Ch. 11 Fourier Analysis
Part I

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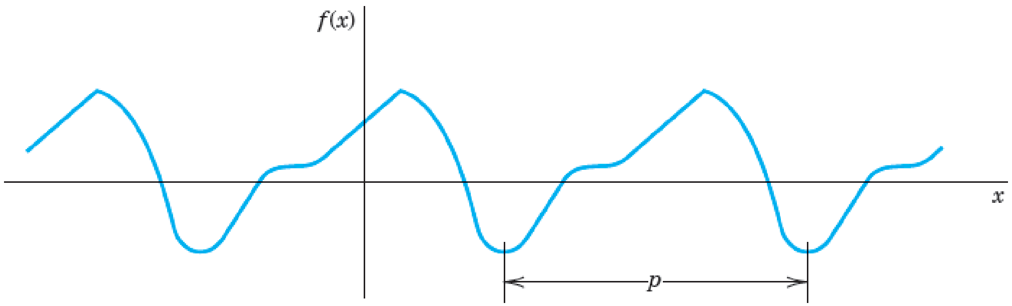
※ 본 강의 자료는 이규열, 장범선,노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

11.1 Fourier Series

☑ Periodic Function (주기함수) *f (x)*

- *f(x)* is defined for all real *x* (perhaps except at some points)
- There is some positive number *p*, such that *f(x+p) = f (x)* for all *x*.

  p : A period of *f (x)*



- Familiar periodic functions : cosine and sine
- Example of functions that are not periodic : x, x^2, x^3, e^x, *coshx*, *lnx*
- *f(x)* has period *p*, for any integer *n* = 1, 2, 3, …, ***f(x+np) = f(x)*** for all *x*.
- If *f(x)* and *g(x)* have period *p*, then *af(x)+bg(x)* with any constants *a* and *b* also has the period *p*

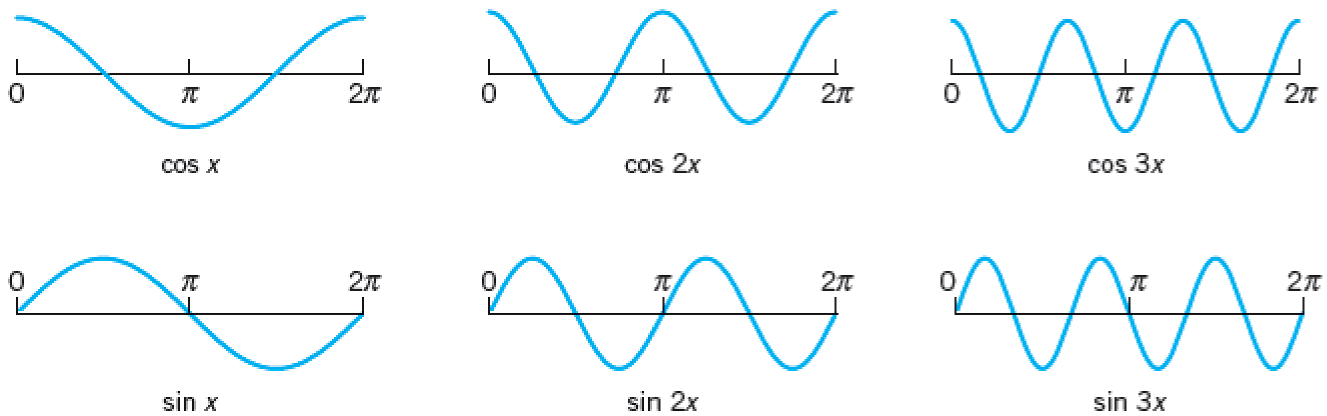
11.1 Fourier Series

☑ Trigonometric Series (삼각급수, 삼각함수급수)

- Trigonometric System : 1, cos *x*, sin *x*, cos 2*x*, sin 2*x*, …, cos *nx*, sin *nx*, …
- **Trigonometric Series :**

  $a_0, a_1, b_1, a_2, b_2, \ldots$ are constants, called the coefficients of the series
- If the coefficients are such that the series converges, its sum will be a function of period 2π.

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$



Cosine and sine functions having the period 2π (the first few members of the trigonometric system in the above equation)

11.1 Fourier Series

☑ Fourier Series (퓨리에급수)

: The trigonometric system whose the coefficients are given by the Euler formulas

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

- Fourier Coefficients of *f(x)* are given by the following Euler Formulas.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 1,\ 2,\ \cdots$$

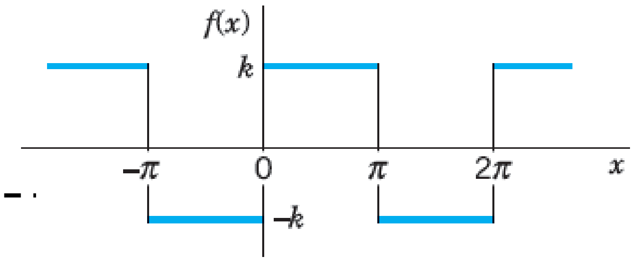
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1,\ 2,\ \cdots$$

우함수: $f(-x) = f(x)$ => y축 대칭
기함수: $f(-x) = -f(x)$ => 원점 대칭

☑ Ex.1 Periodic Rectangular Wave (주기적인 직사각형파)

Find the Fourier coefficients of the periodic function f(x). The formula is

$$f(x) = \begin{cases} -k & (-\pi < x < 0) \\ k & (0 < x < \pi) \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$



Sol) Since $f(x)$ is odd (기함수) and cos x is even (우함수), $f(x)$ cos x is odd function

Since $f(x)$ is odd and sin x is odd, $f(x)$ sin x is even function

$$\int_0^{\pi} f(x)dx = k \cdot \pi \quad \text{and} \quad \int_{-\pi}^{0} f(x)dx = -k \cdot \pi$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} f(x)dx = \int_0^{\pi} f(x)dx + \int_{-\pi}^{0} f(x)dx = 0 \qquad \therefore \quad a_0 = 0$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} f(x)\cos x dx = 0 \qquad \therefore \quad a_n = 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 1,\ 2,\ \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1,\ 2,\ \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} k\sin nx dx = \frac{2}{\pi}k\left.\frac{-\cos nx}{n}\right|_0^{\pi} = \frac{2k}{n\pi}(1 - \cos n\pi) = \begin{cases} \dfrac{4k}{n\pi} & n: \text{ odd} \\ 0 & n: \text{even} \end{cases}$$

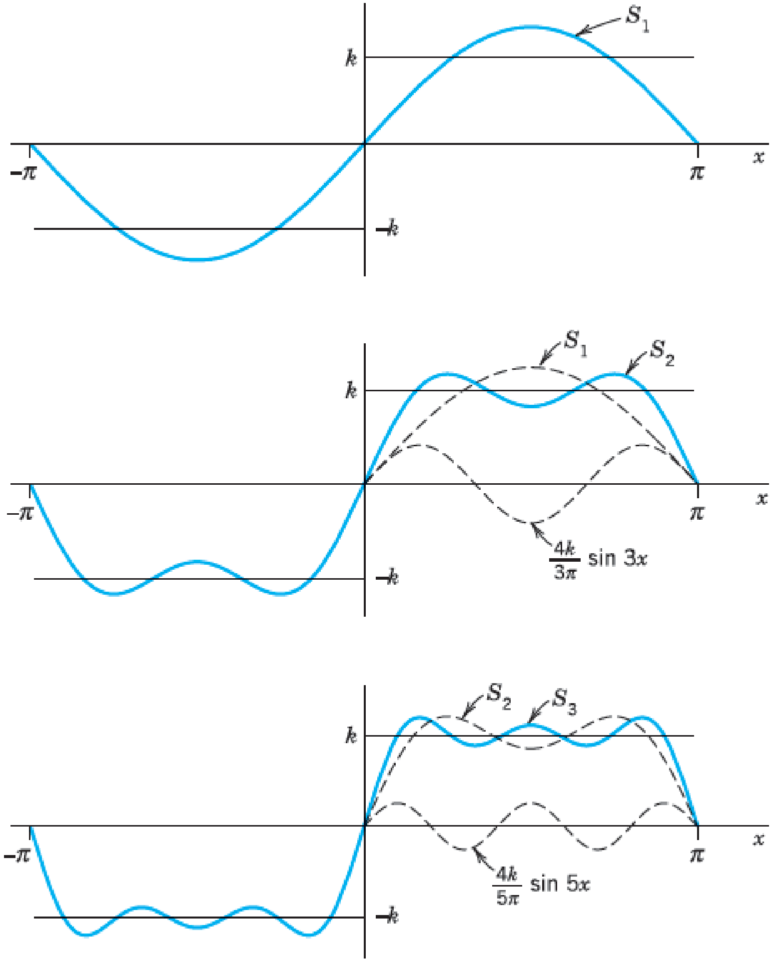
11.1 Fourier Series

Sol) Fourier series of $f(x)$:

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right)$$

$$\Rightarrow \quad 1 - \frac{1}{3} + \frac{1}{5} - + \cdots = \frac{\pi}{4}$$



The first three partial sums of the corresponding Fourier series

☑ **Theorem 1 Orthogonality of the Trigonometric System (삼각함수계의 직교성)**

The trigonometric system is orthogonal on the interval $-\pi \le x \le \pi$; that is, the integral of the product of any two functions over that interval is 0, so that for any integers *n* and *m*,

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0 \qquad (n \neq m \text{ or } n = m)$$

☑ Proof

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx = 0 \quad \textit{when } m \neq n$$

$$= \pi \quad \textit{when } m = n$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx = 0 \quad \textit{when } m \neq n$$

$$= \pi \quad \textit{when } m = n$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x\, dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x dx = 0 + 0$$

$$\textit{when } m \neq n \textit{ or } m = n$$

11.1 Fourier Series

☑ Proof of Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 1,\ 2,\ \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1,\ 2,\ \cdots$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0 \qquad (n \neq m \ \text{ or } \ n = m)$$

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right) = 2\pi a_0 \quad \Rightarrow \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

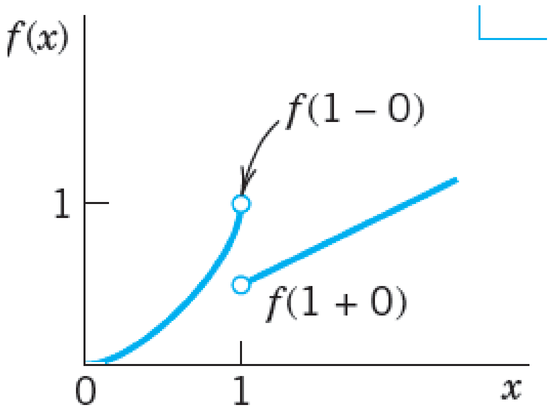
$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)\right]\cos mx\,dx = a_m \pi \quad \Rightarrow \quad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx$$

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)\right]\sin mx\,dx = b_m \pi \quad \Rightarrow \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx$$

11.1 Fourier Series

☑ Theorem 2 Representation by a Fourier Series

- Let *f(x)* be periodic with period 2π and piecewise continuous in the interval $-\pi \le x \le \pi$.
- Furthermore, *let f(x)* have a left-hand derivative and a right-hand derivative at each point of that interval.
- Then the Fourier series of *f(x)* converges.
- Its sum is *f(x)*, except at points where *f(x)* is discontinuous.
- There the sum of the series is the average of the left- and right-hand limits of *f (x)* at x_0.



Left and right hand limits

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x/2 & \text{if } x \geqq 1 \end{cases} \quad \Rightarrow \quad f(1-0) = 1, \quad f(1+0) = \tfrac{1}{2}$$

☑ Fourier series of the function *f*(x) of period 2L:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

⇨

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x\right)$$

Euler formulas

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1, 2, \cdots$$

⇨

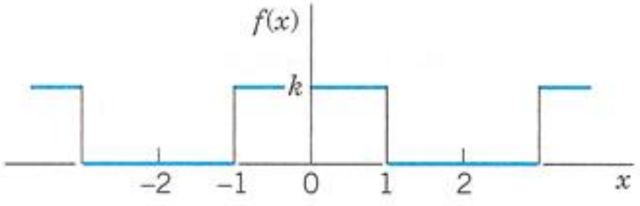
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos \frac{n\pi x}{L} dx, \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \cdots$$

☑ Ex.1 Periodic Rectangular Wave

Find the Fourier series of the function



$$f(x)=\begin{cases}0 & (-2<x<-1)\\ k & (-1<x<1)\\ 0 & (1<x<2)\end{cases} \qquad p=2L=4, \quad L=2$$

$$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n=1,\ 2,\ \cdots$$

$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n=1,\ 2,\ \cdots$$

Sol) $a_0=\frac{1}{4}\int_{-2}^{2} f(x)dx=\frac{1}{4}\int_{-1}^{1} kdx=\frac{1}{4}\cdot 2k=\frac{k}{2}$

$$a_n=\frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx=\frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx=\frac{2k}{n\pi}\sin\frac{n\pi}{2}=\begin{cases}0 & n \text{ is even}\\ \frac{2k}{n\pi} & n=1,\ 5,\ 9,\ \cdots\\ -\frac{2k}{n\pi} & n=3,\ 7,\ 11,\ \cdots\end{cases}$$

Since *f (x)* is even and $\sin\frac{n\pi x}{2}$ is odd, $f(x)\sin\frac{n\pi x}{2}$ is odd $\Rightarrow \int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx=0 \quad \therefore\ b_n=0$

Fourier series : $f(x)=\frac{k}{2}+\frac{2k}{\pi}\left(\cos\frac{\pi}{2}x-\frac{1}{3}\cos\frac{3\pi}{2}x+\frac{1}{5}\cos\frac{5\pi}{2}x-+\cdots\right)$

11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

☑ Summary

- **Even Function (우함수) of Period 2L:**

✓ Fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \quad (f \text{ even})$$

with coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx, \quad n = 1,\ 2,\ \cdots$$

- **Odd Function (기함수) of Period 2L:**

✓ Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \quad (f \text{ odd})$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx, \quad n = 1,\ 2,\ \cdots$$

☑ Theorem 1 Sum and Scalar Multiple

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

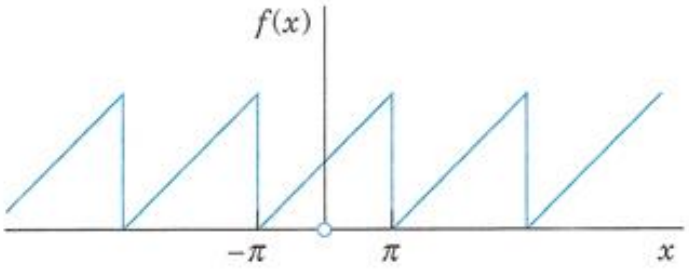
The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

☑ Ex.5 Sawtooth Wave (톱니파)

Find the Fourier series of the function

$f(x) = x + \pi \ (-\pi < x < \pi)$ and $f(x+2\pi) = f(x)$

$f = f_1 + f_2$ where $f_1 = x$ and $f_2 = \pi$



The function *f* (*x*)

Sol) Fourier coefficients of f_1

f_1 is odd $\quad a_n = 0 \ (n = 0, 1, 2, \cdots)$

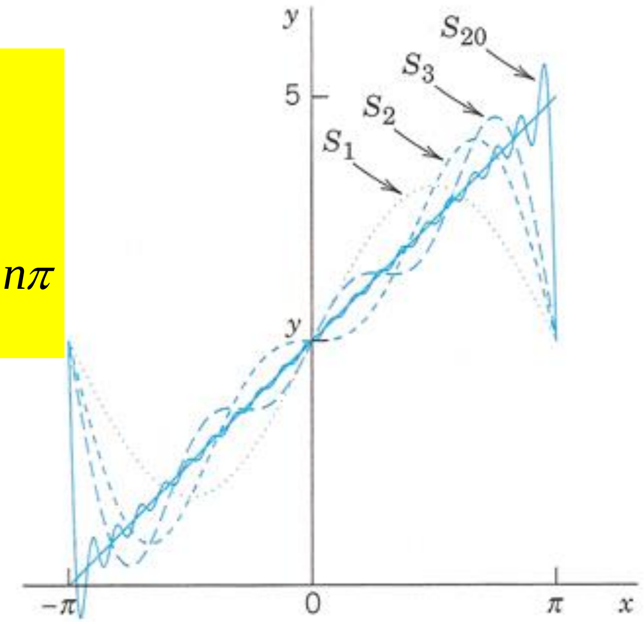
$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\left.\frac{-x\cos nx}{n}\right|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right] = -\frac{2}{n}\cos n\pi$$

Fourier coefficients of f_2

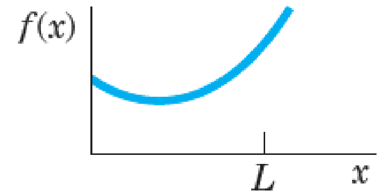
$a_n = b_n = 0$ (n=1, 2,…) and $a_0 = \pi$

$$\therefore \quad f = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right)$$



Partial sums

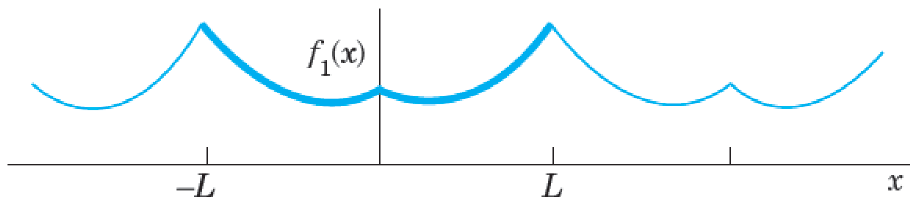
☑ Half-Range Expansions (반구간 전개)



The given function ***f*** **(*x*)**

- Even Periodic Extension

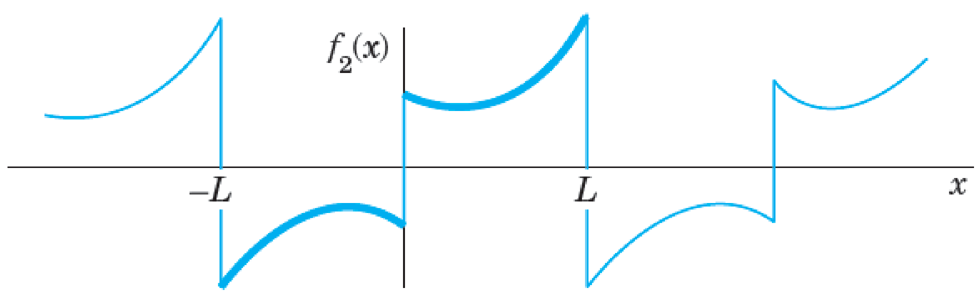
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \quad (f \text{ even})$$



f **(*x*) extended as an even periodic function of period 2*L***

- Odd Periodic Extension

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad (f \text{ odd})$$



f **(*x*) extended as an odd periodic function of period 2*L***

☑ Ex.6 “Triangle” and Its Half-Range Expansions

Find the two half-range expansions of the function

$$f(x)=\begin{cases}\dfrac{2k}{L}x & \left(0<x<\dfrac{L}{2}\right)\\ \dfrac{2k}{L}(L-x) & \left(\dfrac{L}{2}<x<L\right)\end{cases}$$

The given function *f* (*x*)

$$a_0=\frac{1}{L}\int_0^L f(x)\,dx,$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx,\quad n=1,\ 2,\ \cdots$$

Sol) 1. Even periodic extension (주기적 우함수)

$$a_0=\frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2}x\,dx+\frac{2k}{L}\int_{L/2}^{L}(L-x)\,dx\right]=\frac{k}{2}$$

$$a_n=\frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2}x\cos\frac{n\pi}{L}x\,dx+\frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\frac{n\pi}{L}x\,dx\right]=\frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{L}-\cos n\pi-1\right)$$

$$\therefore\ f(x)=\frac{k}{2}-\frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x+\frac{1}{6^2}\cos\frac{6\pi}{L}x+\cdots\right)$$

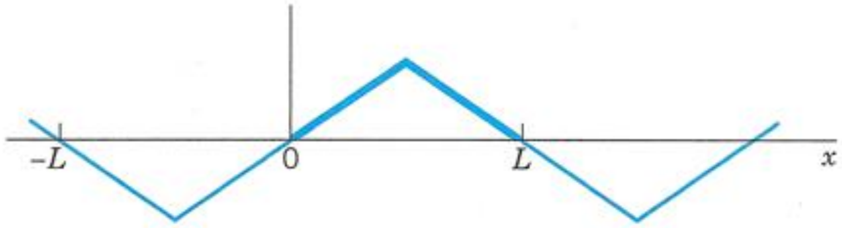
Even extension

2. Odd periodic extension (주기적 기함수)

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx, \quad n = 1,\ 2,\ \cdots$$

$$b_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\sin\frac{n\pi}{L}x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\sin\frac{n\pi}{L}x\,dx\right] = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore\ f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right)$$



Odd extension

11.3 Forced Oscillations

☑ Forced Oscillations $my''+cy'+ky=r(t)$

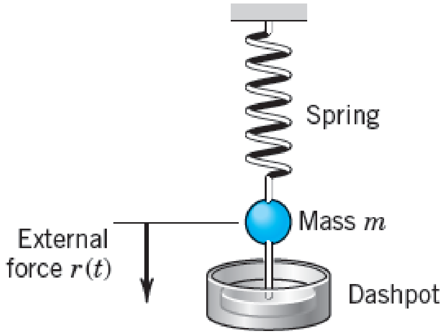
- $r(t)$: not a pure sine or cosine function,

  but is a any other period

- The steady-state solution (정상상태 해): a superposition (중첩) of harmonic oscillations (조화진동) with frequencies equal to that of $r(t)$ and integer multiples of these frequencies.

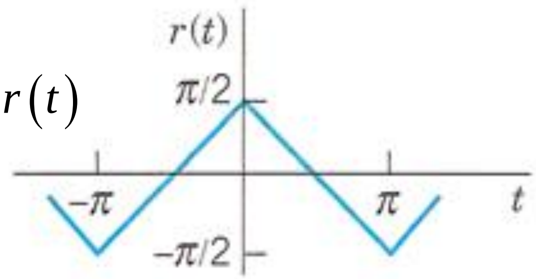
- If one of these frequencies is close to the resonant frequency (공진주파수) of the vibrating system

⇒ the corresponding oscillation may be the dominant part of the response of the system to the external force.



☑ Ex. 1 Forced Oscillations under a Nonsinusoidal Periodic Driving Force

$$y''+0.05y'+25y=r(t),\ r(t)=\begin{cases} t+\dfrac{\pi}{2} & (-\pi<t<0) \\ -t+\dfrac{\pi}{2} & (0<t<\pi) \end{cases} \quad r(t+2\pi)=r(t)$$



Find the steady-state solution y(t).

Sol) Represent *r(t)* by a Fourier series $r(t)=\dfrac{4}{\pi}\left(\cos t+\dfrac{1}{3^2}\cos 3t+\dfrac{1}{5^2}\cos 5t+\cdots\right)$

$$y''+0.05y'+25y=\frac{4}{n^2\pi}\cos nt \qquad (n=1,\ 3,\ \cdots)$$

Steady-state solution $y_n = A_n\cos nt + B_n\sin nt$

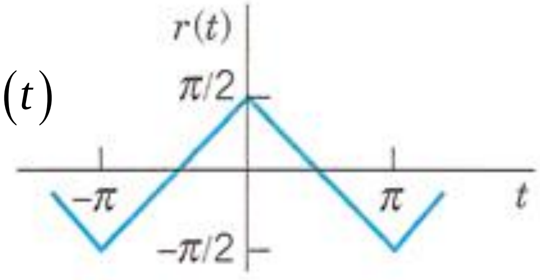
$$(-A_n n^2+0.05nB_n+25A_n)\cos nt+(-B_n n^2-0.05nA_n+25B_n)\sin nt=\frac{4}{n^2\pi}\cos nt \qquad (n=1,\ 3,\ \cdots)$$

$$B_n=\frac{0.05n}{25-n^2}A_n \Rightarrow A_n(25-n^2)+\frac{(0.05n)^2}{25-n^2}A_n=\frac{4}{n^2\pi}$$

$$A_n=\frac{4(25-n^2)}{n^2\pi D_n},\quad B_n=\frac{0.2}{n\pi D_n}\quad \text{where } D_n=(25-n^2)^2+(0.05n)^2$$

☑ Ex. 1 Forced Oscillations under a Nonsinusoidal Periodic Driving Force

$$y''+0.05y'+25y=r(t),\ r(t)=\begin{cases} t+\dfrac{\pi}{2} & (-\pi<t<0) \\ -t+\dfrac{\pi}{2} & (0<t<\pi) \end{cases} \qquad r(t+2\pi)=r(t)$$



Find the steady-state solution y(t).

$$\omega_0=\sqrt{\frac{k}{m}},\ T=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{1}{25}}=1.256$$

continued) Since ODE is linear, the steady state solution to be

$$y=y_1+y_3+y_5+\cdots$$

$$y''+0.05y'+25y=\frac{4}{n^2\pi}\cos nt \quad (n=1,\ \ 3,\ \ \cdots)$$

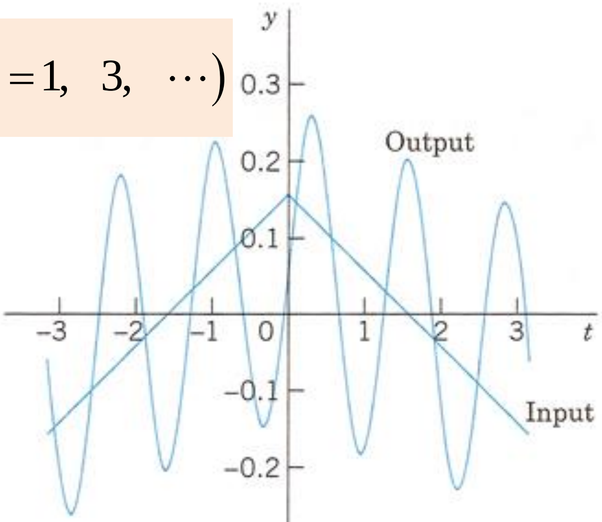
Amplitude of $y_n = A_n\cos nt + B_n\sin nt$ is

$$C_n=\sqrt{A_n^2+B_n^2}=\frac{4}{n^2\pi\sqrt{D_n}},$$ dominating

$C_1=0.0531$ $C_3=0.0088$ $\boxed{C_5=0.2037}$ $C_7=0.0011$ $C_9=0.0003$

Half Period

$C_1:\pi(\approx 3.1)$ $C_3:\pi/3(\approx 1.0)$ $C_5:\pi/5(\approx 0.60)$



Input and steady-state output

11.4 Approximation by Trigonometric Polynomials

☑ Approximation Theory (근사이론)

Approximation Theory concerns the approximation of functions by other functions.

- Idea

  Let $f(x)$ be a function that can be represented by a Fourier series.

  Nth partial sum is an approximation of the given $f(x)$

$$\therefore\ f(x) \approx a_0 + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx\right)$$

- The best approximation of f *by a* trigonometric polynomial of degree N

$$F(x) = A_0 + \sum_{n=1}^{N} \left(A_n \cos nx + B_n \sin nx\right) \quad (N \text{ fixed})$$

are chosen in such a way as to minimize error of the approximation as small as possible.

11.4 Approximation by Trigonometric Polynomials

- Error of such an approximation between f and F on the whole interval $-\pi < x < \pi$

$$E = \int_{-\pi}^{\pi} (f - F)^2 dx$$

$$\therefore \; f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

$$F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx)$$

- Error for $A_n = a_n$, $B_n = b_n$: $E^* \geq 0$

☑ **Theorem 1 Minimum Square Error (최소제곱오차)**

The square error of F relative to f on the interval $-\pi \le x \le \pi$ is minimum if and only if the coefficients of F are the Fourier coefficients of f. This minimum value E^* is given by

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]$$

Bessel's inequality: $2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \le \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx$

Parseval's identity: $2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx$

☑ **Proof of**

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0{}^2 + \sum_{n=1}^{N}\left(a_n{}^2 + b_n{}^2\right)\right]$$

$$f(x) \approx a_0 + \sum_{n=1}^{N}\left(a_n \cos nx + b_n \sin nx\right)$$

$$F(x) = A_0 + \sum_{n=1}^{N}\left(A_n \cos nx + B_n \sin nx\right)$$

$$E = \int_{-\pi}^{\pi} (f - F)^2 dx$$

$$E = \int_{-\pi}^{\pi} f^2 dx - 2\int_{-\pi}^{\pi} fF dx + \int_{-\pi}^{\pi} F^2 dx$$

$$\int_{-\pi}^{\pi} \cos nx^2 dx = \int_{-\pi}^{\pi} \frac{1}{2}(1+\cos 2nx)dx = \frac{1}{2}x + \frac{1}{4n}\cos 2nx\Big|_{-\pi}^{\pi} = \pi$$

$$\int_{-\pi}^{\pi} \sin nx^2 dx = \int_{-\pi}^{\pi} \frac{1}{2}(1-\cos 2nx)dx = \frac{1}{2}x - \frac{1}{4n}\cos 2nx\Big|_{-\pi}^{\pi} = \pi$$

$$\int_{-\pi}^{\pi} F^2 dx = \int_{-\pi}^{\pi}\left[A_0 + \sum_{n=1}^{N}\left(A_n \cos nx + B_n \sin nx\right)\right]^2 dx$$

$$= \pi(2A_0^2 + A_1^2 + \cdots + A_N^2 + B_1^2 + \cdots + B_N^2)$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0 \qquad (n \neq m \ \text{ or } \ n = m)$$

$$\int_{-\pi}^{\pi} fF dx = \pi(2A_0 a_0 + A_1 a_1 + \cdots + A_N a_N + B_1 b_1 + \cdots + B_N b_N)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$\therefore E = \int_{-\pi}^{\pi} f^2 dx - 2\pi\left[2A_0 a_0 + \sum_{n=1}^{N}\left(A_n a_n + B_n b_n\right)\right] + \pi\left[2A_0{}^2 + \sum_{n=1}^{N}\left(A_n{}^2 + B_n{}^2\right)\right]$$

We now take $A_n = a_n, B_n = b_n$

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0{}^2 + \sum_{n=1}^{N}\left(a_n{}^2 + b_n{}^2\right)\right]$$

$$E = \int_{-\pi}^{\pi} f^2 dx - 2\pi\left[2A_0 a_0 + \sum_{n=1}^{N}\left(A_n a_n + B_n b_n\right)\right] + \pi\left[2A_0^{\ 2} + \sum_{n=1}^{N}\left(A_n^{\ 2} + B_n^{\ 2}\right)\right]$$

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^{\ 2} + \sum_{n=1}^{N}\left(a_n^{\ 2} + b_n^{\ 2}\right)\right]$$

$$E - E^* = \pi\left\{2(A_0 - a_0)^2 + \sum_{n=1}^{N}[(A_n - a_n)^2 + (B_n - b_n)^2]\right\}$$

$$E - E^* \geq 0 \quad \Rightarrow E \geq E^*$$

$$E = E^* \text{ if and only if } A_0 = a_0, A_n = a_n, B_n = b_n$$

☑ Theorem 1 Minimum Square Error

The square error of F relative to *f* on the interval $-\pi \leq x \leq \pi$ is minimum if and only if the coefficients of *F* are the Fourier coefficients of *f*. This minimum value *E** is given by

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^{\ 2} + \sum_{n=1}^{N}\left(a_n^{\ 2} + b_n^{\ 2}\right)\right]$$

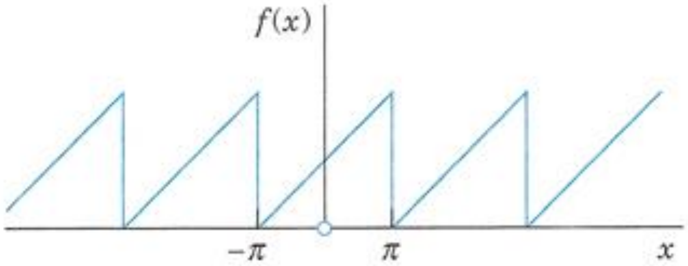
$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right]$$

☑ Ex. 5 Sawtooth Wave

Find the Fourier series of the function

$f(x) = x + \pi$ $(-\pi < x < \pi)$ and $f(x+2\pi) = f(x)$

$f = f_1 + f_2$ where $f_1 = x$ and $f_2 = \pi$



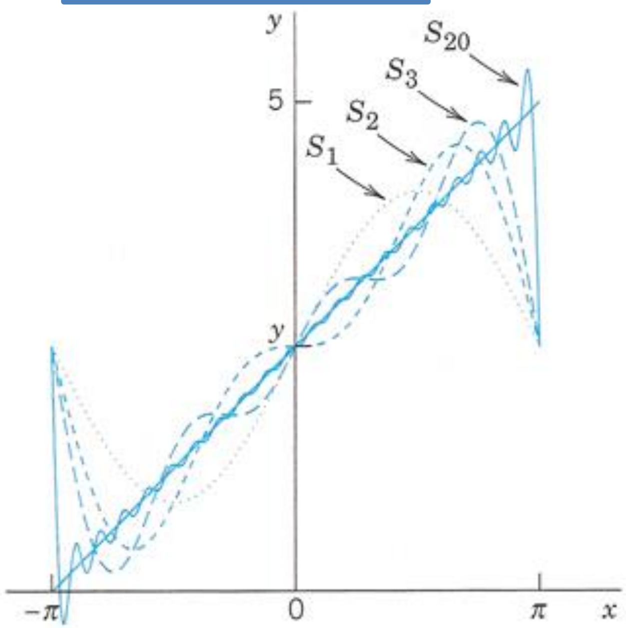
The function *f* (*x*)

Sol) Solution was

$$\therefore \quad f = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right)$$

$$E^* = \int_{-\pi}^{\pi} (x+\pi)^2 dx - \pi\left[2\pi^2 + 4\sum_{n=1}^{N}\frac{1}{n^2}\right]$$

N	E^*	N	E^*	N	E^*	N	E^*
1	8.1045	6	1.9295	20	0.6129	70	0.1782
2	4.9629	7	1.6730	30	0.4120	80	0.1561
3	3.5666	8	1.4767	40	0.3103	90	0.1389
4	2.7812	9	1.3216	50	0.2488	100	0.1250
5	2.2786	10	1.1959	60	0.2077	1000	0.0126



Partial sums

11.5 Sturm-Liouville (스트름-리우빌) Problems. Orthogonal Functions

☑ Frourier series: Trigonometric system (삼각함수계)

- Generalized using other orthogonal systems (삼각함수계가 아닌)? "Yes"

☑ Sturm-Liouville equation:

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0, \quad a \le x \le b$$

Sturm-Liouville boundary conditions:

$$k_1 y(a) + k_2 y'(a) = 0 \ \ at \ \ x = a$$
$$l_1 y(b) + l_2 y'(b) = 0 \ \ at \ \ x = b$$

- Homogeneous equation & Homogeneous condition

  ⇒ Homogenous boundary Value Problem

☑ Goal

✓ y = 0: Trivial solution

✓ Find nontrivial solution y (Eigenvalues, Eigenfunctions)

Review on Ordinary Differential Equation

▪**Review**

Linear Equations		General solutions
$y' + \alpha y = 0$		$y = c_1 e^{-\alpha x}$
$y'' + \alpha^2 y = 0$	$\alpha > 0$	$y = c_1 \cos \alpha x + c_2 \sin \alpha x$
$y'' - \alpha^2 y = 0$	$\alpha > 0$	$\begin{cases} y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}, or \\ y = c_1 \cosh \alpha x + c_2 \sinh \alpha x) \end{cases}$ ← When x is an infinite or half finite interval; ← When x is a finite interval

Cauchy-Euler Equation		General solutions $x > 0$
$x^2 y'' + xy' - \alpha^2 y = 0$	$\alpha \geq 0$	$\begin{cases} y = c_1 x^{-\alpha} + c_2 x^{\alpha}, \alpha \neq 0 \\ y = c_1 + c_2 \ln x, \quad \alpha = 0 \end{cases}$

Linear Equations

Review on Ordinary Differential Equation

▪**Review**

Parametric Bessel equation $\nu = 0$	General solutions $x > 0$
$x^2 y'' + y' + \alpha^2 x^2 y = 0$	$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$

Legendre's equation $n = 0, 1, 2, \ldots$	Particular solutions are polynomials
$(1-x^2)y'' - 2xy' + n(n+1)y = 0$	$y = P_0(x) = 1,$ $y = P_1(x) = x,$ $y = P_2(x) = \frac{1}{2}(3x^2 - 1),$ $\vdots$

11.5 Sturm-Liouville (스트름-리우빌) Problems. Orthogonal Functions

☑ **Solve :** $\left[p(x)y'\right]'+\left[q(x)+\lambda r(x)\right]y=0, \quad a\le x\le b$

Subject to :

$$k_1 y(a)+k_2 y'(a)=0 \;\; at \;\; x=a$$

$$l_1 y(b)+l_2 y'(b)=0 \;\; at \;\; x=b$$

☑ **Ex.1 Trigonometric Functions as Eigenfunctions. Vibrating String**

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$y'' + \lambda y = 0,\ y(0) = 0,\ y(\pi) = 0$

$p = 1,\ q = 0,\ r = 1$ and $a = 0,\ b = \pi,\ k_1 = l_1 = 1,\ k_2 = l_2 = 0$

Sol) Case 1. Negative eigenvalue $\lambda = -\nu^2$

A general solution of the ODE: $y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x}$

Apply the boundary conditions ($c_1 = c_2 = 0$): $\boldsymbol{y \equiv 0}$

Case 2. λ = 0 Situation is similar: $\boldsymbol{y \equiv 0}$

☑ Ex.1 Trigonometric Functions as Eigenfunctions. Vibrating String

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0,\ \ y(0) = 0,\ \ y(\pi) = 0$$

continued)

Case 3. Positive eigenvalue $\lambda = \nu^2$

A general solution: $y(x) = A\cos \nu x + B\sin \nu x$

Apply the first boundary condition: $y(0) = A = 0$

Apply the second boundary condition:

$y(\pi) = B\sin \nu\pi = 0$ thus $\nu = 0, \pm 1, \pm 2, \ldots$

For $\nu = 0$, $y \equiv 0$

For $\lambda = \nu^2 = 1, 4, 9, 16, \ldots$, $\mathbf{y(x) = \sin \nu x\ (\nu = 0, 1, 2, 3, 4, \ldots)}$

Eigenvalues: $\lambda = \nu^2\ (\nu = 0, 1, 2, 3, 4, \ldots)$

Eigenfunctions: $y(x) = \sin \nu x\ (\nu = 0, 1, 2, 3, 4, \ldots)$

: infinitely many eigenvalues & orthogonality

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0 \qquad (n \neq m)$$

11.5 Sturm-Liouville (스트름-리우빌) Problems. Orthogonal Functions

☑ Orthogonal Functions

- Functions $y_1(x), y_2(x)$ defined on some interval $a \le x \le b$ are called **orthogonal** in this interval with respect to the **weight function (가중함수)** $r(x) > 0$ if for all m and all n different from m,

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x)\, dx = 0 \quad (m \neq n)$$

- The **norm** $\|y_m\|$ of y_m is defined by

$$\|y_m\| = \sqrt{(y_m, y_n)} = \sqrt{\int_a^b r(x) y_m{}^2(x)\, dx}$$

- Note that this is the square root of the integral with $n = m$.
- The functions $y_1, y_2, \ldots$ are called **orthonormal** (정규직교) on $a \le x \le b$ if they are orthogonal on this interval and all have norm 1 (Fourier series와 달리 $r(x)$가 있어 가능).

* 직교성 (orthogonality): 서로 다른 것끼리는 공통점이 없다라는 뜻. 선형대수학에서 두 벡터 사이의 내적이 0이라는 것으로 정의하며, 한 벡터가 다른 벡터의 성분을 조금도 가지고 있지 않다는 것을 말함
* 정규직교성 (orthonormarlity): 선형대수학에서 두 단위 벡터 (크기가 1) 사이의 내적이 0이라는 것

11.5 Sturm-Liouville (스트름-리우빌) Problems. Orthogonal Functions

☑ Theorem 1 Orthogonality of Eigenfunctions (고유함수의 직교성)

- Suppose that the functions *p*, *q*, *r*, and *p′* in the Sturm-Liouville equation are real-valued and continuous and $r(x) > 0$ on the interval $a \le x \le b$.

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0, \quad a \le x \le b$$

$$k_1 y(x) + k_2 y'(x) = 0 \ at \ x = a \longleftarrow \text{first}$$

$$l_1 y(x) + l_2 y'(x) = 0 \ at \ x = b \longleftarrow \text{second}$$

- Let $y_m(x)$ and $y_n(x)$ be eigenfunctions of the Sturm-Liouville problem that correspond to different eigenvalues λ_m and λ_n, respectively.
- Then y_m, y_n are orthogonal on that interval with respect to the weight functions $r(x)$,

$$\int_a^b r(x)\, y_m(x) y_n(x)\, dx = 0$$

- If $p(a) = 0 \Rightarrow$ the first boundary condition can be dropped from the problem.
- If $p(b) = 0 \Rightarrow$ the second boundary condition can be dropped.
- If $p(a) = p(b) \Rightarrow$ the boundary condition can be replaced by the "**periodic boundary conditions (주기적 경계조건)**"

$$y(a) = y(b), \ y'(a) = y'(b)$$

11.5 Sturm-Liouville (스트름-리우빌) Problems. Orthogonal Functions

☑ **Proof of** $\int_a^b r(x)\,y_m(x)y_n(x)\,dx = 0$

$$\left[p(x)y'\right]' + \left[q(x)+\lambda r(x)\right]y = 0,\quad a \le x \le b$$

$$(py_m')' + (q+\lambda_m r)y_m = 0 \qquad \times y_n$$

$$-)\ (py_n')' + (q+\lambda_n r)y_n = 0 \qquad \times -y_m$$

$$(\because (py_n')y_m' - (py_m')y_m = 0)$$

⇨ $(\lambda_m - \lambda_n) r y_m y_n = y_m (py_n')' - y_n (py_m')' = \left[(py_n')y_m - (py_m')y_n\right]'$

$$(\lambda_m - \lambda_n)\int_a^b r y_m y_n\,dx = \left[p(y_n' y_m - y_m' y_n)\right]_a^b \qquad (a<b)$$

Right side=

$$= p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)]$$
$$- p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)]$$

if **Right side** =0,
since $\lambda_m \neq \lambda_n$

⇨ $\int_a^b r(x)\,y_m(x)y_n(x)\,dx = 0$

11.5 Sturm-Liouville (스트름-리우빌) Problems. Orthogonal Functions

☑ **Proof of** $\int_a^b r(x)y_m(x)y_n(x)\,dx = 0$ ⇨

Goal is to prove Right Side

$$= p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)] - p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)]$$ =0

- ✓ Case 1, $p(a) = p(\mathrm{b}) = 0$, $\Rightarrow$ Right side = 0,
- ✓ Case 2, $p(a) = 0,\ p(b) \neq 0$

Right side = $p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)]$

Use Boundary Conditions

~~$k_1 y(x) + k_2 y'(x) = 0 \ at \ x = a$~~

$l_1 y(x) + l_2 y'(x) = 0 \quad at \ x = b$

⇨

$$l_1 y_n(b) + l_2 y_n'(b) = 0$$

$$l_1 y_m(b) + l_2 y_m'(b) = 0$$

First boundary condition can be dropped

As k_1, k_2 **are not both zero**

$$\det\begin{bmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{bmatrix} = y_m(b)y_n'(b) - y_m'(b)y_n(b) = 0$$

⇨ Right side = $p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)] = 0$

☑ **Proof of** $\int_a^b r(x)y_m(x)y_n(x)\,dx = 0$ ⇨ Goal is to prove Right Side

$$= p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)] - p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)] = 0$$

✓ Case 3, $p(a) \neq 0,\ p(b) = 0$

Right side = $-p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)]$

Use Boundary Conditions

$$k_1 y(x) + k_2 y'(x) = 0 \text{ at } x = a$$

$$~~l_1 y(x) + l_2 y'(x) = 0 \text{ at } x = b~~$$

⇨

$$k_1 y_n(a) + k_2 y_n'(a) = 0$$

$$k_1 y_m(a) + k_2 y_m'(a) = 0$$

Second boundary condition can be dropped

As k_1, k_2 **are not both zero**

$$\det\begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix} = y_m(a)y_n'(a) - y_m'(a)y_n(a) = 0$$

⇨ Right side $= -p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)] = 0$

☑ **Proof of** $\int_a^b r(x)\,y_m(x)y_n(x)\,dx = 0$ ⇨ Goal is to prove Right Side

$$= p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)] - p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)] = 0$$

✓ Case 4, $p(a) \neq 0,\ p(b) \neq 0$

Right side $= p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)] - p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)]$

Use Boundary Conditions

$$k_1 y(x) + k_2 y'(x) = 0 \ \ at \ \ x = a$$
$$l_1 y(x) + l_2 y'(x) = 0 \ \ at \ \ x = b$$

⇨ Proceed as Case 2 & Case 3

As k_1, k_2 **are not both zero**

$$\det = 0 \Rightarrow y\ (a)y_n'(a) - y_m'(a)y_n(a) = 0 \ \Rightarrow\ k_2[y_n'(a)y_m(a) - y_m'(a)y_n(a)] = 0$$

$$\det = 0 \Rightarrow y_m(b)y_n'(b) - y_m'(b)y_n(b) = 0 \ \Rightarrow\ k_2[y_n'(b)y_m(b) - y_m'(b)y_n(b)] = 0$$

⇨ Right side = 0

11.5 Sturm-Liouville (스트름-리우빌) Problems. Orthogonal Functions

☑ **Proof of** $\int_a^b r(x)\,y_m(x)y_n(x)\,dx = 0$ ⇨ Goal is to prove Right Side

$$= p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)] - p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)]$$

=0

✓ Case 5, $p(a) = p(b)$

Right side ⇨ $p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b) - y_n'(a)y_m(a) + y_m'(a)y_n(a)]$

"periodic boundary conditions" $y(\mathrm{a}) = y(\mathrm{b}),\ y'(\mathrm{a}) = y'(\mathrm{b})$

⇨ Right side = 0

11.5 Sturm-Liouville (스트름-리우빌) Problems. Orthogonal Functions

☑ **Conclusively**

Case 1, $p(a) = p(\mathrm{b}) = 0$

Case 2, $p(a) \neq 0, p(b) = 0$

Case 3, $p(a) = 0, p(b) \neq 0$

Case 4, $p(a) \neq 0, p(b) \neq 0$

Case 5, $p(a) = p(b)$ with periodic boundary conditions $y(\mathrm{a}) = y(\mathrm{b}), y'(\mathrm{a}) = y'(\mathrm{b})$

$$
\begin{aligned}
(\lambda_m - \lambda_n)\int_a^b r y_m y_n dx &= \left[p(y_n' y_m - y_m' y_n) \right]_a^b \\
&= p(b)[y_n'(b) y_m(b) - y_m'(b) y_n(b)] \\
&\quad - p(a)[y_n'(a) y_m(a) - y_m'(a) y_n(a)] \\
&= 0
\end{aligned}
$$

11.5 Sturm-Liouville (스트름-리우빌) Problems. Orthogonal Functions

☑ Example 3 Application of Theorem 1. Vibrating String

$$y'' + \lambda y = 0, y(0) = 0, y(\pi) = 0$$

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0$$

$$k_1 y(x) + k_2 y'(x) = 0 \quad at \quad x = a$$

$$l_1 y(x) + l_2 y'(x) = 0 \quad at \quad x = b$$

- The ODE in Example 1 is a Sturm-Liouville equation with

$$p(x) = 1,\ q = 0,\ r = 1$$

- The solutions are
  Eigenvalues: $\lambda = v^2$ ($v = 0, 1, 2, 3, 4, \ldots$)
  Eigenfunctions: $y(x) = \sin vx$ ($v = 0, 1, 2, 3, 4, \ldots$)
  : infinitely many eigenvalues & orthogonality

- It satisfy Theorem 1 since eigenfunctions are orthogonal on the interval $0 \le x \le \pi$.

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0 \qquad (n \neq m)$$

☑ Example 4 Orthogonality of the Legendre Polynomials

$$\left[p(x)y'\right]' + \left[q(x) + \lambda r(x)\right]y = 0, \quad a \le x \le b$$

Legendre's equation may be written

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \Rightarrow \left[(1-x^2)y'\right]' + \lambda y = 0 \qquad \lambda = n(n+1)$$

$p(x) = (1-x^2),\ q = 0,\ r = 1$ $\quad p(1) = p(-1) = 0$ ➡ We can use these as boundary condition. Thus, we need no boundary conditions!

a singular Sturm-Liouville problem on the interval $-1 \le x \le 1$

$$n = 0 \Rightarrow \lambda = 0, \quad n = 1 \Rightarrow \lambda = 2, \quad n = 2 \Rightarrow \lambda = 6, \quad n = 3 \Rightarrow \lambda = 12,$$

- The Legendre polynomials $P_n(x)$: solutions of the problem ⇒ eigenfuntions
- From Theorem 1, they are orthogonal

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \qquad (m \ne n)$$