and is exact in *D*.

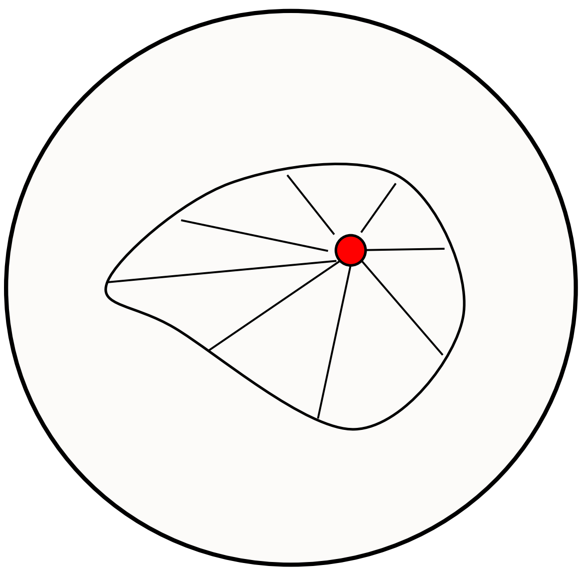
⇨ $\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$ is exact if and only if there is f in *D* such that $\mathbf{F} = \text{grad } f$.

10.2 Path Independence of Line Integrals

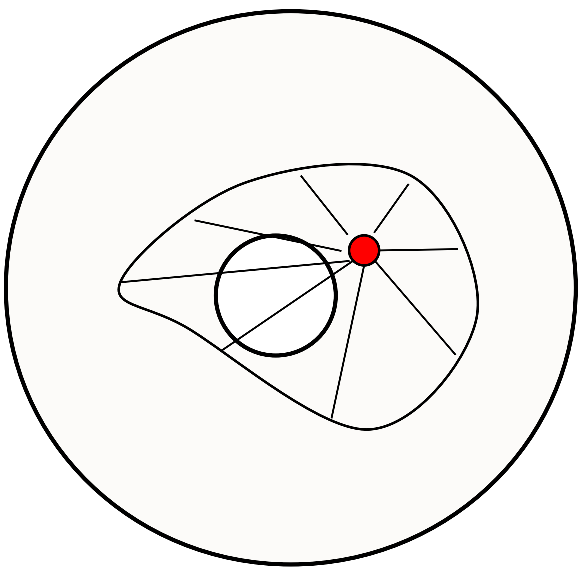
- Simple connected (단순 연결됨)

A domain D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

it can shrink to a point in D continuously



it can't shrink to a point in D continuously



10.2 Path Independence of Line Integrals

☑ Theorem 3 Criterion for Exactness (완전성) and Path Independence (경로 독립성)

- Let F_1, F_2, F_3 in the line integral, $\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ be continuous and have continuous first partial derivatives in a domain *D* in space.

a. If the differential form $\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$ is exact in *D*

- and thus line integral is path independent (from Theorem 3*),-

then in *D*, $\text{curl}\,\mathbf{F} = 0$; in components $\dfrac{\partial F_3}{\partial y} = \dfrac{\partial F_2}{\partial z}, \dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x}, \dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$.

$$\left(\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left[\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z}, \dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}, \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} \right] \right)$$

Proof a) From Theorem 3*, **F** = grad *f* ➡ curl **F** = curl (grad *f*) = 0

10.2 Path Independence of Line Integrals

☑ Theorem 3 Criterion for Exactness and Path Independence

b. If $\text{curl } \mathbf{F} = 0$ holds in *D* and *D* is simply connected,

then $\mathbf{F} \cdot d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$ is exact in D

and thus line integral is path independent.

Proof b)

To prove this, we need "Stokes's theorem" that will be presented later.

[Review] 1.4 Exact ODEs, Integrating Factors

❖ **Exact Differential Equation (완전미분 방정식)**:

The ODE $M(x, y)dx + N(x, y)dy = 0$ whose the differential form $M(x, y)dx + N(x, y)dy$ is exact **(완전미분)**, that is, this form is the differential $du = \underbrace{\boxed{\frac{\partial u}{\partial x}}}_{M(x,y)} dx + \underbrace{\boxed{\frac{\partial u}{\partial y}}}_{N(x,y)} dy$ of $u(x, y)$

- If ODE is an exact differential equation, then

$$M(x,y)dx + N(x,y)dy = 0 \quad \Rightarrow \quad du = 0 \quad \Rightarrow \quad u(x,y) = c$$

❖ Condition for exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \left(\because \quad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial N}{\partial x} \right)$

❖ Solve the exact differential equation.

$$M(x,y) = \frac{\partial u}{\partial x} \quad \Rightarrow \quad u(x,y) = \int M(x,y)dx + k(y) \quad \Rightarrow \quad \frac{\partial u}{\partial y} = N(x,y) \quad \Rightarrow \quad \frac{dk}{dy} \;\; \& \;\; k(y)$$

$$N(x,y) = \frac{\partial u}{\partial y} \quad \Rightarrow \quad u(x,y) = \int N(x,y)dy + l(x) \quad \Rightarrow \quad \frac{\partial u}{\partial x} = M(x,y) \quad \Rightarrow \quad \frac{dl}{dx} \;\; \& \;\; l(x)$$

☑ Ex.3 Exactness and Independence of Path. Determination of a Potential

Show that the differential form under the integral sign of

$$I = \int_C \left[2xyz^2 dx + \left(x^2 z^2 + z\cos yz \right) dy + \left(2x^2 yz + y\cos yz \right) dz \right]$$

is exact, so that we have independence of path in any domain, and

find the value of I from A: (0, 0, 1) to B: (1, π/4, 2).

Solution)

Exactness: $\left(F_3\right)_y = 2x^2 z + \cos yz - yz\sin yz = \left(F_2\right)_z$,

$\left(F_1\right)_z = 4xyz = \left(F_3\right)_x$,

$\left(F_2\right)_x = 2xz^2 = \left(F_1\right)_y$

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

☑ Ex.3 Exactness and Independence of Path. Determination of a Potential

Show that the differential form under the integral sign of

$$I = \int_C \left[2xyz^2 dx + \left(x^2 z^2 + z\cos yz \right) dy + \left(2x^2 yz + y\cos yz \right) dz \right]$$

A: (0, 0, 1) to B: (1, π/4, 2)

To find f $\quad \mathbf{F} = \operatorname{grad} f \quad \left(F_1 = \frac{\partial f}{\partial x},\ F_2 = \frac{\partial f}{\partial y},\ F_3 = \frac{\partial f}{\partial z} \right)$

$$f = \int F_2 dy = \int \left(x^2 z^2 + z\cos yz \right) dy = x^2 yz^2 + \sin yz + g(x, z)$$

$$f_x = 2xyz^2 + g_x = F_1 = 2xyz^2 \quad \Rightarrow \quad g_x = 0 \quad \Rightarrow \quad g = h(z)$$

$$f_z = 2x^2 yz + y\cos yz + h' = F_3 = 2x^2 yz + y\cos yz \quad \Rightarrow \quad h' = 0 \quad \Rightarrow \quad h = \text{const}$$

(If we assume h = 0)

$$\therefore \ f = x^2 yz^2 + \sin yz, \quad f(B) - f(A) = 1 \cdot \frac{\pi}{4} \cdot 4 + \sin\frac{\pi}{2} - 0 = \pi + 1$$

☑ Example

Show that the vector field

F=(y^2+5)**i**+(2xy-8)**j** is a gradient field.

Find a potential function for **F**

(**F** = grad ϕ).

Solution)

$$\mathbf{F}=(y^2+5)\mathbf{i}+(2xy-8)\mathbf{j}$$
$$=P(x,y)\mathbf{i}+Q(x,y)\mathbf{j}$$

$$\frac{\partial P}{\partial y}=2y,\ \frac{\partial Q}{\partial x}=2y$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$ (exact)

∴Vector field **F** is a gradient field.

$$P=\frac{\partial \phi}{\partial x}=y^2+5$$

$$Q=\frac{\partial \phi}{\partial y}=2xy-8$$

$$\phi=\int(y^2+5)dx=y^2x+5x+g(y)$$

$$\frac{\partial \phi}{\partial y}=2xy+g'(y)$$

$$\therefore g'(y)=-8,\ g(y)=-8y+C$$

$$\phi=y^2x+5x-8y+C$$

$$\nabla\phi=\frac{\partial \phi}{\partial x}\mathbf{i}+\frac{\partial \phi}{\partial y}\mathbf{i}=(y^2+5)\mathbf{i}+(2xy-8)\mathbf{j}$$

10.2 Path Independence of Line Integrals

☑ Ex.4 On the Assumption of Simple Connectedness

Let $F_1 = -\dfrac{y}{x^2+y^2}$, $F_2 = \dfrac{x}{x^2+y^2}$, $F_3 = 0$ (not defined in the origin).

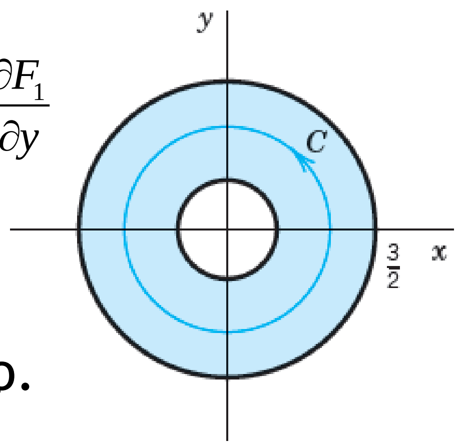
Differentiation show that $\dfrac{\partial F_3}{\partial y} = \dfrac{\partial F_2}{\partial z}$, $\dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x}$, $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ is satisfied in any domain of the *xy*-plane not containing the origin in the domain D: $\dfrac{1}{2} < \sqrt{x^2+y^2} < \dfrac{3}{2}$

Solution)

1) F_1 and F_2 do not depend on z, and $F_3 = 0$ $\Rightarrow$ $\dfrac{\partial F_3}{\partial y} = \dfrac{\partial F_2}{\partial z}$, $\dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x}$

By differentiation:

$$\frac{\partial F_2}{\partial x} = \frac{x^2+y^2-x\cdot 2x}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2} = -\frac{x^2+y^2-y\cdot 2y}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2} = \frac{\partial F_1}{\partial y}$$



2) D is not simply connected

$\Rightarrow$ the integral on any closed curve in D is not zero.

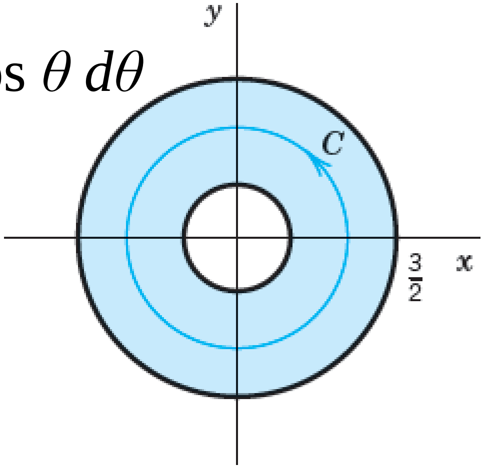
☑ Ex.4 On the Assumption of Simple Connectedness

Let $F_1 = -\dfrac{y}{x^2+y^2}, \quad F_2 = \dfrac{x}{x^2+y^2}, \quad F_3 = 0$

---

3) For example, on the circle $x^2 + y^2 = 1$,

$x = r\cos\theta,\ y = r\sin\theta,\ r=1 \Rightarrow dx = -\sin\theta\, d\theta,\ dy = \cos\theta\, d\theta$



$$I = \int_c (F_1 dx + F_2 dy) = \int_c \frac{-ydx + xdy}{x^2+y^2}$$

$$= \int_c \frac{\sin^2\theta d\theta + \cos^2\theta d\theta}{1} = \int_c \frac{d\theta}{1} = \int_0^{2\pi} \frac{d\theta}{1} = \boxed{2\pi}$$

≠0 (integral is not zero)

"Since *D* is not simply connected, we cannot apply Theorem 3 and I is not independent of path in *D*."

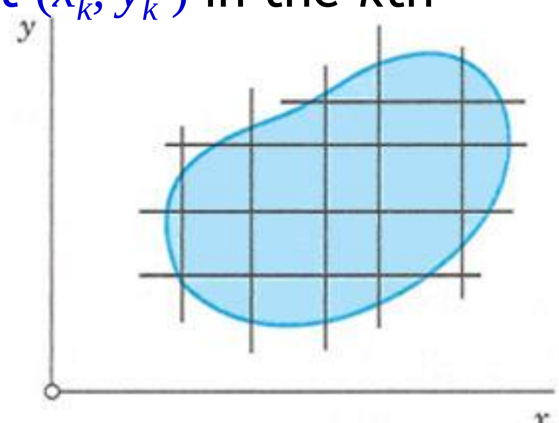
10.3 Calculus Review: Double Integrals

☑ Double integral (이중적분)

: Volume of the region between the surface defined by the function and the plane

☑ Definition of the double integral

- Subdivide the region R by drawing parallels to the x- and y-axes.
- Number the rectangles that are entirely within *R* from 1 to *n*.
- In each such rectangle we choose a point (x_k, y_k) in the kth rectangle, whose area is ΔA_k

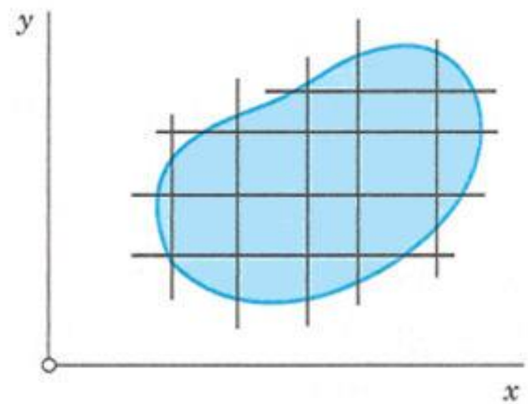


Subdivision of a region *R*

10.3 Calculus Review: Double Integrals

- The length of the maximum diagonal of the rectangles approaches zero as n approaches infinity.
- We form the sum $J_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$
- Assuming that $f(x, y)$ is continuous in *R* and *R* is bounded by finitely many smooth curves,
- one can show that this sequence $J_{n_1}, J_{n_2}, \cdots$ converges and its limit is independent of the choice of subdivisions and corresponding points (x_k, y_k).
- This limit is called the **double integral of** $f(x, y)$ over the region R.

$$\iint_R f(x,y)\,dxdy \quad \text{or} \quad \iint_R f(x,y)\,dA$$



Subdivision of a region ***R***

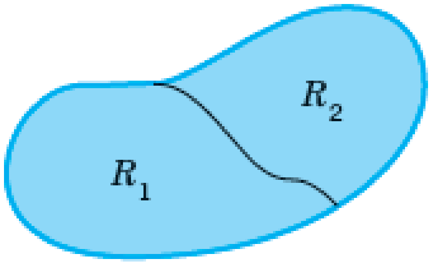
10.3 Calculus Review: Double Integrals

☑ Properties of double integrals

$$\iint_R kf\,dxdy = k\iint_R f\,dxdy \qquad (k \text{ constant})$$

$$\iint_R (f+g)\,dxdy = \iint_R f\,dxdy + \iint_R g\,dxdy$$

$$\iint_R f\,dxdy = \iint_{R_1} f\,dxdy + \iint_{R_2} f\,dxdy \qquad (\text{See Figure})$$



☑ Mean Value Theorem

- *R* is simply connected, then there exists at least one point (x_0, y_0) in *R* such that we have

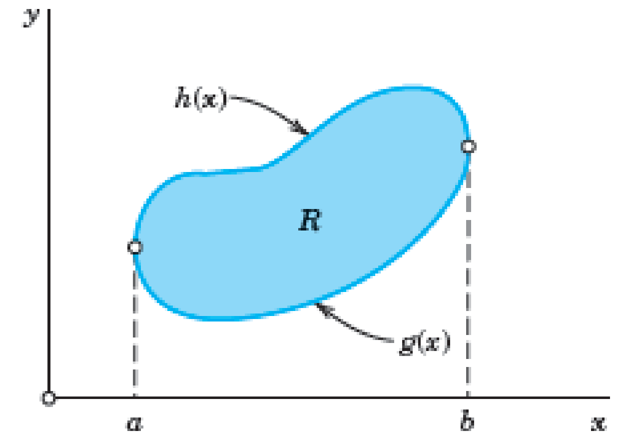
$$\iint_R f(x,y)\,dxdy = f(x_0, y_0)\,A$$

where *A* is the area of *R*.

10.3 Calculus Review: Double Integrals

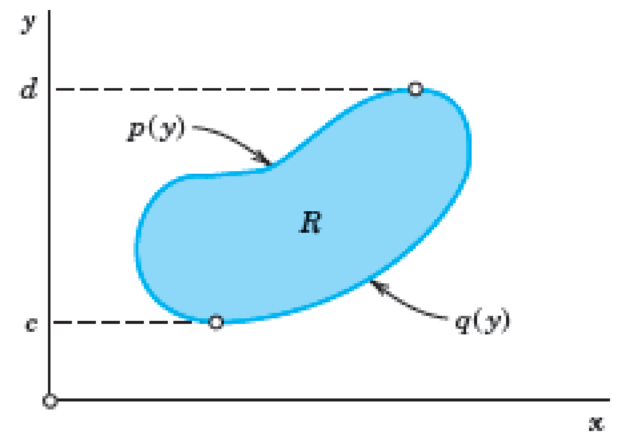
☑ Evaluation of Double Integrals by Two Successive Integrations

$$\iint\limits_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy \right] dx$$



Evaluation of a double integral

$$\iint\limits_R f(x,y)\,dx\,dy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)\,dx \right] dy$$



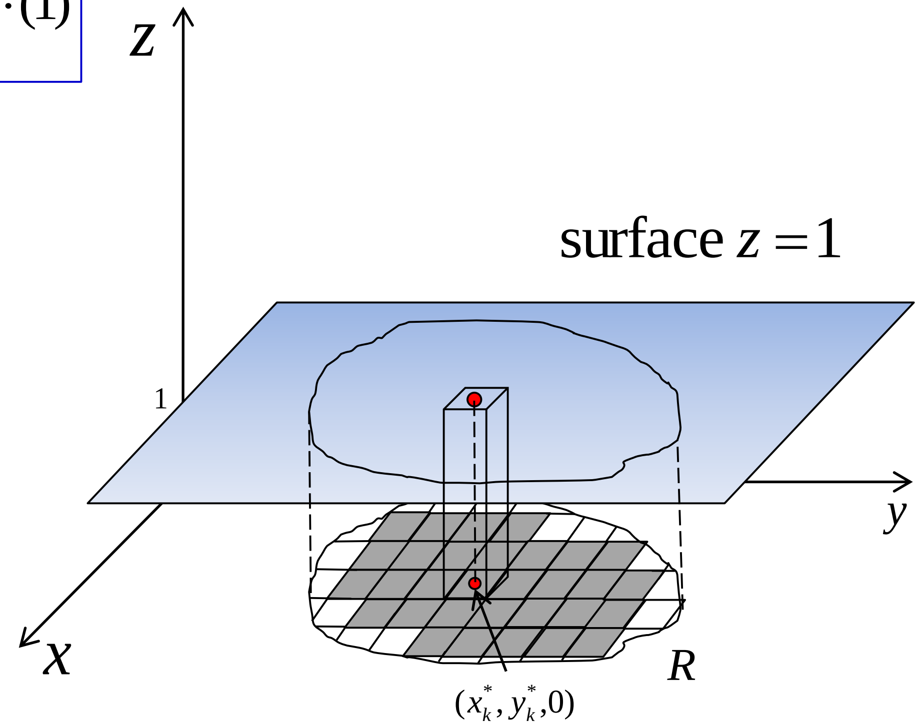
Evaluation of a double integral

☑ Applications of Double Integrals: Area

when $f(x, y) = 1$ on R, then $\lim_{\|\mathrm{P}\| \to 0} \sum_{k=1}^{n} \Delta A$ simply give the **area** A of the region

$$\boxed{\iint_R f(x, y)\, dA = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k \cdots (1)}$$

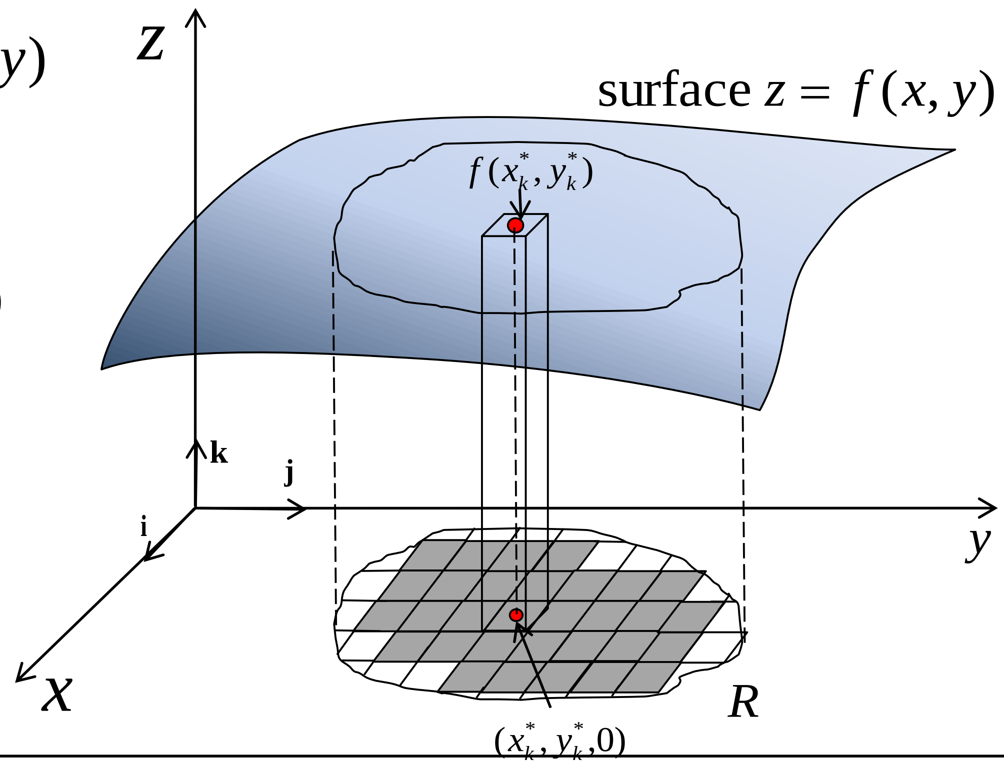
$$A = \iint_R dA$$



10.3 Calculus Review: Double Integrals

☑ Applications of Double Integrals: Volume

If $f(x,y)>0$ on *R*, then the product $f(x_k^*, y_k^*)\Delta A_k$ give the volume of rectangular prism. The summation of volume $\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ is approximation to the **volume** *V*, of the solid *above* the region R and *below* the surface $z = f(x,y)$



The limit of this sum as $\|P\| \to 0$

$$V = \iint_R f(x,y)\,dA$$

10.3 Calculus Review: Double Integrals

☑ Applications of Double Integrals

- Let $f(x, y)$ be the density (= mass per unit area) of a distribution of mass in *xy*-plane
- Total mass *M* in *R*: $$M = \iint_R f(x,y)\,dxdy$$
- Center of gravity of the mass in *R*:

$$\bar{x} = \frac{1}{M}\iint_R xf(x,y)\,dxdy, \quad \bar{y} = \frac{1}{M}\iint_R yf(x,y)\,dxdy$$

- Moments of inertia of the mass in *R* about the *x*- and *y*-axes

$$I_x = \iint_R y^2 f(x,y)\,dxdy, \quad I_y = \iint_R x^2 f(x,y)\,dxdy$$

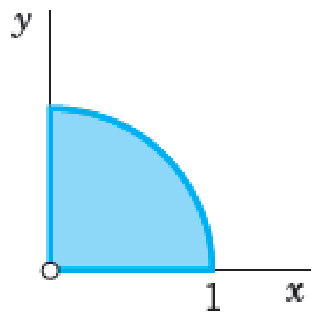
- Polar moment of inertia about the origin of mass in *R*:

$$I_0 = I_x + I_y = \iint_R \left(x^2 + y^2\right) f(x,y)\,dxdy$$

10.3 Calculus Review: Double Integrals

☑ Change of Variables in Double Integrals. Jacobian

$$M = \iint_R f(x,y)\,dxdy$$

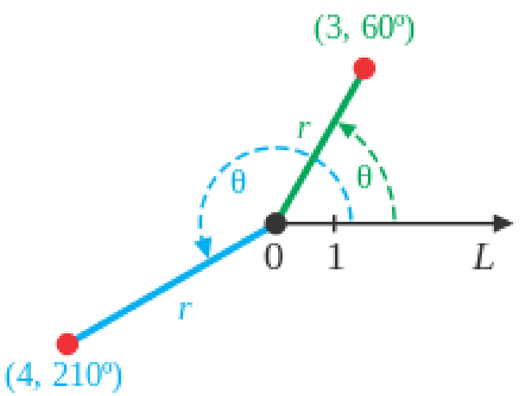


How to apply????

Change of variables !!

$$M = \iint_R dxdy = \int_0^{\pi/2}\int_0^1 r\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{2}\,d\theta = \frac{\pi}{4}$$

Polar coordinate (극좌표계)



10.3 Calculus Review: Double Integrals

☑ Change of Variables in Double Integrals. Jacobian

- A change of variables in double integrals from *x*, *y* to *u* ,*v*

$$\iint_R f(x,y)\,dxdy = \iint_{R^*} f\left(x(u,v), y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

- Jacobian: $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$

- Polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\iint_R f(x,y)\,dxdy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

10.3 Calculus Review: Double Integrals

☑ Ex. 1 Evaluate the double integral over the square *R*

$$\iint_R \left(x^2+y^2\right)dxdy$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$$

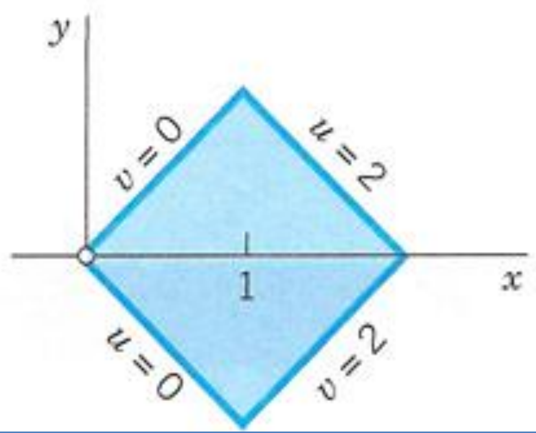
Sol) Transformation $x+y=u,\ x-y=v$ $\left(x=\frac{1}{2}(u+v),\ \ y=\frac{1}{2}(u-v)\right)$

Q : solve it

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\[1mm] \frac{1}{2} & -\frac{1}{2}\end{vmatrix}=-\frac{1}{2}$$

$$\iint_R f(x,y)\,dxdy=\iint_{R^*} f\left(x(u,v),y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

$$\Rightarrow \therefore \iint_R \left(x^2+y^2\right)dxdy=\int_0^2\int_0^2 \frac{1}{2}\left(u^2+v^2\right)\frac{1}{2}\,dudv=\frac{8}{3}$$



Region *R* in Example 1

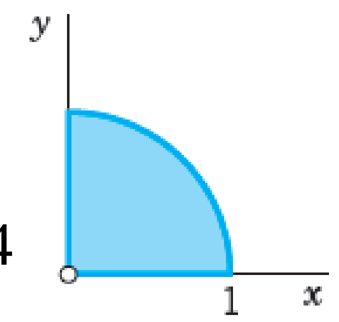
☑ Ex.2 Double Integrals in Polar Coordinates.

- Let $f(x, y) = 1$ be the mass density in the region, Find the total mass, the center of gravity, and the moments of inertia I_x, I_y, I_0.

Sol)

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r, \qquad \iint_R f(x,y)\,dxdy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\,rdrd\theta$$

$$M = \iint_R dxdy = \int_0^{\pi/2}\int_0^1 rdrd\theta = \int_0^{\pi/2}\frac{1}{2}d\theta = \frac{\pi}{4}$$



$$\bar{x} = \frac{4}{\pi}\int_0^{\pi/2}\int_0^1 r\cos\theta\, rdr\, d\theta = \frac{4}{\pi}\int_0^{\pi/2}\frac{1}{3}\cos\theta d\theta = \frac{4}{3\pi} = 0.4244$$

$$\bar{y} = \frac{4}{3\pi}$$

$$\bar{x} = \frac{1}{M}\iint_R xf(x,y)\,dxdy, \quad \bar{y} = \frac{1}{M}\iint_R yf(x,y)\,dxdy$$

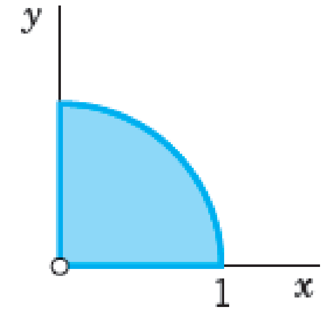
☑ Ex.2 Double Integrals in Polar Coordinates.

- Let $f(x, y) = 1$ be the mass density in the region, Find the total mass, the center of gravity, and the moments of inertia I_x, I_y, I_0.

Sol)

$$I_x = \int_R \int y^2 dxdy \frac{4}{\pi} = \int_0^{\pi/2} \int_0^1 r^2 \sin^2\theta \, r dr \, d\theta = \int_0^{\pi/2} \frac{1}{4} \sin^2\theta d\theta$$

$$= \int_0^{\pi/2} \frac{1}{8}(1-\cos 2\theta) d\theta = \frac{1}{8}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{16} = 0.1963$$



$$I_y = \frac{\pi}{16}$$

$$I_0 = I_x + I_y = \frac{\pi}{8}$$

$$I_x = \iint_R y^2 f(x,y) dxdy, \quad I_y = \iint_R x^2 f(x,y) dxdy$$

10.4 Green's Theorem in the Plane

☑ Theorem 1 Green's Theorem in the Plane

- Let *R* be a closed bounded region in the *xy*-plane whose boundary *C* consists of finitely many smooth curves.
- Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continuous and have continuous partial derivatives $\dfrac{\partial F_1}{\partial y}$ and $\dfrac{\partial F_2}{\partial x}$ everywhere in some domain containing *R*. Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C \left(F_1 dx + F_2 dy \right)$$

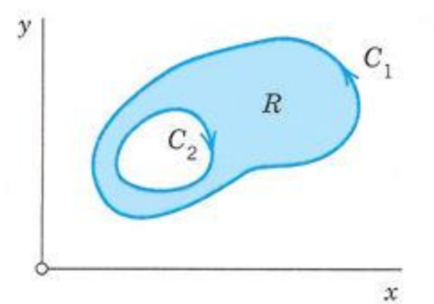
Here we integrate along the entire boundary *C* of *R* in such a sense that ***R* is on the left** as we advance in the direction of integration.

- Vectorial form

$$\left(\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left[\cancel{\frac{\partial F_3}{\partial y}} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \cancel{\frac{\partial F_3}{\partial x}}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \right)$$

$$\mathbf{F} = [F_1, F_2] = F_1 \mathbf{i} + F_2 \mathbf{j}$$

$$\Rightarrow \quad \iint_R (\operatorname{curl} \mathbf{F}) \bullet \mathbf{k} \, dxdy = \oint_C \mathbf{F} \bullet d\mathbf{r}$$



Region *R* whose boundary *C* consists of two parts

☑ Ex. 1 Verification of Green's Theorem in the Plane

$F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ and C the circle $x^2 + y^2 = 1$.

---

1) $$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \iint_R \left[(2y+2) - (2y-7) \right] dxdy = 9 \iint_R dxdy = 9\pi$$

(Circular disk R has area π.)

2) We must orient C counterclockwise ⇒ $\mathbf{r}(t) = [\cos t, \sin t]$, $\mathbf{r}'(t) = [-\sin t, \cos t]$

$$F_1 = y^2 - 7y = \sin^2 t - 7\sin t, \quad F_2 = 2xy + 2x = 2\cos t \sin t + 2\cos t$$

$$\Rightarrow \oint_C (F_1 x' + F_2 y') dt = \int_0^{2\pi} \left[\left(\sin^2 t - 7\sin t \right)\left(-\sin t \right) + 2\left(\cos t \sin t + \cos t \right)\left(\cos t \right) \right] dt$$

$$= \int_0^{2\pi} \left(-\sin^3 t + 7\sin^2 t + 2\cos^2 t \sin t + 2\cos^2 t \right) dt$$

$$= 0 + 7\pi - 0 + 2\pi = 9\pi$$

10.4 Green's Theorem in the Plane

☑ Proof of Green's Theorem

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \boxed{\frac{\partial F_1}{\partial y}} \right) dxdy = \oint_C \left(\boxed{F_1 dx} + F_2 dy \right)$$

$$\oint_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} \boxed{- \frac{\partial F_1}{\partial y} \Bigg) dA}$$

Proof) $R: g_1(x) \le y \le g_2(x), \quad a \le x \le b$

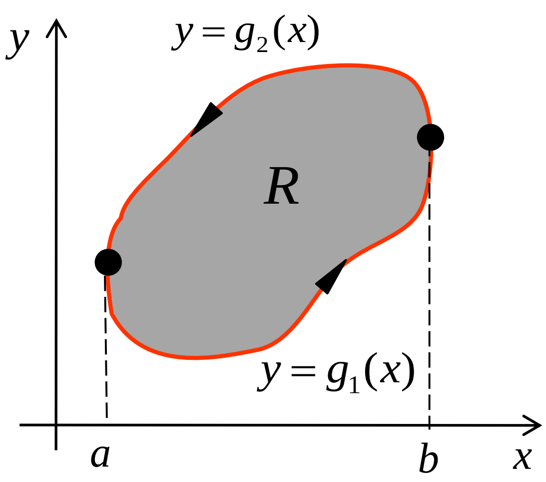
$$-\iint_R \frac{\partial F_1}{\partial y} dA = -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial F_1}{\partial y} dydx$$

$$= -\int_a^b [F_1(x, g_2(x)) - F_1(x, g_1(x))] dx$$

$$= \int_a^b F_1(x, g_1(x)) dx - \int_a^b F_1(x, g_2(x)) dx$$

$$= \int_a^b F_1(x, g_1(x)) dx + \int_b^a F_1(x, g_2(x)) dx$$

$$= \oint_C F_1(x, y) dx$$



☑ Proof of Green's Theorem

$$\iint_R \left(\boxed{\frac{\partial F_2}{\partial x}} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C \left(F_1 dx + \boxed{F_2 dy} \right)$$

$$\oint_C F_1 dx + F_2 dy = \iint_R \left(\boxed{\frac{\partial F_2}{\partial x}} - \frac{\partial F_1}{\partial y} \right) dA$$

Proof) $R: h_1(y) \le x \le h_2(y), \quad c \le y \le d$

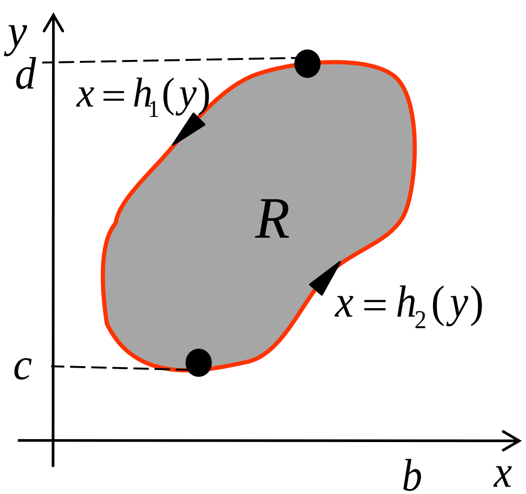
$$\iint_R \frac{\partial F_2}{\partial x} dA = \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial F_2}{\partial x} dxdy$$

$$= \int_c^d [F_2(h_2(y), y) - F_2(h_1(y), y)] dy$$

$$= \int_c^d F_2(h_2(y), y) dy - \int_c^d F_2(h_1(y), y) dy$$

$$= \int_c^d F_2(h_2(y), y) dy + \int_d^c F_2(h_1(y), y) dy$$

$$= \oint_C F_2(x, y) dy$$

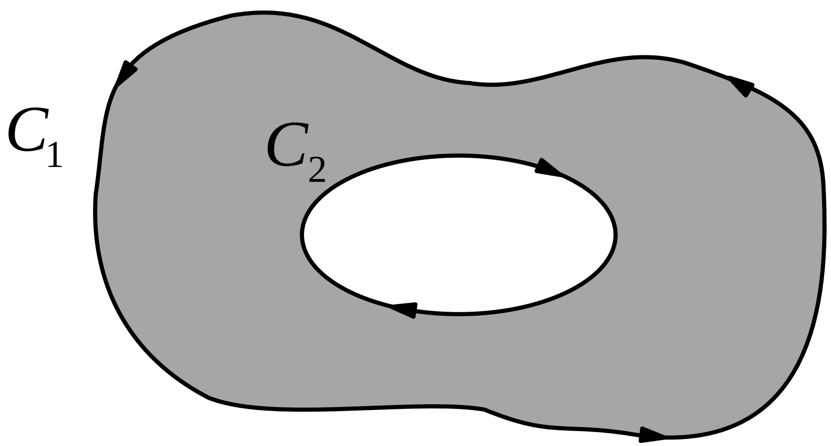


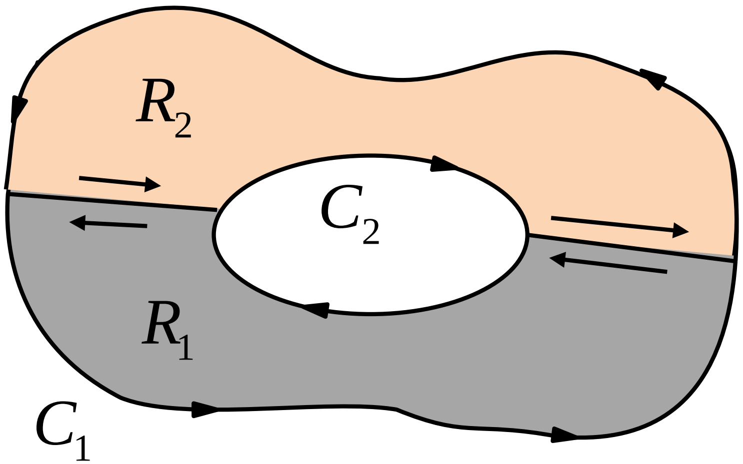
Region with Holes

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \iint_{R_1} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA + \iint_{R_2} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$= \oint_{C_1} F_1 dx + F_2 dy + \oint_{C_2} F_1 dx + F_2 dy$$

$$= \oint_C F_1 dx + F_2 dy \qquad (C = C_1 \cup C_2)$$





☑ Some Applications of Green's Theorem

- **Ex. 2** Area of a Plane Region as a Line Integral Over the Boundary

R의 넓이 (=A)

$$F_1 = 0, \quad F_2 = x \quad \Rightarrow \iint_R dxdy = \oint_C xdy$$

$$F_1 = -y, \quad F_2 = 0 \quad \Rightarrow \iint_R dxdy = -\oint_C ydx$$

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

- **Ex. 3** Area of a Plane Region in Polar Coordinates

Polar coordinates $x = r\cos\theta, y = r\sin\theta$

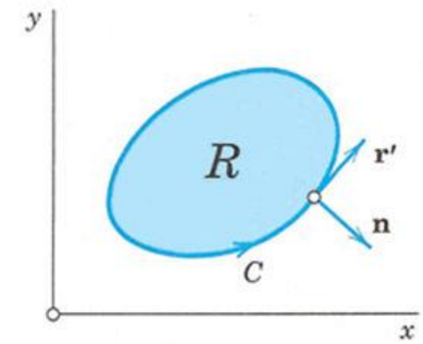
$$\Rightarrow dx = \cos\theta\, dr - r\sin\theta\, d\theta,\ dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

$$= \frac{1}{2}\oint_C \left[(r\cos\theta)(\sin\theta dr + r\cos\theta d\theta) - (r\sin\theta)(\cos\theta dr - r\sin\theta d\theta)\right] = \frac{1}{2}\oint_C r^2 d\theta$$

☑ **Ex. 4 Transformation of a Double Integral of the Laplacian of a Function ($\nabla^2 w$) into a Line Integral of Its Normal Derivative ($\frac{\partial w}{\partial n}$)**

$w(x,y)$ is continuous and has continuous first and second partial derivatives in a domain of the *xy*-plane containing a region *R* of the type indicated in Green's theorem.



We set $F_1=-\frac{\partial w}{\partial y},\quad F_2=\frac{\partial w}{\partial x}$

1. $\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\nabla^2 w \;\Rightarrow\; \iint_R \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy=\iint_R \nabla^2 w\,dxdy$

2. $\oint_C (F_1dx+F_2dy)=\oint_C \left(F_1\frac{dx}{ds}+F_2\frac{dy}{ds}\right)ds=\oint_C \left(-\frac{\partial w}{\partial y}\frac{dx}{ds}+\frac{\partial w}{\partial x}\frac{dy}{ds}\right)ds$

$$\mathbf{r}'\bullet\mathbf{n}=\left[\frac{dx}{ds},\frac{dy}{ds}\right]\bullet\left[-\frac{dy}{ds},\frac{dx}{ds}\right]=0$$

$$here,\ \frac{\partial w}{\partial x}\frac{dy}{ds}-\frac{\partial w}{\partial y}\frac{dx}{ds}=\left[\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right]\left[\frac{dy}{ds},-\frac{dx}{ds}\right]=(\text{grad}\,w)\bullet\mathbf{n}=\frac{\partial w}{\partial n}$$

$$D_{\mathbf{u}}f(x,y)=\nabla f(x,y)\cdot\mathbf{u}$$

$$=\oint_C \frac{\partial w}{\partial n}ds$$

$$\therefore\quad \iint_R \nabla^2 w\,dxdy=\oint_C \frac{\partial w}{\partial n}ds$$

10.4 Green's Theorem in the Plane

$$F_1 = -\frac{\partial w}{\partial y}, \quad F_2 = \frac{\partial w}{\partial x}$$

☑ **Ex. 4 Transformation of a Double Integral of the Laplacian of a Function ($\nabla^2 w$) into a Line Integral of Its Normal Derivative ($\frac{\partial w}{\partial n}$)**

$$\iint_R \nabla^2 w\,dxdy = \iint_R \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\,dxdy = \iint_R \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)dxdy$$

$$= \iint_R \frac{\partial}{\partial x}\underbrace{\boxed{\left(\frac{\partial w}{\partial x}\right)}}_{F_2} - \frac{\partial}{\partial y}\underbrace{\boxed{\left(-\frac{\partial w}{\partial y}\right)}}_{F_1}dxdy = \oint_C \left(\underbrace{\boxed{-\frac{\partial w}{\partial y}}}_{F_1}dx + \underbrace{\boxed{\frac{\partial w}{\partial x}}}_{F_2}dy\right)$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy = \oint_C (F_1 dx + F_2 dy)$$

$$= \oint_C \left(-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds}\right)ds = \oint_C \left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right]\left[\frac{dy}{ds}, -\frac{dx}{ds}\right]ds = \oint_C (\operatorname{grad} w)\bullet \mathbf{n}\,ds$$

$$= \oint_C \frac{\partial w}{\partial n}\,ds$$

$$\therefore \quad \iint_R \nabla^2 w\,dxdy = \oint_C \frac{\partial w}{\partial n}ds$$

"결국 앞 페이지의 F_1, F_2는 과정을 쉽게 하기 위해 도입한 것이지, 위 식이 성립하는데 어떤 전제조건도 되지 않는다. 즉, 본 식은 일반적으로 어느 scalar w에 대해서 성립한다."

Example Flow of a Compressible Fluid.

비압축성 유체(Incompressible fluid)라고 가정하면,

☑ Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{V}) = 0 \quad \xrightarrow{(\rho = const)} \quad \nabla \bullet \mathbf{V} = 0 \quad \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right)$$

(divergence)

☑ Velocity Potential

$$\mathbf{V} = \mathrm{grad}\,\phi \qquad \mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

$$+$$

$$\nabla \bullet \mathbf{V} = 0 \quad \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right)$$

$$||$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi = 0$$

Example Flow of a Compressible Fluid.

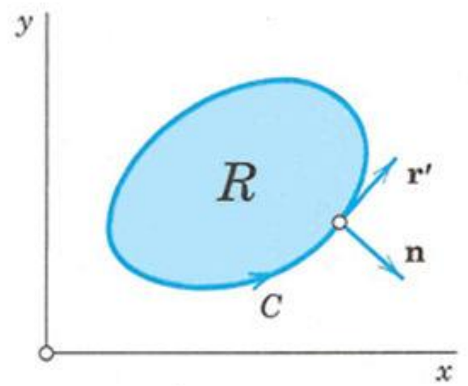
$$\therefore \quad \iint_R \nabla^2 w\,dxdy = \oint_C \frac{\partial w}{\partial n} ds$$

☑ Physical Meaning

- Incompressible fluid when continuity equation is satisfied,

$$\nabla^2 \phi = 0$$

$$\therefore \quad \iint_R \nabla^2 \phi\,dxdy = 0$$



C의 normal 방향으로의 유체의 속도

$$\oint_C \boxed{\frac{\partial \phi}{\partial n}} ds = 0$$

경계 C를 통해 단위 시간당 들어오고 나온 유체의 양=0

Example Flow of a Compressible Fluid.

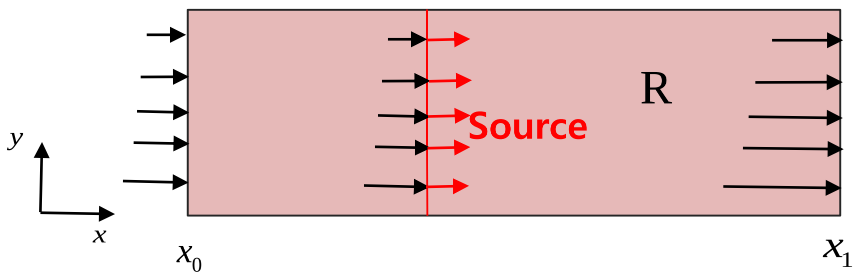
☑ Physical Meaning

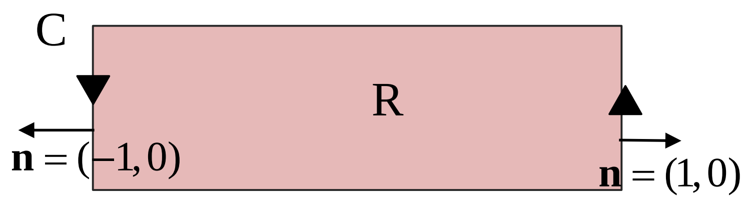
- Incompressible fluid but continuity equation is NOT satisfied,

ex) a source to add flux x direction, no flux in y direction $\dfrac{\partial\phi}{\partial y}=\dfrac{\partial^2\phi}{\partial y^2}=0$

속도u의 변화량

$$u(x_0,y)=\frac{\partial\phi}{\partial x}(x_0,y) \qquad \frac{\partial u}{\partial x}=\frac{\partial^2\phi}{\partial x^2} \qquad u(x_1,y)=\frac{\partial\phi}{\partial x}(x_1,y)$$





경계를 통해 단위 시간당
빠져나가는 유체의 양

$$\oint_C \frac{\partial\phi}{\partial n}ds=\int \boxed{-\frac{\partial\phi}{\partial x}\bigg|_{x_0}+\frac{\partial\phi}{\partial x}\bigg|_{x_1}}\,dy$$
$$=\int -u(x_0,y)+u(x_1,y)dy$$

$$\iint_R \nabla^2\phi dxdy=\int \boxed{\int_{x_0}^{x_1}\frac{\partial^2\phi}{\partial x^2}dxdy}=\int \frac{\partial\phi}{\partial x}\bigg|_{x_0}^{x_1}dy=\int \boxed{u(x_1,y)-u(x_0,y)}dy$$

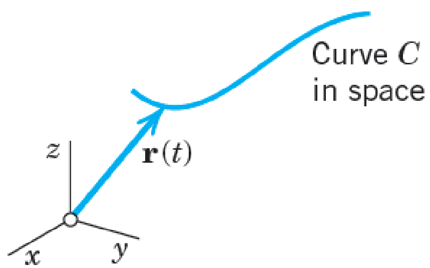
속도의 변화량의 적분

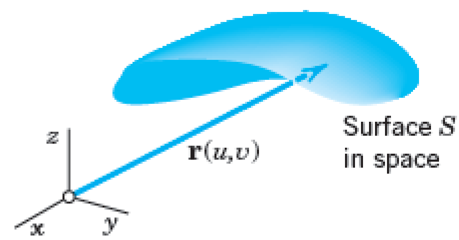
양 경계에서의 단위시간당 속도차
= 생성되는 유체의 양

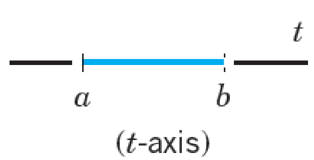
10.5 Surfaces for Surface Integrals

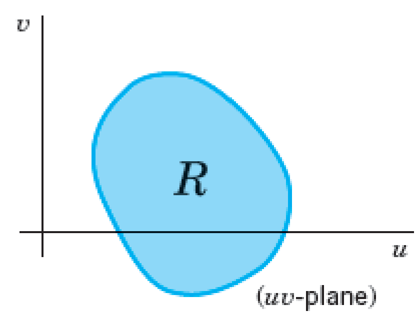
☑ Representation of Surfaces: $z = f(x, y)$ or $g(x, y, z) = 0$

- Curve *C*: $r = \mathbf{r}(t)$, where $a \le t \le b$
- Surface *S*: $\mathbf{r}(u,v) = [x(u,v)\ ,\ y(u,v)\ ,\ z(u,v)] = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$
  where (u,v) varies in some region *R* of the *uv*-plane









Parametric representation of a curve

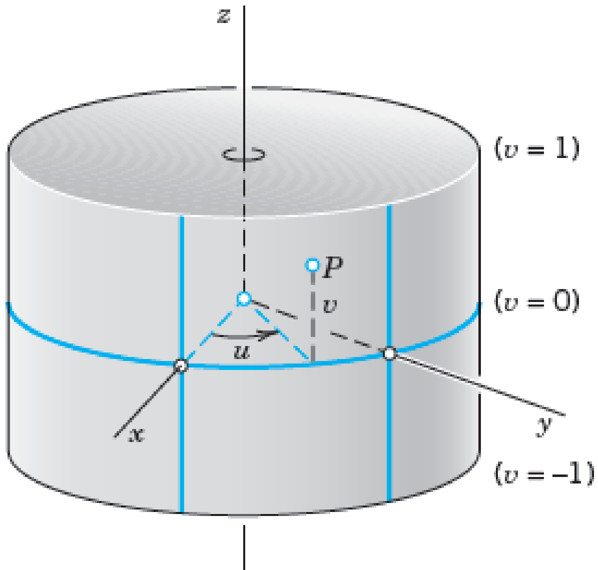
Parametric representation of a surface

☑ Ex. 1 Parametric Representation of a Cylinder

- The circular cylinder $x^2 + y^2 = a^2$, $-1 \le z \le 1$, has radius a, height 2, and the z-axis as axis
- Parametric representation:

  $$\mathbf{r}(u,v) = [a \cos u,\, a \sin u,\, v] = a \cos u\, \mathbf{i} + a \sin u\mathbf{j} + v\mathbf{k}$$

- The parameters u, v vary in the rectangle $R: 0 \le u \le 2\pi$, $-1 \le v \le 1$ in the uv-plane
- The components of $\mathbf{r}$ are $x = a \cos u$, $y = a \sin u$, $z = v$
- The curves $u = \text{const}$ are vertical straight lines.
- The curves $v = \text{const}$ are parallel circles.



Parametric representation of a cylinder

"cylinder surface를 u, v 사각형 구간으로 변환해야 2차원 적분이 쉽다."

10.5 Surfaces for Surface Integrals

☑ Ex. 2 Parametric Representation of a Sphere

A sphere $x^2 + y^2 + z^2 = a^2$ can be represented in the form

$\mathbf{r}(u,v) = a \cos v \cos u \,\mathbf{i} + a \cos v \sin u \,\mathbf{j} + a \sin v \,\mathbf{k}$

where the parameters u, v vary in the rectangle

$$R: 0 \le u \le 2\pi, \quad -\frac{\pi}{2} \le v \le \frac{\pi}{2}$$

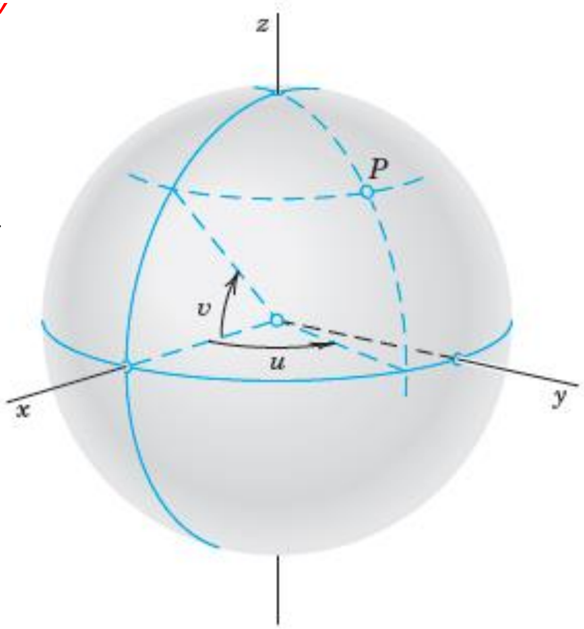
"u, v 사각형 구간"

Another parametric representation is

$\mathbf{r}(u,v) = a \cos u \sin v \,\mathbf{i} + a \sin u \sin v \,\mathbf{j} + a \cos v \,\mathbf{k}$

where

$$R: 0 \le u \le 2\pi, \quad 0 \le v \le \pi$$



Parametric representation of a sphere

10.5 Surfaces for Surface Integrals

☑ Tangent Plane and Surface Normal

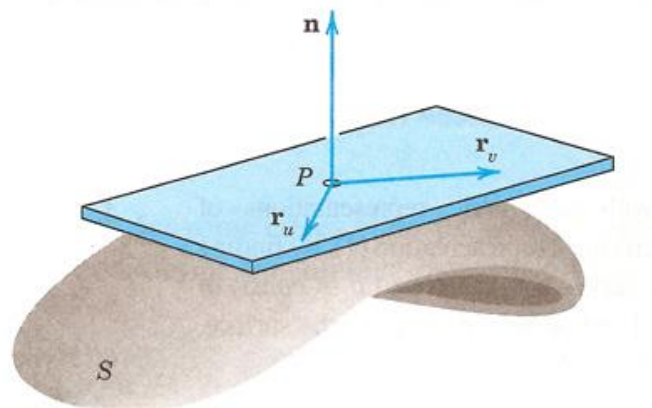
- **Tangent Plane:** A Plane which is formed by the tangent vectors of all the curves on a surface *S* through a point *P* of *S*
- Normal Vector: A vector perpendicular to the tangent plane
- *S*: $\mathbf{r} = \mathbf{r}(u, v)$ and *C*: $\tilde{\mathbf{r}}(t) = \mathbf{r}(u(t), v(t))$

scalar

✓ Tangent vector: $\tilde{\mathbf{r}}'(t) = \dfrac{d\tilde{\mathbf{r}}}{dt} = \dfrac{\partial \mathbf{r}}{\partial u}\dfrac{du}{dt} + \dfrac{\partial \mathbf{r}}{\partial v}\dfrac{dv}{dt} = \dfrac{\partial \mathbf{r}}{\partial u}\boxed{u'} + \dfrac{\partial \mathbf{r}}{\partial v}\boxed{v'}$

✓ $\dfrac{\partial \mathbf{r}}{\partial u}$: tangent vector along u direction at P of a curve $\mathbf{r}(u)$ when $v = \text{const}$ like $\mathbf{r}'(\text{t})$

✓ $\dfrac{\partial \mathbf{r}}{\partial v}$: tangent vector along v direction at P of a curve $\mathbf{r}(\text{v})$ when $v = \text{const like } \mathbf{r}'(\text{t})$



Tangent plane and normal vector

10.5 Surfaces for Surface Integrals

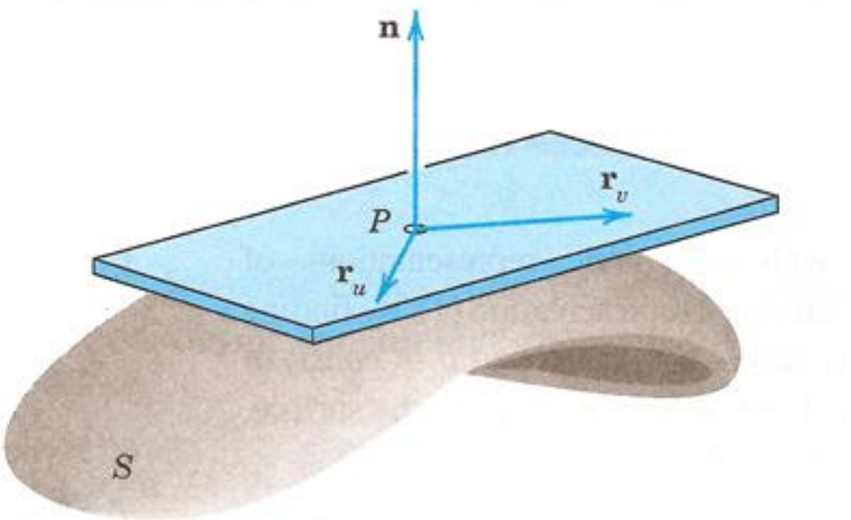
☑ Tangent Plane and Surface Normal

- **Tangent Plane:** A Plane which is formed by the tangent vectors of all the curves on a surface *S* through a point *P* of *S*
- Normal Vector: A vector perpendicular to the tangent plane
- Normal vector:

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$$

- Unit normal vector:

$$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N} = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\mathbf{r}_u \times \mathbf{r}_v$$



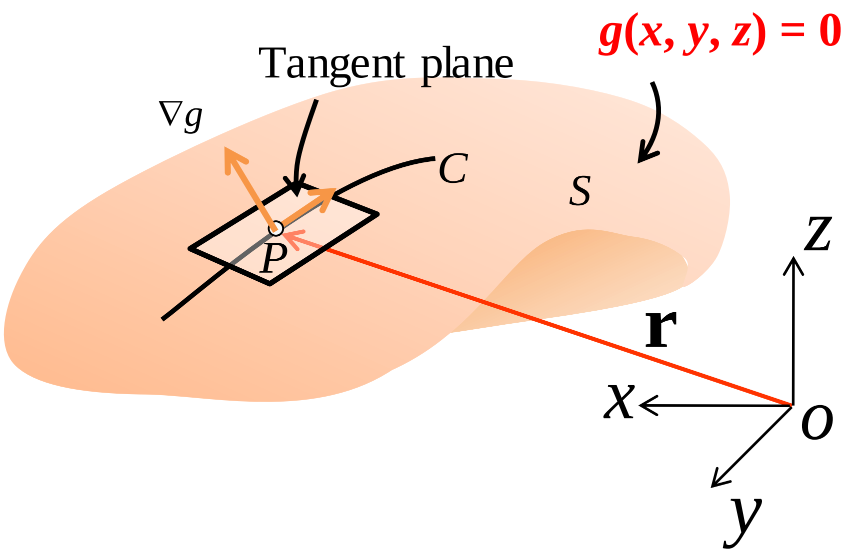
Tangent plane and normal vector

10.5 Surfaces for Surface Integrals

☑ Tangent Plane (접평면) and Surface Normal (곡면 법선)

- S is represented by $g(x, y, z) = 0$
- S is a smooth surface if its surface normal depends continuously on the point of S
- S is piecewise smooth if it consists of finitely many smooth portions.

$$\mathbf{n} = \frac{1}{|\operatorname{grad} g|} \operatorname{grad} g$$



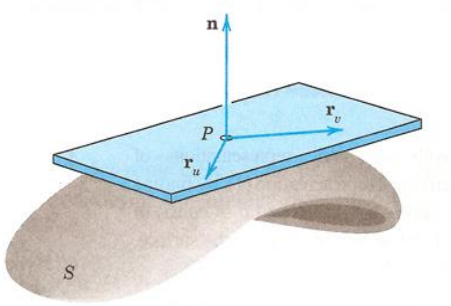
Normal vector for surface $g(x, y, z) = 0$

10.5 Surfaces for Surface Integrals

☑ Theorem 1 Tangent Plane and Surface Normal

- If a surface S is given by $\mathbf{r}(u,v)=[x(u,v)\,,\,y(u,v)\,,\,z(u,v)]$ with continuous $\mathbf{r}_u$ and $\mathbf{r}_v$ satisfying $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ at every point of S,
- then S has at every point P a **unique tangent plane** passing through P and spanned by $\mathbf{r}_u$ and $\mathbf{r}_v$,
- and a **unique normal** whose direction depends continuously on the points of S. A normal vector is given by $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ and the corresponding unit normal vector by

$$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N} = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\mathbf{r}_u \times \mathbf{r}_v$$



☑ Ex. 4 Unit Normal Vector of a Sphere

The sphere $g(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$ has the unit normal vector

$$\mathbf{n}(x,y,z) = \frac{1}{|\operatorname{grad} g|}\operatorname{grad} g = \left[\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right] = \frac{x}{a}\mathbf{i} + \frac{y}{a}\mathbf{j} + \frac{z}{a}\mathbf{k}$$

☑ Surface Integral

$$S : \mathbf{r}(u,v) = [x(u,v), y(u,v), z(u,v)] = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

- Normal vector: $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$
- Unit normal vector: $\mathbf{n} = \dfrac{1}{|\mathbf{N}|}\mathbf{N} = \dfrac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\mathbf{r}_u \times \mathbf{r}_v$
- **Surface integral over S:** $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{N}(u,v)\,du\,dv$

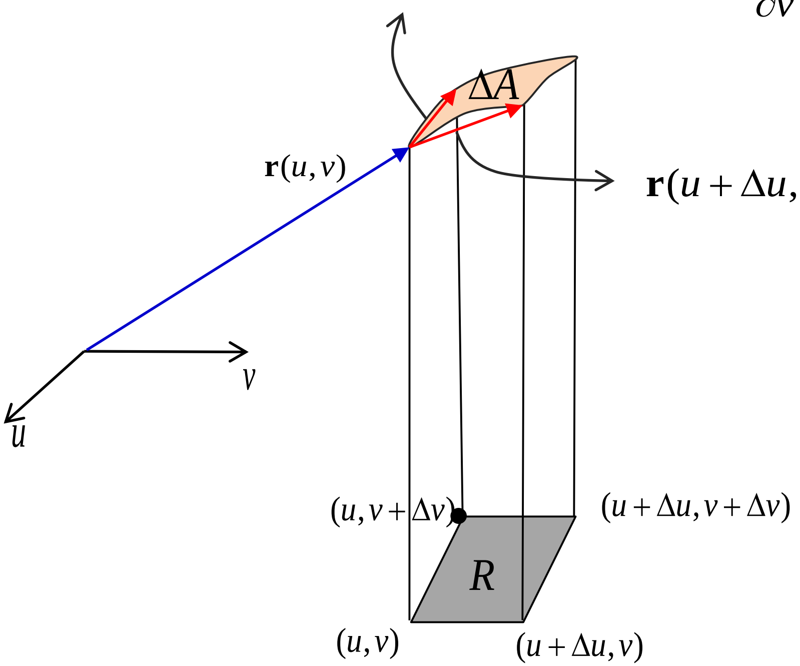
$|\mathbf{N}| = |\mathbf{r}_u \times \mathbf{r}_v|$: the area of the parallelogram with sides $\mathbf{r}_u$ and $\mathbf{r}_v$
(평행사변형)
$dA = |\mathbf{N}|\,du\,dv$

$$\therefore \mathbf{n}\,dA = \mathbf{n}|\mathbf{N}|\,du\,dv = \mathbf{N}\,du\,dv$$

10.6 Surface Integrals

$$\boxed{\mathbf{n}dA = \mathbf{n}\left|\mathbf{N}\right| dudv = \mathbf{N}dudv\ ?}$$

$$\mathbf{r}(u, v+\Delta v) - \mathbf{r}(u,v) = \frac{\partial \mathbf{r}}{\partial v}\Delta v$$



$$\mathbf{r}(u+\Delta u, v) - \mathbf{r}(u,v) = \frac{\partial \mathbf{r}}{\partial u}\Delta u$$

$$\Delta A = \left|\frac{\partial \mathbf{r}}{\partial u}\Delta u \times \frac{\partial \mathbf{r}}{\partial v}\Delta v\right| = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|\Delta u \Delta v$$

$$dA = \left|\mathbf{r}_u \times \mathbf{r}_v\right| dudv, \quad \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$$

$$\boxed{\therefore\ dA = \left|\mathbf{N}\right| dudv}$$

10.6 Surface Integrals

☑ Surface Integral (면적분)

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_R \mathbf{F}\big(\mathbf{r}(u,v)\big)\cdot\mathbf{N}(u,v)\,du\,dv$$

$\mathbf{F}\bullet\mathbf{n}$: the normal component of $\mathbf{F}$

When $\mathbf{F} = \rho\mathbf{v}$ (density x velocity vector of the flow)

⇨ flux across S = mass of fluid crossing S per unit time

- In components

Here, α, β, γ are the angles between **n** and the coordinate axes.

$$\mathbf{F} = [F_1, F_2, F_3], \quad \mathbf{N} = [N_1, N_2, N_3], \quad \mathbf{n} = [\cos\alpha, \cos\beta, \cos\gamma]$$

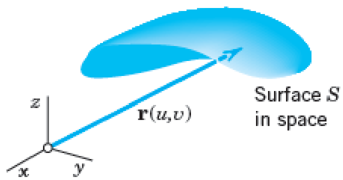
$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dA = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dA$$

$$= \iint_{\boxed{R}} (F_1N_1 + F_2N_2 + F_3N_3)\,du\,dv$$

$$\cos\alpha = \frac{\mathbf{n}\cdot\mathbf{i}}{|\mathbf{n}||\mathbf{i}|} = \mathbf{n}\cdot\mathbf{i} = n_1$$

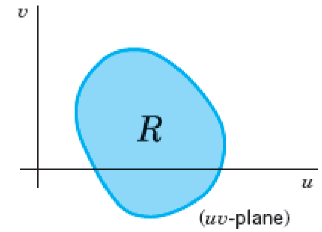
$$\cos\beta = \frac{\mathbf{n}\cdot\mathbf{j}}{|\mathbf{n}||\mathbf{j}|} = \mathbf{n}\cdot\mathbf{j} = n_2$$

$$\cos\gamma = \frac{\mathbf{n}\cdot\mathbf{k}}{|\mathbf{n}||\mathbf{k}|} = \mathbf{n}\cdot\mathbf{k} = n_3$$



$$\cos\alpha\,dA = dydz, \ \cos\beta\,dA = dzdx, \ \cos\gamma\,dA = dxdy$$

$$= \iint_{\boxed{S}} (F_1dydz + F_2dzdx + F_3dxdy)$$



10.6 Surface Integrals

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{N}(u,v)\,du\,dv$$

☑ Ex. 1 Flux Through a Surface

- Compute the flux of water through the parabolic cylinder: $S: y=x^2,\ 0 \le x \le 2,\ 0 \le z \le 3$
  velocity vector: $\mathbf{v} = \mathbf{F} = [3z^2, 6, 6xz]$ (m/sec)
  $\mathbf{F} = \rho\mathbf{v}$, the density ρ = 1gm/cm³ = 1ton/m³

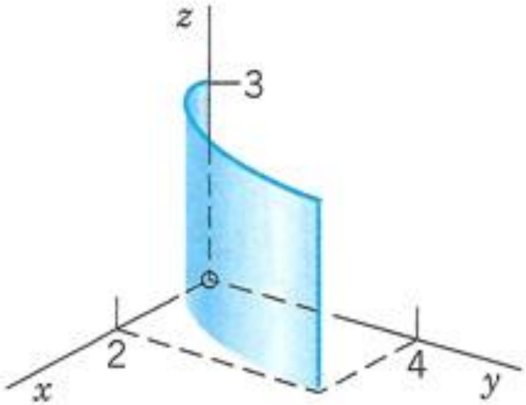
Sol) Representation S: $\mathbf{r} = [u, u^2, v]$ $(0 \le u \le 2,\ 0 \le v \le 3)$

By differentiation and by the definition of the cross product

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [1,\ 2u,\ 0]\times[0,\ 0,\ 1] = [2u,\ -1,\ 0]$$

$$\therefore\ \mathbf{F}(S)\bullet\mathbf{N} = 6uv^2 - 6$$

By integration



$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dA = \int_0^3\int_0^2 (6uv^2-6)\,du\,dv = \int_0^3 \left(3u^2v^2-6u\right)\Big|_{u=0}^{2} dv$$

$$= \int_0^3 (12v^2-12)\,dv = \left(4v^3-12v\right)\Big|_{v=0}^{3} = 72\left[\mathrm{m}^3/\mathrm{sec}\right]$$

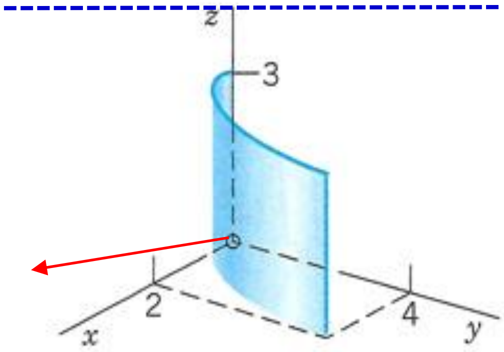
10.6 Surface Integrals

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{N}(u,v)\,du\,dv$$

☑ Ex. 1 Flux Through a Surface

- Compute the flux of water through
  the parabolic cylinder: $S: y=x^2,\ 0 \le x \le 2,\ 0 \le z \le 3$
  velocity vector: $\mathbf{v} = \mathbf{F} = [3z^2, 6, 6xz]$ (m/sec)
  $\mathbf{F} = \rho\mathbf{v}$, the density ρ = 1gm/cm³ = 1ton/m³

Sol) Representation S: $\mathbf{r} = [u, u^2, v]$ $(0 \le u \le 2,\ 0 \le v \le 3)$



$$\boxed{\iint_S \mathbf{F}\bullet\mathbf{n}\,dA = \iint_S \left(F_1\,dydz + F_2\,dzdx + \cancel{F_3\,dxdy}\right)}$$

$$\mathbf{N} = |\mathbf{N}|\mathbf{n} = |\mathbf{N}|[\cos\alpha, \cos\beta, \cos\gamma] = [2u, -1, 0],\ \cos\alpha > 0,\ \cos\beta < 0,\ \cos\gamma = 0$$

$-\pi/2<\alpha<0 \Rightarrow \cos\alpha>0,\qquad \pi/2<\beta<\pi \Rightarrow \cos\beta<0,\qquad \gamma=\pi/2 \Rightarrow \cos\gamma=0$

$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dA = \int_0^3\int_0^4 3z^2\,dydz - \int_0^2\int_0^3 6\,dzdx = \int_0^3 4(3z^2)\,dydz - \int_0^2 6\cdot 3\ dx = 72$$

10.6 Surface Integrals

☑ Ex. 2 Surface Integral

$\mathbf{F} = [x^2, 0, 3y^2]$, $x + y + z = 1$. Evaluate $\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \bullet \mathbf{N}(u,v)\, du\, dv$

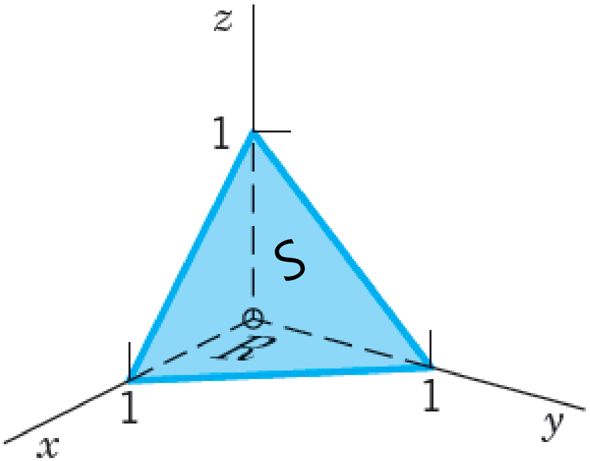
Sol) Representation S ⇨ Representation R (곡면 S의 xy-plane으로의 투영)

$x = u,\ y = v,$ ⇨ $z = 1 - x - y = 1 - u - v$

R

$\mathbf{r}(u, v) = [u, v, 1 - u - v]$

$0 \le u \le 1 - v,\ 0 \le v \le 1$



$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [1,\ 0,\ -1] \times [0,\ 1,\ -1] = [1,\ 1,\ 1]$$

$$\mathbf{F}(S) \bullet \mathbf{N} = [u^2, 0, 3v^2] \bullet [1,\ 1,\ 1] = u^2 + 3v^2$$

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iint_R (u^2 + 3v^2)\, du\, dv = \int_0^1 \int_0^{1-v} (u^2 + 3v^2)\, du\, dv$$

$$= \int_0^1 \left[\frac{1}{3}(1-v)^3 + 3v^2(1-v) \right] dv = \frac{1}{3}$$

10.6 Surface Integrals

☑ Orientation (방향) of Surfaces

- The value of the integral depends on the choice of the unit normal vector **n**.
- An oriented surface *S* (방향을 가진 곡면, 유향곡면): a surface *S* on which we have chosen one of the two possible unit normal vectors in a continuous fashion
- If we change the orientation of *S*, this means that we replace **n** with –**n.**

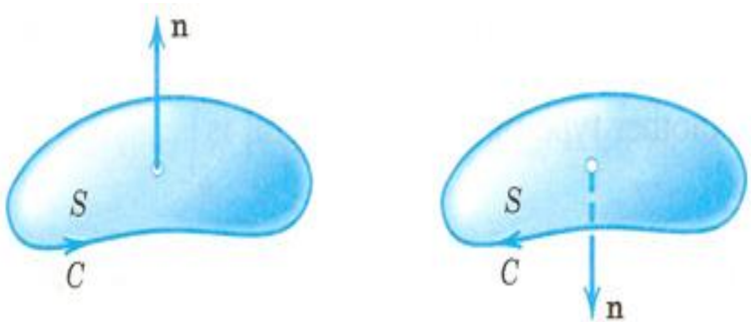
☑ Theorem 1 Change of Orientation in a Surface Integral

The replacement of **n** by **-n** corresponds to the multiplication of the integral by -1

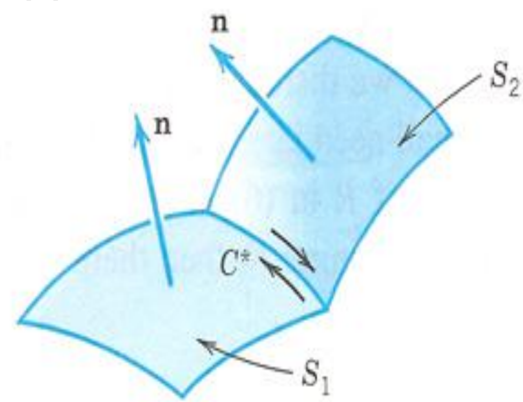
10.6 Surface Integrals

☑ Orientation of Piecewise Smooth Surfaces

- S is orientable (방향을 가질 수 있는) if the positive normal direction can be continued in a unique and continuous way to the entire surface.
- For a smooth orientable surface S with boundary curve *C* we may associate each of the two possible orientations of *S* with an orientation of *C*.
- A piecewise smooth surface is orientable (방향을 가질 수 있는) if we can orient each smooth piece of *S* so that along each curve C* which is a common boundary of two pieces S_1 and S_2.
- The positive direction of C^* relative to S_1 is opposite to the direction of C^* relative to S_2.



(A) Smooth surface

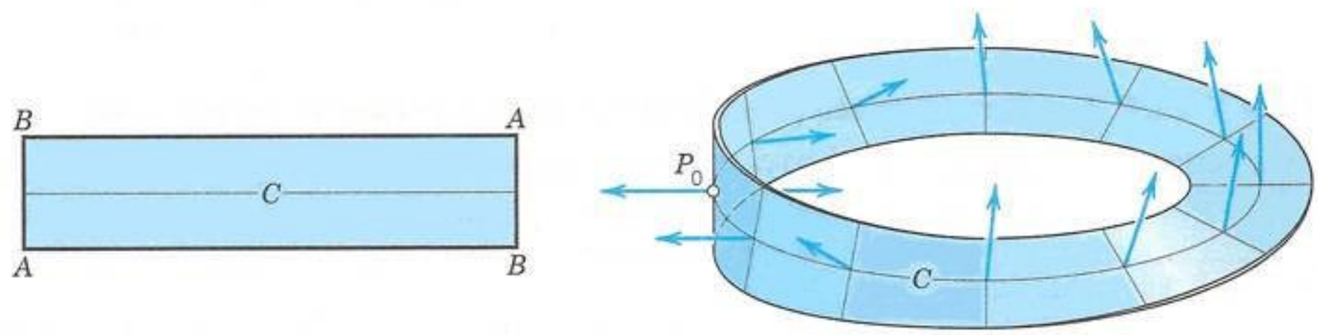


(B) Piecewise smooth surface

Orientation of a surface

☑ Nonorientable (방향을 가질 수 없는) Surfaces

- A sufficiently small piece of a smooth surface is always orientable. This may not hold for entire surfaces. Ex. Möbius strip



Möbius strip

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_R \mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{N}(u,v)\,du\,dv$$

☑ Surface Integrals Without Regard to Orientation

- Another type of surface integral disregarding the orientation

$$\iint_S G(\mathbf{r})\,dA = \iint_R G(\mathbf{r}(u,v))\left|\mathbf{N}(u,v)\right|du\,dv$$

Here $dA = |\mathbf{N}|\,du\,dv = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$ is the element of area of *S*.

- Mean value theorem for surface integrals

If *R* is simply connected and **G**(*r*) is continuous in a domain containing *R*, then there is a point in R such that

$$\iint_S G(\mathbf{r})\,dA = G(\mathbf{r}(u_0,v_0))\,A \qquad (\text{A: Area of } S)$$

- Area of A: $A(S) = \iint_S dA = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$

☑ Ex. 4 Area of a Sphere (구의 겉넓이)

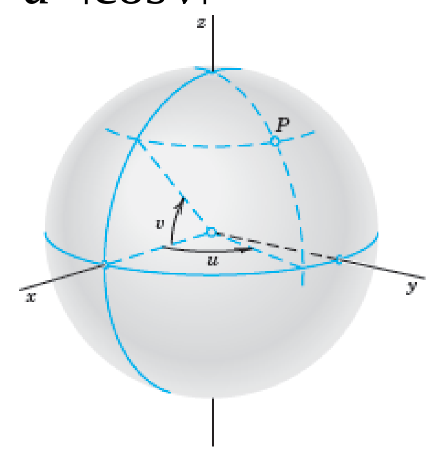
For a sphere $\mathbf{r}(u, v) = [a \cos v \cos u, a \cos v \sin u, a \sin v]$,

Sol) $0 \le u \le 2\pi$, $-\pi/2 \le v \le \pi/2$, we obtain by direct calculation

$$\mathbf{r}_u \times \mathbf{r}_v = \left[a^2 \cos^2 v \cos u, \quad a^2 \cos^2 v \sin u, \quad a^2 \cos v \sin v \right]$$

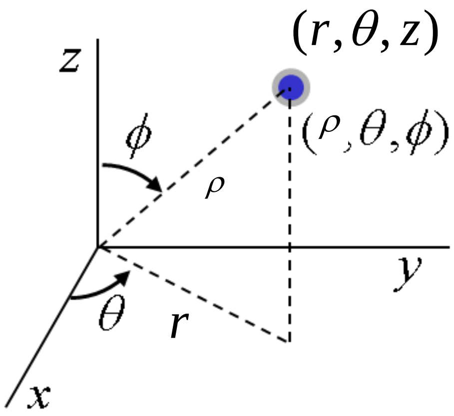
Using $\cos^2 u + \sin^2 u = 1$, $\cos^2 v + \sin^2 v = 1$

$$\left| \mathbf{r}_u \times \mathbf{r}_v \right| = a^2 \left(\cos^4 v \cos^2 u + \cos^4 v \sin^2 u + \cos^2 v \sin^2 v \right)^{1/2} = a^2 \left| \cos v \right|$$



$$\therefore A(S) = a^2 \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \left| \cos v \right| du dv = 2\pi a^2 \int_{-\pi/2}^{\pi/2} \cos v dv = 4\pi a^2$$

10.6 Surface Integrals



Spherical coordinates

Spherical → Cylindrical

$$r = \rho \sin\phi \qquad \theta = \theta \qquad z = \rho\cos\phi$$

Spherical → Cartesian

$$x = r\sin\phi\cos\theta \qquad y = r\sin\phi\sin\theta \qquad z = r\cos\phi$$

Cartesian → Spherical

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\iint_S G(\mathbf{r})dA = \iint_R G(\mathbf{r}(u,v))|\mathbf{N}(u,v)|dudv$$

☑ Representations $z = f(x, y)$

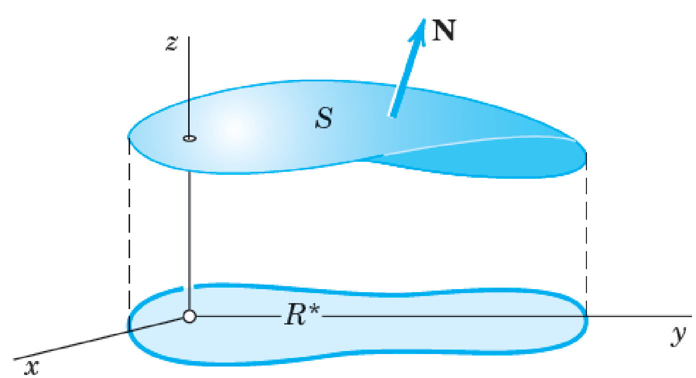
- If a surface S is given by $z = f(x, y)$

$$|\mathbf{N}| = |\mathbf{r}_u \times \mathbf{r}_v| = \|[1, 0, f_u] \times [0, 1, f_v]\| = \|[-f_u, -f_v, 1]\| = \sqrt{1 + f_u^{\ 2} + f_v^{\ 2}}$$

- Surface integral: $$\iint_S G(\mathbf{r})dA = \iint_{R^*} G(x, y, f(x, y))\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy$$

Here, R^*: projection of S into the xy-plane

- Area: $$A(S) = \iint_{R^*} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy$$

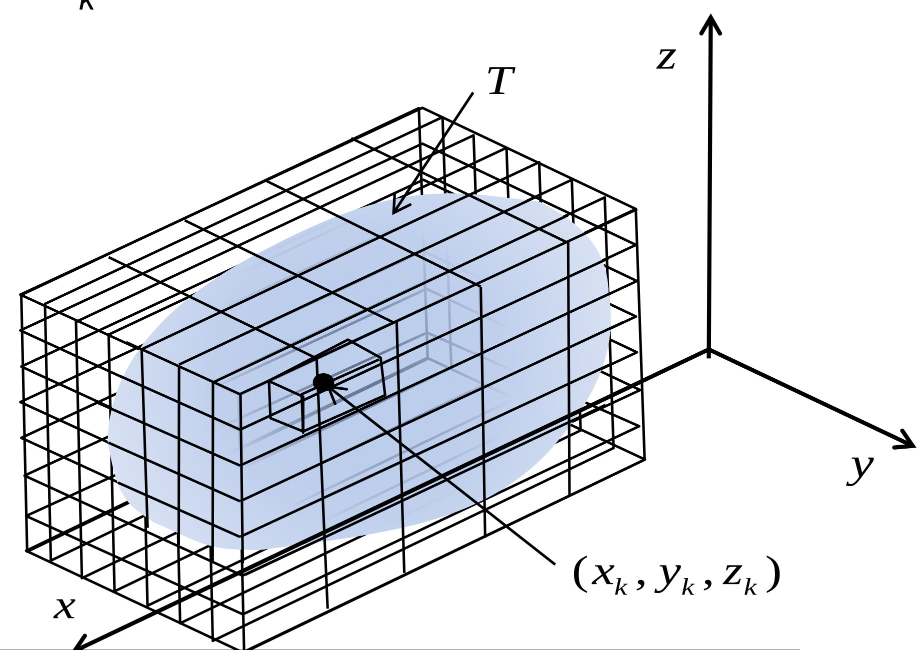


10.7 Triple Integrals. Divergence Theorem of Gauss

☑ Triple integral for an integral of a function $f(x, y, z)$

- We subdivide T by planes parallel to the coordinate planes.
- We consider those boxes of the subdivision that lie entirely inside T, and number them from 1 to n.
- In each such box we choose an arbitrary point, say, (x_k, y_k, z_k) in box k.
- The maximum length of all edges of those n boxes approaches zero as n approaches infinity.
- The volume of box k we denote by ΔV_k. We now form the sum

$$J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta V_k$$



10.7 Triple Integrals. Divergence Theorem of Gauss

☑ Theorem 1 Divergence Theorem of Gauss (발산이론)

Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $\mathbf{F}(x,y,z)$ be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then

$$\iiint_T \operatorname{div} \mathbf{F} dV = \iint_S \mathbf{F} \bullet \mathbf{n} dA$$

In components of $\mathbf{F}=[F_1, F_2, F_3]$ and of the outer unit normal vector $\mathbf{n} = [\cos\alpha, \quad \cos\beta, \quad \cos\gamma]$ of S, formula becomes

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_S \left(F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma \right) dA = \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy \right)$$

Proof)

$$\iiint_T \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dxdydz=\iint_S (F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma)dA$$

$$=\iint_S (F_1 dydz+F_2 dzdx+F_3 dxdy)$$

This equation is true if and only if the integrals of each component on both sides are equal

(3) $$\iiint_T \frac{\partial F_1}{\partial x}dxdydz=\iint_S F_1\cos\alpha\, dA=\iint_S F_1 dydz$$

(4) $$\iiint_T \frac{\partial F_2}{\partial y}dxdydz=\iint_S F_2\cos\beta\, dA=\iint_S F_2 dxdz$$

(5) $$\iiint_T \frac{\partial F_3}{\partial z}dxdydz=\iint_S F_3\cos\gamma\, dA=\iint_S F_3 dxdy$$

Proof continued)

(5) $$\iiint_T \frac{\partial F_3}{\partial z}\,dxdydz = \iint_S F_3 \cos\gamma \, dA$$

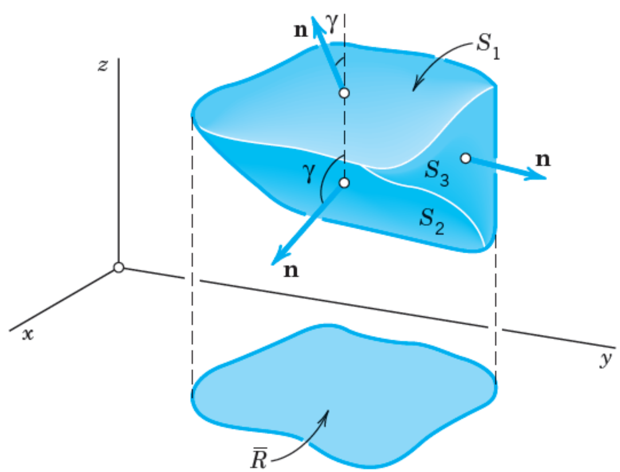
We first prove (5) for a special region *T* that is bounded by a piecewise smooth orientable surface S and has the property that any straight line parallel to any one of the coordinate axes and intersecting *T* has at most one segment (or a single point)

It implies that *T* can be represented in the form

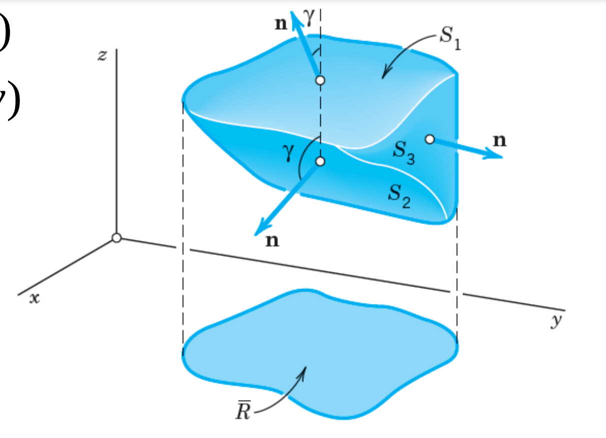
(6) $$g(x, y) \le z \le h(x, y)$$

$S_1 : h(x, y)$

$S_2 : g(x, y)$



10.7 Triple Integrals. Divergence Theorem of Gauss



Proof continued)

(5) $$\iiint_T \frac{\partial F_3}{\partial z} dxdydz = \iint_S F_3 \cos\gamma \, dA$$

$$\iiint_T \frac{\partial F_3}{\partial z} dxdydz = \iint_{\bar{R}} \left[\int_{g(x,y)}^{h(x,y)} \frac{\partial F_3}{\partial z} dz \right] dxdy$$

$$= \iint_{\bar{R}} F_3[x,y,h(x,y)]dxdy - \iint_{\bar{R}} F_3[x,y,g(x,y)]dxdy$$

We can decide the sign of the integral because $\cos\gamma < 0$ **on S_2, and** $\cos\gamma > 0$ **on S_1**

$$\iint_S F_3 \cos\gamma dA = \iint_S F_3 dxdy = +\iint_{\bar{R}} F_3[x,y,h(x,y)]dxdy - \iint_{\bar{R}} F_3[x,y,g(x,y)]dxdy$$

Therefore, we prove (5). In the same manner, (3), (4) can be proven.

$$\iiint_T \frac{\partial F_3}{\partial z} dxdydz = \iint_S F_3 \cos\gamma \, dA$$

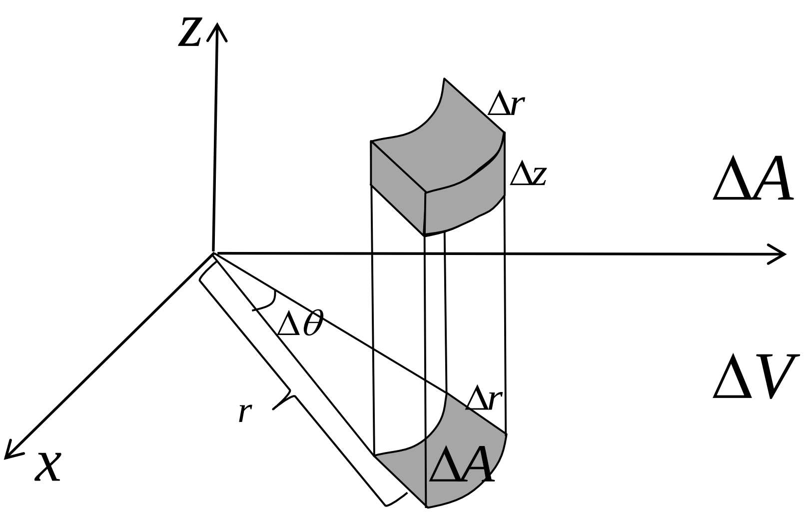
10.7 Triple Integrals. Divergence Theorem of Gauss

☑ Ex. 1 Evaluation of a Surface Integral by the Divergence Theorem

Evaluate $I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdxdy \right)$ where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ $(0 \le z \le b)$ and the circular disks $z = 0$ and $z = b$ $(x^2 + y^2 \le a^2)$.

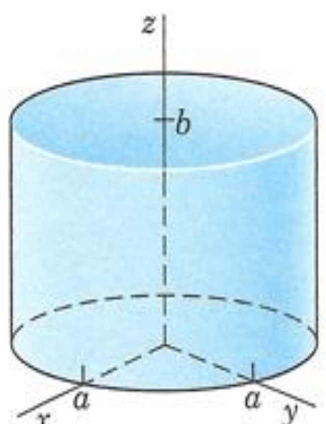
Sol) $F_1 = x^3,\ F_2 = x^2 y,\ F_3 = x^2 z \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = 3x^2 + x^2 + x^2 = 5x^2$

Polar coordinates ($dxdydz = rdrd\theta dz$)



$$\Delta A = (r\Delta\theta)\cdot\Delta r$$

$$\Delta V = \Delta A\,\Delta z = r\Delta r\Delta\theta\,\Delta z$$



Surface S in Example 1

☑ Ex. 1 Evaluation of a Surface Integral by the Divergence Theorem

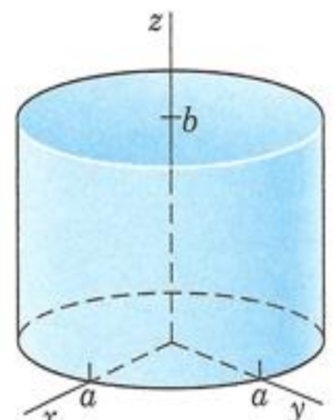
Evaluate $I = \iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdxdy \right)$ where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ $(0 \le z \le b)$ and the circular disks $z = 0$ and $z = b$ $(x^2 + y^2 \le a^2)$.

Sol) $F_1 = x^3, \ F_2 = x^2 y, \ F_3 = x^2 z \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = 3x^2 + x^2 + x^2 = 5x^2$

Polar coordinates ($dxdydz = rdrd\theta dz$)

$$I = \iiint_T 5x^2 dxdydz = \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \left(5r^2 \cos^2 \theta\right) rdrd\theta dz$$

$$= 5 \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \frac{a^4}{4} \cos^2 \theta d\theta dz = 5 \int_{z=0}^{b} \frac{a^4 \pi}{4} dz = \frac{5\pi}{4} a^4 b$$



Surface S in Example 1

☑ Coordinate Invariance of the Divergence (발산의 좌표계 불변)

- Mean value theorem for triple integrals

For any continuous function $f(x, y, z)$ in a bounded and simply connected region *T* there is a point $Q{:}(x_0, y_0, z_0)$ in *T* such that

$$\iiint_T f(x,y,z)\,dV = f(x_0,y_0,z_0)\,V(T) \qquad (V(T) = \text{volume of } T\,)$$

$$\text{set } f = \operatorname{div}\mathbf{F} \quad \Rightarrow \quad \operatorname{div}\mathbf{F}(x_0,y_0,z_0) = \frac{1}{V(T)}\iiint_T \operatorname{div}\mathbf{F}\,dV = \frac{1}{V(T)}\iint_{S(T)} \mathbf{F}\bullet\mathbf{n}\,dA$$

- Choose a point $P{:}(x_1, y_1, z_1)$ in *T* and let *T* shrink down onto *P* such that maximum distance *d*(*T*) of the points of *T* from *P* goes to zero.
- Then $Q{:}(x_0, y_0, z_0)$ must approach *P*.

$$\operatorname{div}\mathbf{F}(P) = \lim_{d(T)\to 0}\frac{1}{V(T)}\iint_{S(T)} \mathbf{F}\bullet\mathbf{n}\,dA$$

☑ Theorem 2 Invariance of the Divergence

- The divergence of a vector function *F* with continuous first partial derivatives in a region *T* is independent of the particular choice of Cartesian coordinates. For any *P* in *T* it is given by

$$\operatorname{div}\mathbf{F}(P) = \lim_{d(T)\to 0} \frac{1}{V(T)} \iint_{S(T)} \mathbf{F} \bullet \mathbf{n}\, dA$$

⇨ *Definition of the divergence*

10.8 Further Applications of the Divergence Theorem

☑ Ex. 1 Fluid Flow. Physical Interpretation of the Divergence

- An intuitive interpretation of the divergence of a vector

- The flow of an incompressible fluid of constant density ρ = 1 which is steady (does not vary with time).
- Such a flow is determined by the field of its velocity vector **v**(P) at any point P.
- Let S be the boundary surface of a region T in space, and **n** be the outer unit normal vector of S.

✓ The total mass of fluid that flows across S from T to the outside per unit time

$$\iint_S \mathbf{v} \bullet \mathbf{n} dA$$

✓ The average flow out of T: $\frac{1}{V}\iint_S \mathbf{v} \bullet \mathbf{n} dA$

10.8 Further Applications of the Divergence Theorem

☑ Ex. 1 Fluid Flow. Physical Interpretation of the Divergence

- The flow is steady and the fluid is incompressible

⇨ the amount of fluid flowing outward must be continuously supplied.

$\frac{1}{V}\iint_S \mathbf{v}\bullet\mathbf{n}dA \neq 0$ ⇒ there must be sources in T, that is, points where fluid is produced or disappears.

- Let T shrink down to a fixed point *P* in T, we obtain the source intensity at P

$$\operatorname{div}\mathbf{v}(P) = \lim_{d(T)\to 0}\frac{1}{V(T)}\iint_{S(T)} \mathbf{v}\bullet\mathbf{n}dA$$

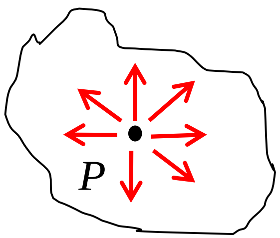
⇨ The divergence of the velocity vector **v** of a steady incompressible flow is the source intensity (생성 강도) of the flow at the corresponding point.

- If no sources in T $\iint_S \mathbf{v}\bullet\mathbf{n}dA = 0$

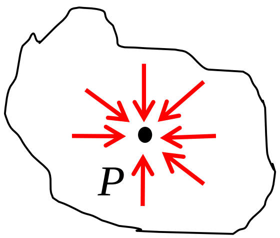
[Reference] Source & Sink

$\nabla \cdot \mathbf{F} = 0$: incompressible flow

$\nabla \cdot \mathbf{F} \neq 0$: compressible flow



Source
: Net outward flow
$(\text{div } \mathbf{F}(P) > 0)$

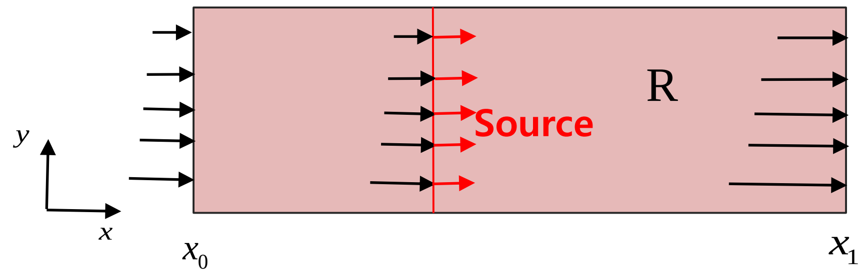


Sink
: Net inward flow
$(\text{div } \mathbf{F}(P) < 0)$

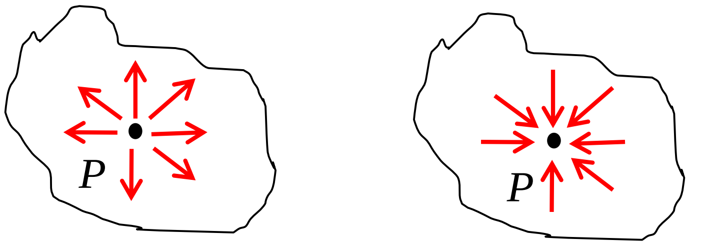
Generate a body shape by using Source and Sink

속도u의 변화량

$$u(x_0, y) = \frac{\partial \phi}{\partial x}(x_0, y) \qquad \frac{\partial u}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} \qquad u(x_1, y) = \frac{\partial \phi}{\partial x}(x_1, y)$$



[Reference] Source & Sink

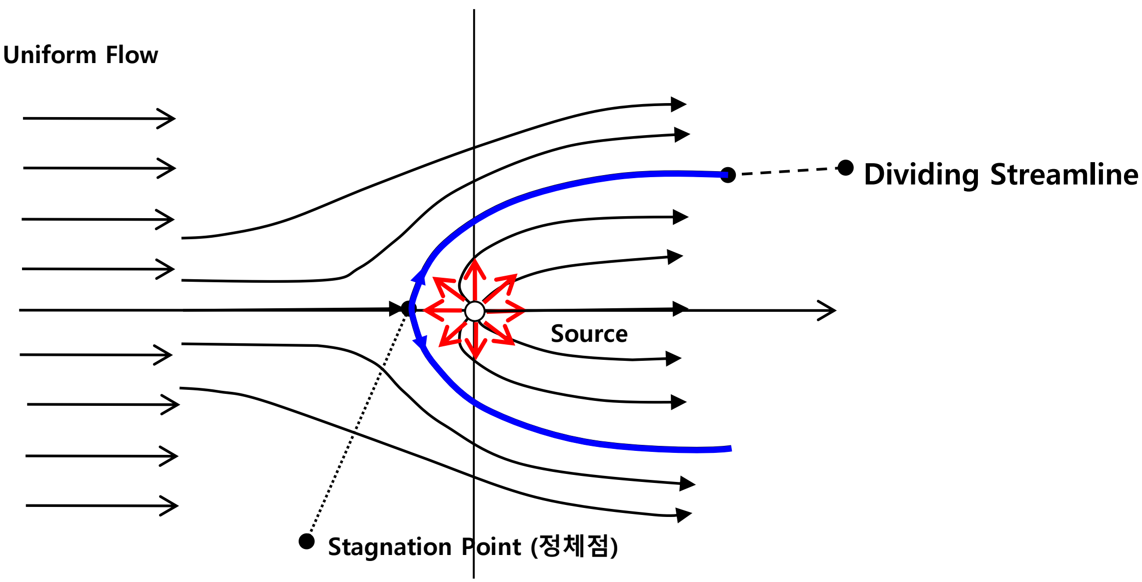


Source
: **Net outward flow**
$(\text{div } \mathbf{F}(P) > 0)$

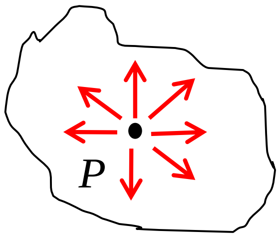
Sink
: **Net inward flow**
$(\text{div } \mathbf{F}(P) < 0)$

Generate a body-like shape by using Source and Sink

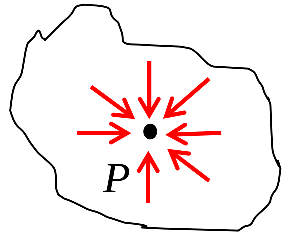
Half Body: Uniform Flow + Source



[Reference] Source & Sink



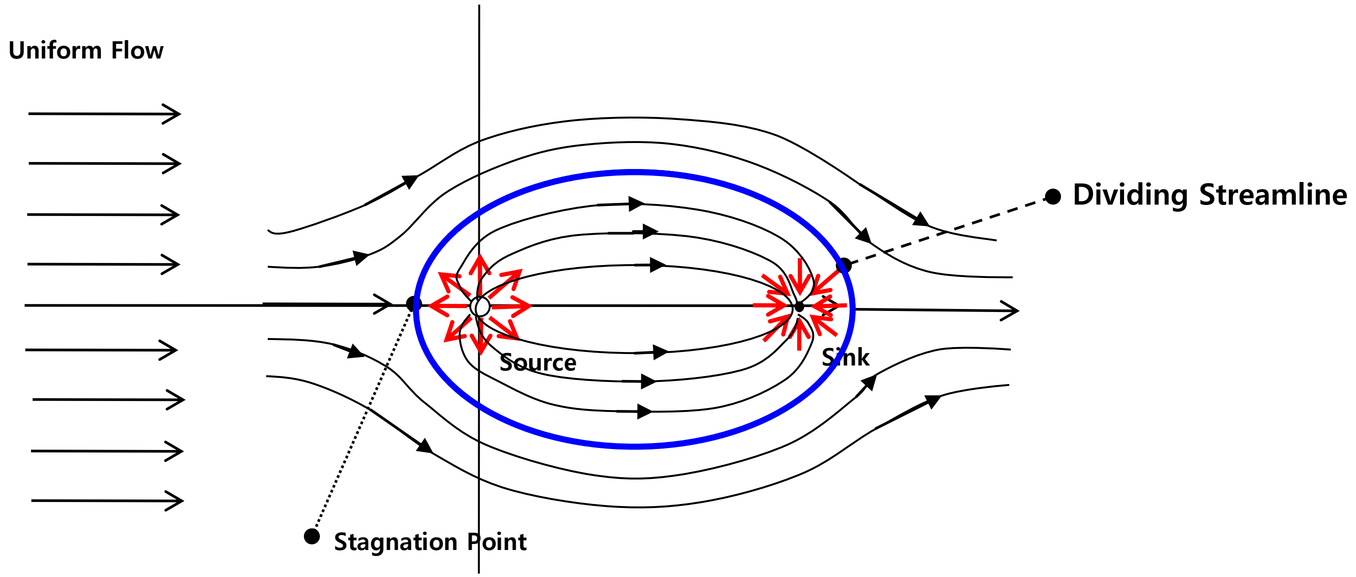
Source
: Net outward flow
$(\mathrm{div}\ \mathbf{F}(P) > 0)$



Sink
: Net inward flow
$(\mathrm{div}\ \mathbf{F}(P) < 0)$

Generate a body-like shape by using Source and Sink

Rankine Ovoid: Uniform Flow + Source + Sink



10.8 Further Applications of the Divergence Theorem

☑ Potential Theory. Harmonic Functions (조화함수)

- Laplace's equation: $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

- Potential theory: The theory of solutions of Laplace's equation

- Harmonic function

  : A solution of Laplace's equation with continuous second-order partial derivatives

☑ Theorem 1 A Basic Property of Harmonic Functions

Let $f(x,y,z)$ be a harmonic function in some domain *D* is space. Let *S* be any piecewise smooth closed orientable surface in *D* whose entire region it encloses belongs to *D*. Then the integral of the normal derivative of *f* taken over *S* is zero.

10.8 Further Applications of the Divergence Theorem

☑ Ex. 4 Green's Theorems

Let f and g be scalar functions such that $\mathbf{F} = f \,\mathrm{grad}\, g$ satisfies the assumptions of the divergence theorem in some region T. Then

$$\mathrm{div}\,\mathbf{F} = \mathrm{div}\left(f\,\mathrm{grad}\,g\right) = \mathrm{div}\left(\left[f\frac{\partial g}{\partial x},\ f\frac{\partial g}{\partial y},\ f\frac{\partial g}{\partial z}\right]\right)$$

$$= \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2}\right) + \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial y^2}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial^2 g}{\partial z^2}\right) = f\ \nabla^2 g + \mathrm{grad}\,f \bullet \mathrm{grad}\,g$$

Divergence theory

$$\iiint_T \mathrm{div}\,\mathbf{F}\,dV = \iint_S \boxed{\mathbf{F}\bullet\mathbf{n}}\,dA \quad\Rightarrow\quad \mathbf{F}\bullet\mathbf{n} = \mathbf{n}\bullet\mathbf{F} = \mathbf{n}\bullet\left(f\,\mathrm{grad}\,g\right) = \left(\mathbf{n}\bullet\mathrm{grad}\,g\right)f$$

- Green's first formula:

$$\mathbf{n}\bullet\mathrm{grad}\,g = \frac{\partial g}{\partial n}\ \text{(directional derivative)} \Rightarrow \iiint_T \left(f\ \nabla^2 g + \mathrm{grad}\,f \bullet \mathrm{grad}\,g\right)dV = \iint_S f\,\frac{\partial g}{\partial n}\,dA$$

- **For $\mathbf{F} = g\ \mathrm{grad}\ f$**

$$\Rightarrow \iiint_T \left(g\ \nabla^2 f + \mathrm{grad}\,g \bullet \mathrm{grad}\,f\right)dV = \iint_S g\,\frac{\partial f}{\partial n}\,dA$$

- Green's second formula:

$$\iiint_T \left(f\ \nabla^2 g - g\nabla^2 f\right)dV = \iint_S \left(f\,\frac{\partial g}{\partial n} - g\,\frac{\partial f}{\partial n}\right)dA$$

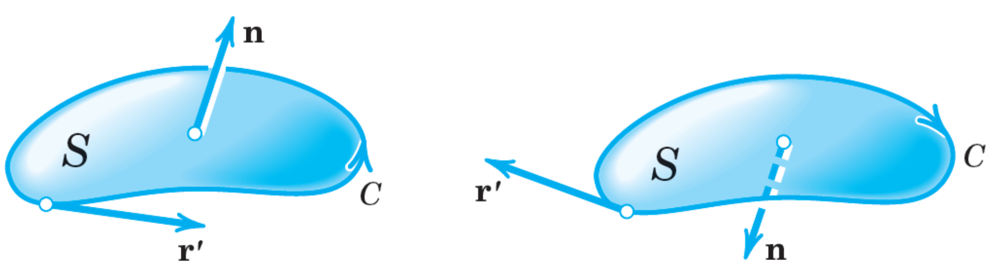
10.9 Stokes's Theorem

☑ Theorem 1 Stokes's Theorem

- *S*: a piecewise smooth oriented surface in space
  the boundary of *S* be a piecewise smooth simple closed curve C.
- *F*(*x*,*y*,*z*): a continuous vector function that has continuous first partial derivatives in a domain in space containing *S*.

$$\iint_S (\operatorname{curl} \mathbf{F}) \bullet \mathbf{n}\, dA = \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C \mathbf{F} \bullet \mathbf{r}'(s)\, ds$$

✓ Here $\mathbf{n}$: a unit normal vector of *S*
✓ **r**' = d**r**/ds is the unit tangent vector
✓ *s*: the arc length of *C*



10.9 Stokes's Theorem

☑ Theorem 1 Stokes's Theorem

- In components, formula becomes

$$\iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) N_3 \right] du\,dv = \oint_{\bar{C}} \left(F_1 dx + F_2 dy + F_3 dz \right)$$

- Here

$$\mathbf{F} = [F_1, F_2, F_3],\ \mathbf{N} = [N_1, N_2, N_3],\ \mathbf{n}dA = \mathbf{N}du\,dv,\ \mathbf{r}'ds = [dx, dy, dz]$$

- *R* is the region with boundary curve $\bar{C}$ in the *uv*-plane corresponding to *S* represented by **r**(*u*,*v*).

10.9 Stokes's Theorem

☑ Green's Theorem: Double Integrals ⇔ Line Integrals

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C \left(F_1 dx + F_2 dy \right)$$

☑ Gauss's Theorem (Divergence Theorem): Triple Integrals ⇔ Surface Integrals

$$\iiint_T \operatorname{div} \mathbf{F} dV = \iint_S \mathbf{F} \bullet \mathbf{n} dA$$

☑ Stokes's Theorem: Surface Integrals ⇔ Line Integrals (Generalization of Green's Theorem in the Plane)

$$\iint_S (\operatorname{curl} \mathbf{F}) \bullet \mathbf{n} dA = \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C \mathbf{F} \bullet \mathbf{r}'(s) ds$$

☑ Ex. 1 Verification of Stokes's Theorem

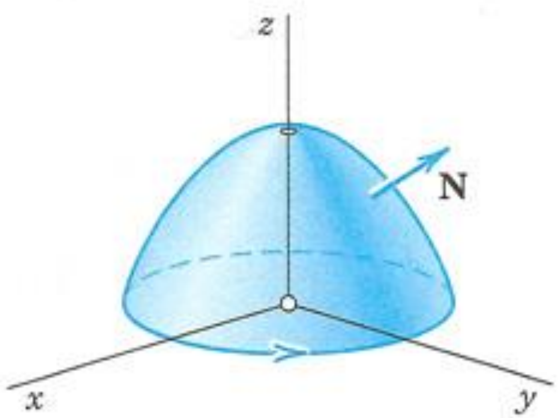
Let us first get used to it by verifying it for $\mathbf{F} = [y, z, x]$ and S the paraboloid $z = f(x,y) = 1 - (x^2 + y^2)$, $z \geq 0$

Case 1. The curve C is the circle $\mathbf{r}(s) = [\cos s, \sin s, 0]$

Its unit tangent vector: $\mathbf{r}'(s) = [-\sin s, \cos s, 0]$

The function F on C: $\mathbf{F}(\mathbf{r}(s)) = [\sin s, 0, \cos s]$

$$\therefore \oint_C \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(s)) \bullet \mathbf{r}'(s) ds = \int_0^{2\pi} \left[(\sin s)(-\sin s + 0 + 0) \right] ds = -\pi$$



Surface S

$$\iint_S (\operatorname{curl} \mathbf{F}) \bullet \mathbf{n} \, dA = \oint_C \mathbf{F} \bullet \mathbf{r}'(s) \, ds$$

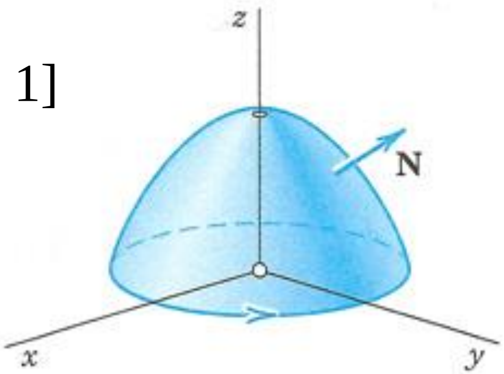
10.9 Stokes's Theorem

☑ Ex. 1 Verification of Stokes's Theorem

Let us first get used to it by verifying it for $\mathbf{F} = [y, z, x]$ and S the paraboloid $z = f(x,y) = 1 - (x^2 + y^2)$, $z \geq 0$

Case 2. The surface integral

$F_1 = y$, $F_2 = z$, $F_3 = x \Rightarrow \text{curl}\,\mathbf{F} = \text{curl}[F_1, F_2, F_3] = \text{curl}[y, z, x] = [-1, -1, -1]$



A normal vector of S: $\mathbf{N} = \text{grad}(z - f(x, y)) = [2x, 2y, 1]$

$(\text{curl}\,\mathbf{F})\bullet\mathbf{N} = -2x - 2y - 1$

$$\therefore \iint_S (\text{curl}\,\mathbf{F}) \bullet \mathbf{n}\,dA = \iint_R (\text{curl}\,\mathbf{F}) \bullet \mathbf{N}\,dxdy = \iint_R (-2x - 2y - 1)\,dxdy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \left(-2r(\cos\theta + \sin\theta) - 1\right) r\,dr\,d\theta = \int_{\theta=0}^{2\pi} \left(-\frac{2}{3}(\cos\theta + \sin\theta) - \frac{1}{2}\right) d\theta = 0 + 0 - \frac{1}{2}(2\pi) = -\pi$$

10.9 Stokes's Theorem

☑ Proof

$$\iint_R \left[\left(\boxed{\frac{\partial F_3}{\partial y}} - \boxed{\frac{\partial F_2}{\partial z}} \right) N_1 + \left(\boxed{\frac{\partial F_1}{\partial z}} - \boxed{\frac{\partial F_3}{\partial x}} \right) N_2 + \left(\boxed{\frac{\partial F_2}{\partial x}} - \boxed{\frac{\partial F_1}{\partial y}} \right) N_3 \right] du\,dv = \oint_{\bar{C}} \left(F_1 dx + F_2 dy + F_3 dz \right)$$

- If the integrals of each component on both sides are equal

$$\iint_R \left(\boxed{\frac{\partial F_1}{\partial z} N_2} \boxed{- \frac{\partial F_1}{\partial y} N_3} \right) du\,dv = \oint_{\bar{C}} F_1 dx$$

$$\iint_R \left(\boxed{-\frac{\partial F_2}{\partial z} N_1} \boxed{+ \frac{\partial F_2}{\partial x} N_3} \right) du\,dv = \oint_{\bar{C}} F_2 dx$$

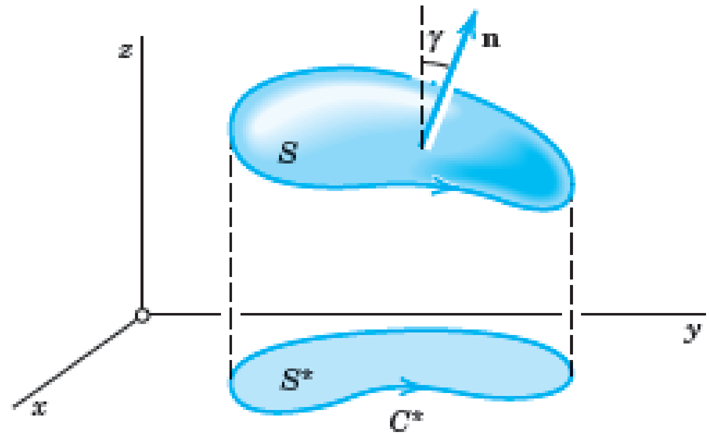
$$\iint_R \left(\boxed{\frac{\partial F_3}{\partial y} N_1} \boxed{- \frac{\partial F_3}{\partial x} N_2} \right) du\,dv = \oint_{\bar{C}} F_3 dx$$

10.9 Stokes's Theorem

☑ Proof

$$\iint_R \left(\frac{\partial F_1}{\partial z} N_2 - \frac{\partial F_1}{\partial y} N_3 \right) dudv = \oint_{\bar{C}} F_1 dx$$

$z = f(x,y)$
$u = x,\ v = y$
$\mathbf{r}(u,v) = \mathbf{r}(x, y) = [x\ ,\ y\ ,\ f(x, y)]$
$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \mathbf{r}_x \times \mathbf{r}_y = [-f_x, -f_y, 1]$



$$= \iint_{S^*} \left(\frac{\partial F_1}{\partial z} (-f_y) - \frac{\partial F_1}{\partial y} \right) dxdy$$

Green Theorem

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C (F_1 dx + F_2 dy)$$

$$\oint_{C^*} F_1 dx = \iint_{S^*} -\frac{\partial F_1}{\partial y} dxdy$$

meanwhile, by chain rule, $-\dfrac{\partial F_1(x, y, f(x, y))}{\partial y} = -\dfrac{\partial F_1(x, y, z)}{\partial y} - \dfrac{\partial F_1(x, y, z)}{\partial z} \boxed{\dfrac{\partial f}{\partial y}}$

$= f_y$

$$\therefore \oint_{C^*} F_1 dx = \iint_{S^*} -\frac{\partial F_1}{\partial y} dxdy = \iint_{S^*} \left(\frac{\partial F_1}{\partial z} (-f_y) - \frac{\partial F_1}{\partial y} \right) dxdy$$

10.9 Stokes's Theorem

☑ Ex 2 Green's Theorem in the Plane (z = 0) as a Special Case of Stokes's Theorem

- $\mathbf{F} = [F_1, F_2]$: continuously differentiable in a domain in the xy-plane containing a simply connected bounded closed region *S* whose boundary *C* is a piecewise smooth simple closed curve.

$$\therefore \quad (\operatorname{curl}\mathbf{F})\bullet\mathbf{n} = (\operatorname{curl}\mathbf{F})\bullet\mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$\iint_S (\operatorname{curl}\mathbf{F})\bullet\mathbf{n}\,dA = \oint_C \mathbf{F}\bullet\mathbf{r}'(s)\,ds = \oint_C (F_1dx + F_2dy + F_3dz)$$

$$\iint_S (\operatorname{curl}\mathbf{F})\bullet\mathbf{n}\,dA = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dA = \oint_C (F_1dx + F_2dy)$$

- The same as Green Theorem

Green Theorem

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy = \oint_C (F_1dx + F_2dy)$$

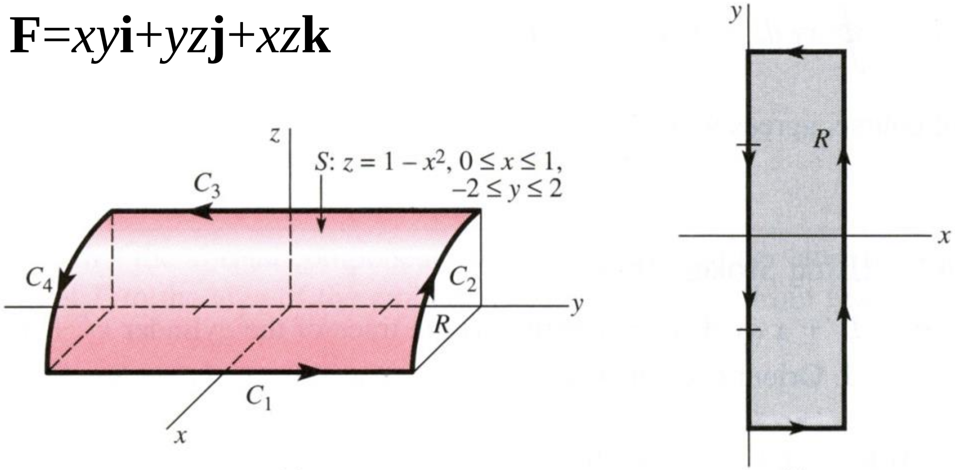
☑Example 1
Verifying Stokes's Theorem

Let S be the part of the cylinder

$z=1-x^2$ for $0 \le x \le 1$, $-2 \le y \le 2$.

Verify Stokes's theorem if

$\mathbf{F}=xy\mathbf{i}+yz\mathbf{j}+xz\mathbf{k}$



(a) (b)

Solution) 1) Surface Integral

$\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$$

$$g(x,y,z) = z + x^2 - 1 = 0$$

$$\mathbf{N} = \nabla g = 2x\mathbf{i} + \mathbf{k}$$

$$\iint_S (\operatorname{curl}\mathbf{F}\bullet\mathbf{n})dA = \iint_R (\operatorname{curl}\mathbf{F})\bullet\mathbf{N}\,dxdy$$

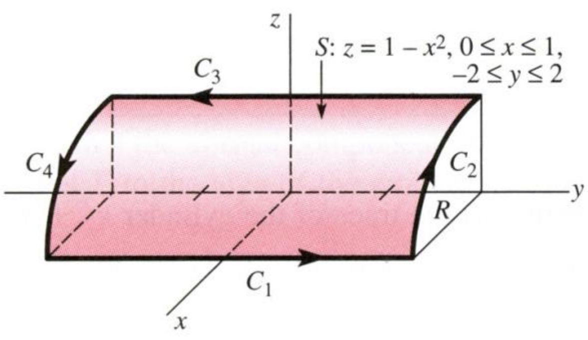
$$= \int_0^1\int_{-2}^2 (-2xy - x)dydx$$

$$= \int_0^1 \left[-xy^2 - xy\right]_{-2}^2 dx$$

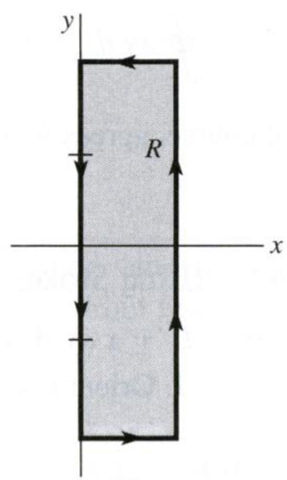
$$= \int_0^1 (-4x)dx = -2$$

☑ Example 1
Verifying Stokes's Theorem

Let S be the part of the cylinder $z=1-x^2$ for $0 \le x \le 1$, $-2 \le y \le 2$. Verify Stokes' theorem if $\mathbf{F}=xy\mathbf{i}+yz\mathbf{j}+xz\mathbf{k}$.



(a)



(b)

Solution) 2) Line Integral

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

$$C_1 : x=1,\ z=0,\ dx=0,\ dz=0$$

$$\int_{C_1} 1\cdot y \cdot 0 + y \cdot 0dy + 1\cdot 0 \cdot 0 = 0$$

$$C_2 : y=2,\ z=1-x^2,\ dy=0,\ dz=-2xdx$$

$$\int_{C_2} 2xdx + 2(1-x^2)0 + x(1-x^2)(-2xdx)$$

$$= \int_1^0 (2x - 2x^2 + 2x^4)dx = -\frac{11}{15}$$

$$C_3 : x=0,\ z=1,\ dx=0,\ dz=0$$

$$\int_{C_3} 0 + ydy + 0 = \int_2^{-2} ydy = 0$$

$$C_4 : y=-2,\ z=1-x^2,\ dy=0,\ dz=-2xdx$$

$$\int_{C_4} -2xdx - 2(1-x^2)0 + x(1-x^2)(-2x)dx$$

$$= \int_0^1 (-2x - 2x^2 + 2x^4)dx = -\frac{19}{15}$$

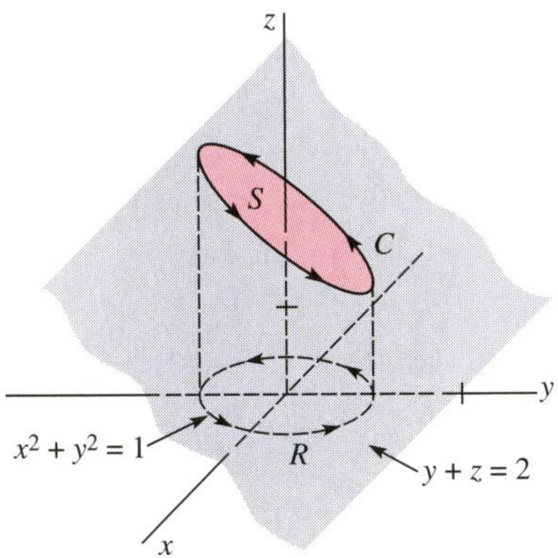
$$\therefore \oint_C xydx + yz + xzdz = 0 - \frac{11}{15} + 0 - \frac{19}{15} = -2$$

$$\iint_S (\text{curl}\,\mathbf{F}) \bullet \mathbf{n}\,dA = \oint_C \mathbf{F} \bullet \mathbf{r}'(s)\,ds = \oint_C (F_1 dx + F_2 dy + F_3 dz)$$

☑ Example 2

Using Stokes's Theorem

Evaluate $\oint_C zdx + xdy + ydz,$ where C is the trace of the cylinder $x^2+y^2=1$ in the plane $y+z=2$. Orient C counterclockwise as viewed from above. See the Figure below



Solution)

$$\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$g(x, y, z) = y + z - 2 = 0$$

$$N = \nabla g = \mathbf{j} + \mathbf{k}$$

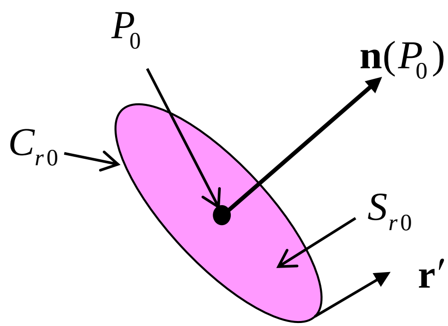
$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\text{curl}\,\mathbf{F} \bullet \mathbf{n})\,dA$$

$$= \iint_R (\text{curl}\,\mathbf{F}) \bullet \mathbf{N}\,dxdy$$

$$= \iint_R \left[(\mathbf{i} + \mathbf{j} + \mathbf{k}) \bullet (\mathbf{j} + \mathbf{k})\right] dA$$

$$= \iint_R 2dA = 2\pi$$

Ex. 4) Physical Interpretation of Curl



C_r is Small circle of radius r centered at P_0

Mean value theorem for surface integrals

$$\oint_{C_{r0}} \mathbf{F}\bullet\mathbf{r}'(s)\,ds = \iint_{S_{r0}} (\operatorname{curl}\mathbf{F})\bullet\mathbf{n}\,dA = (\operatorname{curl}\mathbf{F})\bullet\mathbf{n}(P^*)A_{r0}$$

P^* is a suitable point of S_{r0}.

$$(\operatorname{curl}\mathbf{F})\bullet\mathbf{n}(P^*) = \frac{1}{A_{r0}}\oint_{C_{r0}} \mathbf{F}\bullet\mathbf{r}'ds$$

In case of a fluid motion with velocity vector $\mathbf{F} = \mathbf{v}$,

$\oint_{C_{r0}} \mathbf{v}\bullet\mathbf{r}'ds$: circulation of the flow around C_{r0}.

If we now let r_0 approach zero. $(\operatorname{curl}\mathbf{v})\bullet\mathbf{n}(P) = \lim_{r_0\to 0}\frac{1}{A_{r_0}}\oint_{C_{r0}} \mathbf{v}\bullet\mathbf{r}'ds$

The component of the curl in the positive normal direction
⇨ **specific circulation** (circulation per unit area) of the flow in the surface at the corresponding point

"양의 법선 방향으로의 회전 성분은 그 곡면의 해당 점에서의 유체의 특별한 순환 (단위 넓이당 순환)으로 간주할 수 있다"

Summary

☑ Green's Theorem

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C \left(F_1 dx + F_2 dy\right)$$

$$\therefore \quad \iint_R \nabla^2 \phi \, dxdy = \oint_C \frac{\partial \phi}{\partial n} ds$$

$$\iint_R \nabla^2 \phi \, dxdy = \int \boxed{\int_{x_0}^{x_1} \frac{\partial^2 \phi}{\partial x^2} dxdy} = \int \frac{\partial \phi}{\partial x}\bigg|_{x_0}^{x_1} dy = \int \boxed{u(x_1, y) - u(x_0, y)} dy$$

속도의 변화량의 적분

양 경계에서의 단위시간당 속도차
= 생성되는 유체의 양

경계를 통해 단위 시간당
빠져나가는 유체의 양

$$\oint_C \frac{\partial \phi}{\partial n} ds = \int \boxed{-\frac{\partial \phi}{\partial x}\bigg|_{x_0} + \frac{\partial \phi}{\partial x}\bigg|_{x_1}} dy$$
$$= \int -u(x_0, y) + u(x_1, y) dy$$

- Surface Integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_R \mathbf{F}\left(\mathbf{r}(u,v)\right) \cdot \mathbf{N}(u,v) dudv$$

- **Divergence Theorem of Gauss**

$$\iiint_T \operatorname{div} \mathbf{F} dV = \iint_S \mathbf{F} \bullet \mathbf{n} dA \qquad \iiint_T \nabla^2 \phi \, dV = \iint_S \frac{\partial \phi}{\partial n} dA$$

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \iint_S \left(F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma\right) dA = \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy\right)$$

Green's first formula

$$\iiint_T \left(f \; \nabla^2 g + \operatorname{grad} f \bullet \operatorname{grad} g\right) dV = \iint_S f \frac{\partial g}{\partial n} dA$$

Green's second formula

$$\iiint_T \left(f \; \nabla^2 g - g \nabla^2 f\right) dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n}\right) dA$$

Summary

Green's Theorem

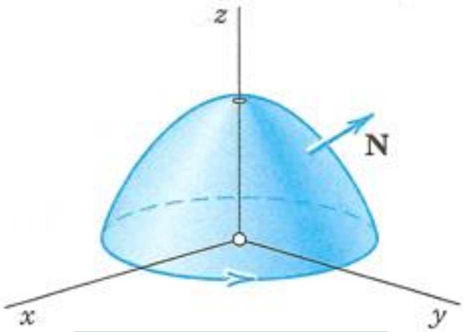
$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C \left(F_1 dx + F_2 dy \right)$$

$$\Rightarrow \quad \iint_R (\operatorname{curl} \mathbf{F}) \bullet \mathbf{k} \, dxdy = \oint_C \mathbf{F} \bullet d\mathbf{r}$$

Stokes's Theorem

$$\iint_S (\operatorname{curl} \mathbf{F}) \bullet \mathbf{n} \, dA = \iint_R (\operatorname{curl} \mathbf{F}) \bullet \mathbf{N} \, dxdy = \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C \mathbf{F} \bullet \mathbf{r}'(s) \, ds$$

$$\iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) N_3 \right] dudv = \oint_{\bar{C}} \left(F_1 dx + F_2 dy + F_3 dz \right)$$



Surface S

Summary

☑ **Green's Theorem: Double Integrals ⇔ Line Integrals**

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C \left(F_1 dx + F_2 dy \right)$$

☑ **Gauss's Theorem (Divergence Theorem): Triple Integrals ⇔ Surface Integrals**

$$\iiint_T \operatorname{div} \mathbf{F} dV = \iint_S \mathbf{F} \bullet \mathbf{n} dA$$

☑ **Stokes's Theorem: Surface Integrals ⇔ Line Integrals (Generalization of Green's Theorem in the Plane)**

$$\iint_S \left(\operatorname{curl} \mathbf{F} \right) \bullet \mathbf{n} dA = \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C \mathbf{F} \bullet \mathbf{r}'(s) ds$$