Ch. 12 Partial Differential Equations (PDEs)

서울대학교
조선해양공학과
서유택
2018.11

※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

[Review] 1.1 Basic Concepts. Modeling

❖ **Differential Equation (미분방정식)**: An equation containing derivatives of an unknown function

Differential Equation
- **Ordinary Differential Equation (상미분 방정식)**
- **Partial Differential Equation (편미분 방정식)**

❖ **Ordinary Differential Equation**: An equation that contains one or several derivatives of an unknown function (y) of one independent variable (x)

ex) $y' = \cos x, \quad y'' + 9y = e^{-2x}, \quad y'y''' - \frac{3}{2}(y')^2 = 0$

❖ **Partial Differential Equation**: An equation involving partial derivatives of an unknown function (u) of two or more variables (x, y)

ex) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

12.1 Basic Concepts of PDEs

☑ **Partial Differential Equation (PDE): An equation involving one or more partial derivatives of an (unknown) function that depends on two or more variables.**

- Order of the PDE: The order of the highest derivative

PDEs

- Linear: The first degree in the unknown function *u* and its partial derivatives
  - Homogeneous: each of its terms contains either *u* or one of its partial derivatives. (no other terms)
  - Nonhomogeneous
- Nonlinear

12.1 Basic Concepts of PDEs

☑ Ex. 1 Important Second-Order PDEs

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ One-dimensional wave equation

(2) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ One-dimensional heat equation

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ Two-dimensional Laplace equation

(4) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$ Two-dimensional Poisson equation

(5) $$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$ Two-dimensional wave equation

(6) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ Three-dimensional Laplace equation

Here c is a positive constant, t is time, x, y, and z are Cartesian coordinates, and dimension is the number of these coordinates in the equation.

12.1 Basic Concepts of PDEs

- **Solution**: Function that has all the partial derivatives appearing in the PDE in some domain D containing R, and satisfies the PDE everywhere in R.

- Ex. $u = x^2 - y^2,\ u = e^x \cos y,\ u = \sin x \cosh y,\ u = \ln\left(x^2 + y^2\right)$ are solution of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$
  - In general, the totality of solutions of a PDE is very large.
  - The unique solution of a PDE corresponding to a given physical problem will be obtained by the use of additional conditions arising from the problem.

- Additional Conditions

  1. Boundary Conditions (경계조건)
  2. Initial Conditions (초기조건)

12.1 Basic Concepts of PDEs

- **Linear O.D.E.**

The dependent variable y and all its derivatives y', y", …, $y^{(n)}$ are of the first degree, that is, the power of each term involving y is 1.
The coefficients a_0, …, a_n of y', y", …, $y^{(n)}$ depend at most on the independent variable x.

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

ex) $my'' + cy' + ky = f(x)$

$y'' + \alpha y = 0$ — where, $y = y(x)$,

$x^2y'' + xy' - \alpha y = 0$ — $m, c, k = \text{constant}$

$xy'' + y' + \alpha^2 y = 0$ — $n = 0,1,2\ldots$

$(1-x^2)y'' - 2xy + n(n+1)y = 0$

$(1-y)y' + 2y = e^x$

$y'' + \sin y = e^x$

$y^{(4)} + y^2 = e^x$

- **Linear P.D.E**

The dependent variable (u) and its partial derivatives appear only to the first power.

General form of a linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

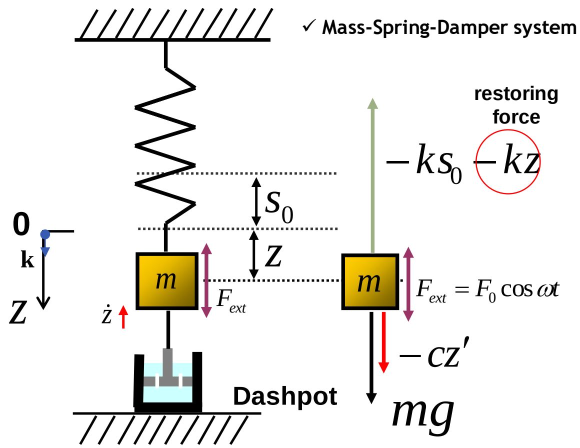
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

12.1 Basic Concepts of PDEs

Linear O.D.E	Linear P.D.E

Ex.) Spring/Mass system

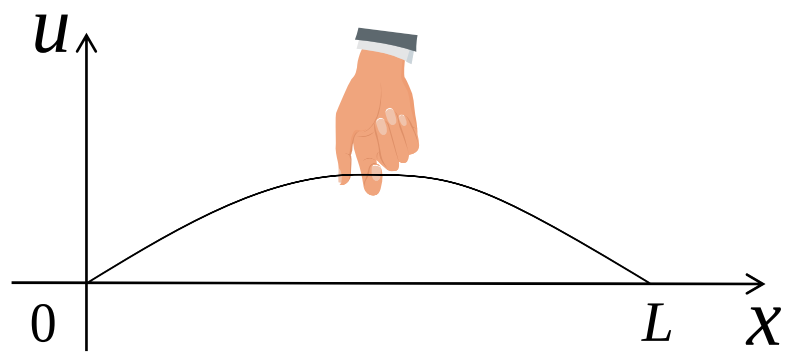
: driven motion with damping



$$m\ddot{z}(t) + c\dot{z}(t) + kz(t) = F_0 \cos\omega t$$

Ex.) Wave Equation

ρ: density, T: tension



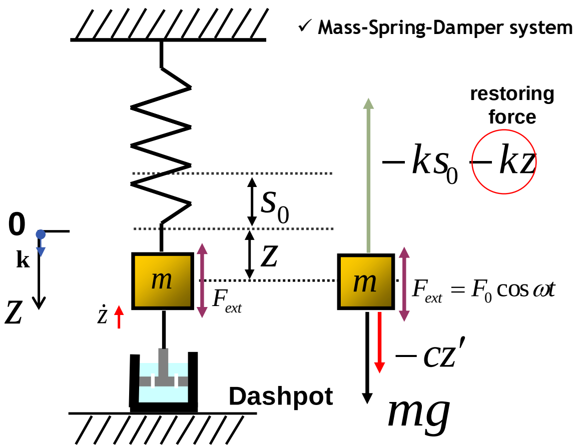
$$u_{xx}(x,t) = \frac{\rho}{T} u_{tt}(x,t)$$

Linear O.D.E		Linear P.D.E
time: t	**independent variable**	x, t: space & time
displacement of the mass at a time: $z = z(t)$	**dependent variable**	$u = u(x,t)$: Displacement of the string on a position(x) at a time(t)
velocity: $\dot{z}(t) = \dfrac{dz(t)}{dt}$		$u_{xx} = \dfrac{\partial^2 u(x,t)}{\partial x^2}$: Second derivative with respect to space
acceleration: $\ddot{z}(t) = \dfrac{d^2 z(t)}{dt^2}$		$u_{tt} = \dfrac{\partial^2 u(x,t)}{\partial t^2}$: acceleration

12.1 Basic Concepts of PDEs

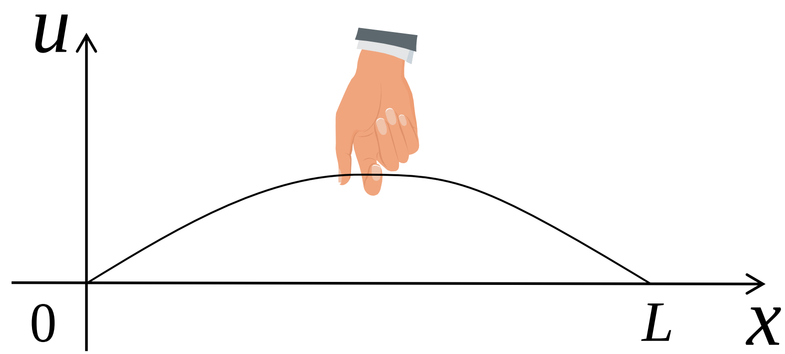
Linear O.D.E	Linear P.D.E
Ex.) Spring/Mass system	**Ex.) Wave Equation**

: driven motion with damping



$$m\ddot{z}(t) + c\dot{z}(t) + kz(t) = F_0 \cos \omega t$$

ρ: density, T: tension



$$u_{xx}(x,t) = \frac{\rho}{T} u_{tt}(x,t)$$

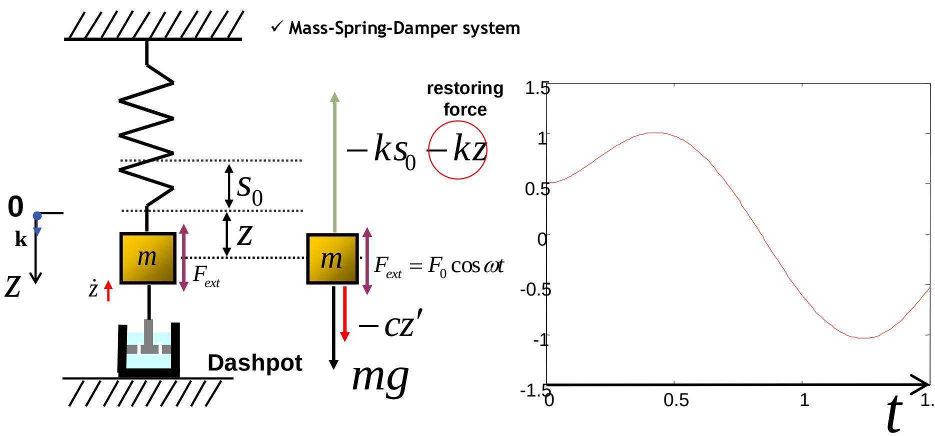
Linear O.D.E		Linear P.D.E
time: t	**independent variable**	x, t: space & time
displacement of the mass at a time: $z = z(t)$	**dependent variable**	$u = u(x,t)$: Displacement of the string on a position(x) at a time(t)
initial condition: $z(0) = a, \dot{z}(0) = b$	**conditions**	initial condition: $u(x,0) = f(x), \frac{\partial u}{\partial t}(x,0) = g(x)$ boundary condition: $u(0,t) = 0, u(L,t) = 0$

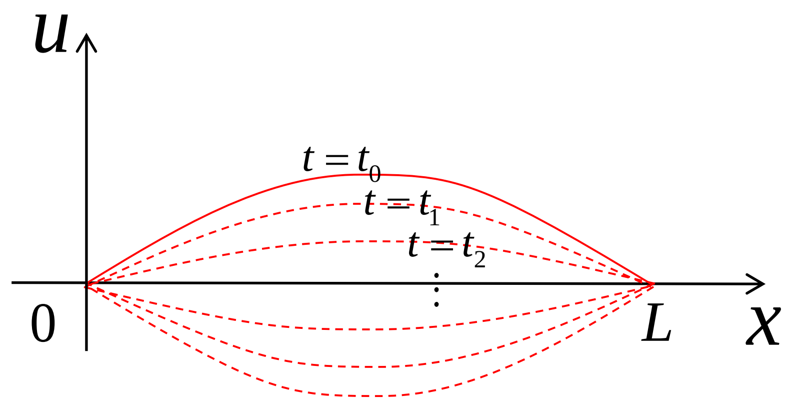
12.1 Basic Concepts of PDEs

Linear O.D.E	Linear P.D.E
Ex.) Spring/Mass system	**Ex.) Wave Equation**

: driven motion with damping

ρ: **density,** T: **tension**





$$m\ddot{z}(t) + c\dot{z}(t) + kz(t) = F_0 \cos \omega t$$

$$u_{xx}(x,t) = \frac{\rho}{T} u_{tt}(x,t)$$

Linear O.D.E			Linear P.D.E
time: t	**independent variable**	x, t	space & time
displacement of the mass at a time: $z = z(t)$	**dependent variable**	$u = u(x,t)$	Displacement of the string on a position(x) at a time(t)
initial condition: $z(0) = a, \dot{z}(0) = b$	**conditions**	$u(x,0) = f(x), \frac{\partial u}{\partial t}(x,0) = g(x)$	initial condition
		$u(0,t) = 0, u(L,t) = 0$	boundary condition

12.1 Basic Concepts of PDEs

$$(1)\ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$(2)\ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

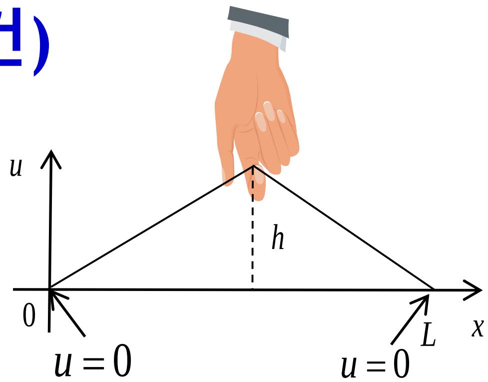
☑ Initial Conditions (초기조건)

- Related to time (t)
- Since solution of equation (1) and (2) *depend on time t*, we can prescribe what happens at t = 0, that is, we can give initial conditions.

$$u(x,0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

☑ Boundary Conditions (경계조건)

- Related to position (x)



$$u(0,t) = 0, \quad u(L,t) = 0, \ t > 0$$

12.1 Basic Concepts of PDEs

☑ Theorem 1 Fundamental Theorem on Superposition

If u_1 and u_2 are solutions of a homogeneous linear PDE in some region *R*, then

$$\mathbf{u} = \mathbf{c}_1\mathbf{u}_1 + \mathbf{c}_2\mathbf{u}_2$$

with any constants $\mathbf{c}_1$ and $\mathbf{c}_2$ is also a solution of that PDE in the region *R*.

12.1 Basic Concepts of PDEs

☑ Ex. 2 Solving $u_{xx} - u = 0$ like an ODE

- Find solutions u of the PDE $u_{xx} - u = 0$ depending on x and y.
- No y-derivatives occur → Solve this PDE like $u'' - u = 0$

$$u = Ae^{x} + Be^{-x} \quad \Rightarrow \quad \therefore u = u(x, y) = A(y)e^{x} + B(y)e^{-x}$$

☑ Ex. 3 Solving $u_{xy} = -u_x$ like an ODE

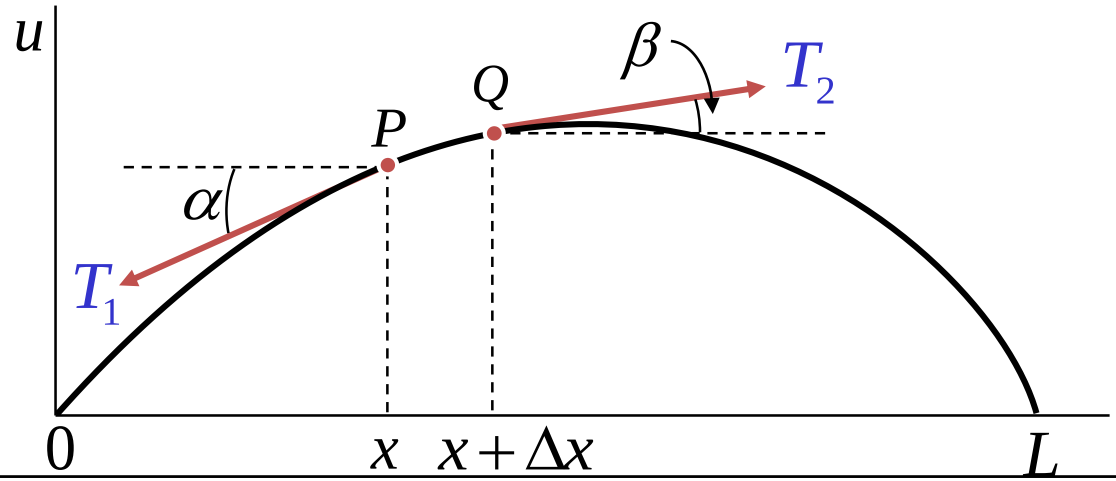
- Find solutions $u = u(x, y)$ of this PDE.
- Setting $u_x = p$ ⇨ $p_y = -p$ ⇨ $p = c(x)e^{-y}$
- By integration with respect to x,

$$u(x, y) = f(x)e^{-y} + g(y), \qquad f(x) = \int c(x)\,dx$$

12.2 Modeling: Vibrating String, Wave Equation

☑ 1-D Wave Equation (파동방정식)

- Drive the equation modeling small transverse vibrations of an elastic string.
- We place the string along the *x*-axis, stretch it to length L, and fasten it at the ends $x = 0$ and $x = L$.
- The problem is to determine the vibration of the string, that is to find its deflection $u(x, t)$ at any point x and at any time $t > 0$.

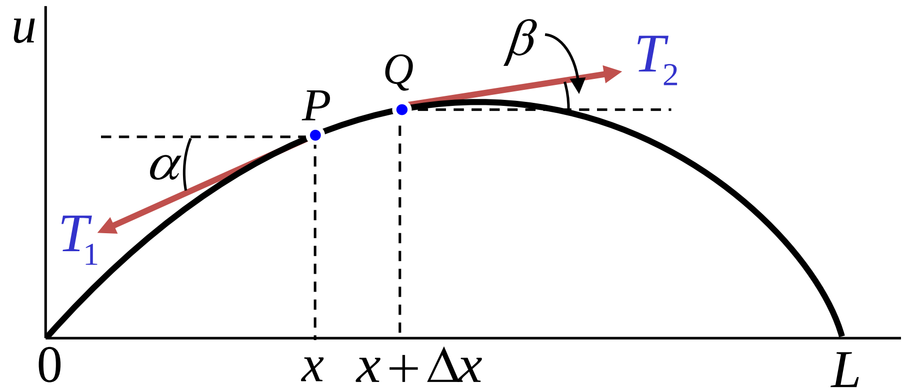


12.2 Modeling: Vibrating String, Wave Equation

☑ 1-D Wave Equation

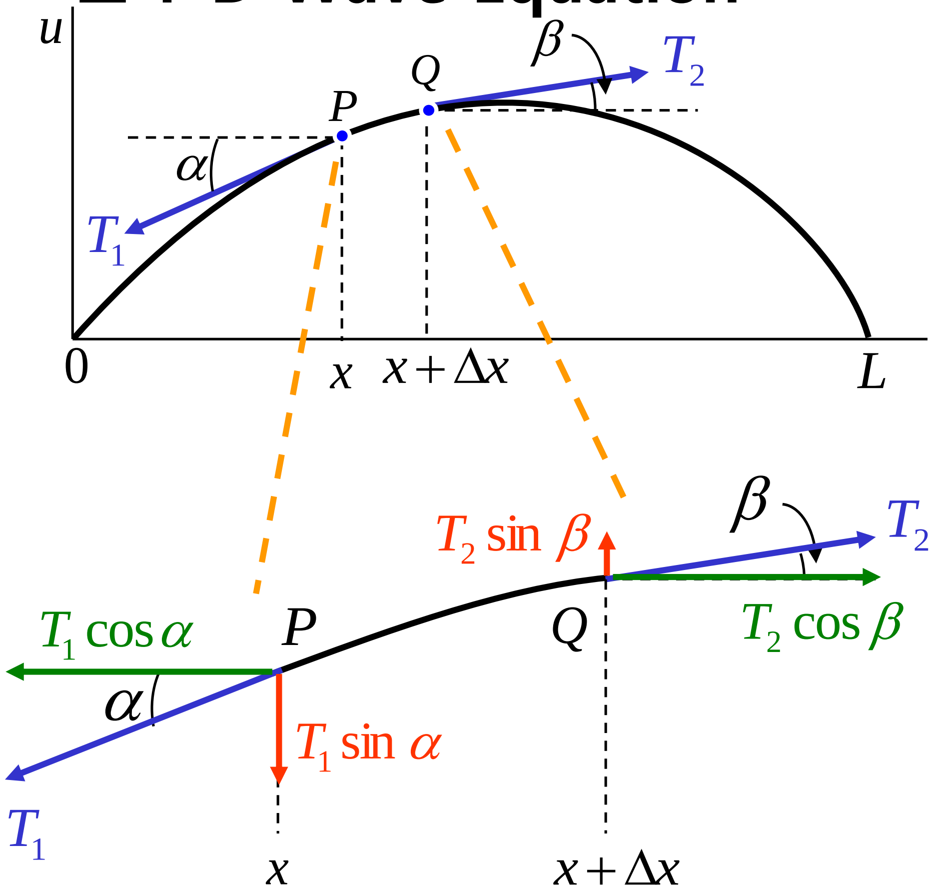
Physical Assumptions

1. The mass of the string per unit length is constant ("homogeneous string"). The string is perfectly elastic and does not offer any resistance to bending.
2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.



12.2 Modeling: Wave Equation

☑ 1-D Wave Equation



T_1, T_2 : tensions at the end point P, Q and they are directed along the tangents at the points

Forces acting on a small portion of the string

- **Point of the string moves vertically. No motion in the horizontal direction**

$$\therefore T_2 \cos\beta - T_1 \cos\alpha = 0$$

$$\therefore T_1 \cos\alpha = T_2 \cos\beta$$
$$= T = \text{constant} \cdots (1)$$

- **Net force acting on a small portion of the string in the vertical direction**

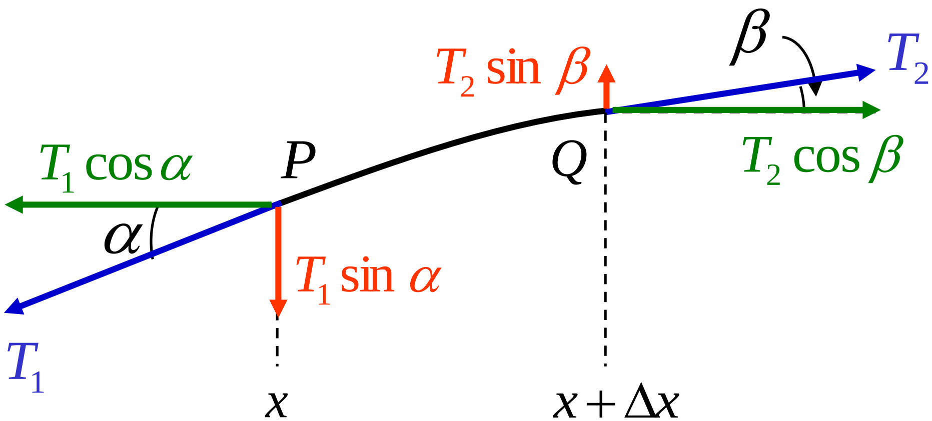
$$\text{net force} = T_2 \sin\beta - T_1 \sin\alpha$$

12.2 Modeling: Wave Equation

$T_1 \cos\alpha = T_2 \cos\beta = T = \text{constant} \cdots (1)$

$\text{net force} = T_2 \sin\beta - T_1 \sin\alpha$

☑ 1-D Wave Equation



- **The mass of the small portion**

$$\Delta m = \rho \Delta x$$

ρ : **mass of the undeflected string per unit length**

Δx : **length of the portion of the undeflected string**

- **The acceleration**

$$a = \frac{\partial^2 u}{\partial t^2}, \quad \text{where } u(x,t): \text{deflection}$$

- **The inertia force of the small portion**

$$\Delta m a = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

- **net force = inertia force**
  Newton's second law

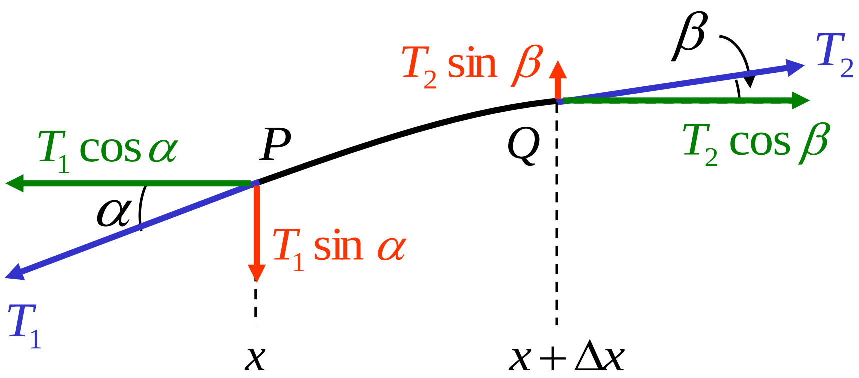
$$\therefore T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

12.2 Modeling: Wave Equation

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{constant} \cdots (1)$$

$$\text{net force} = T_2 \sin\beta - T_1 \sin\alpha$$

☑ 1-D Wave Equation



- **net force = inertia force**

$$\therefore T_2 \sin\beta - T_1 \sin\alpha = \rho\Delta x \frac{\partial^2 u}{\partial t^2}$$

Dividing both sides by (1)

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\therefore \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\because \tan\alpha = \left(\frac{\partial u}{\partial x}\right)_{x}, \quad \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

Letting $\Delta x \to 0$, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{\rho}$$

positive

12.3 Solution by Separating Variables. Use of Fourier Series

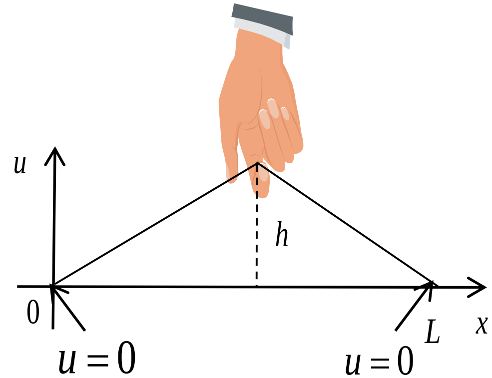
☑ Model of a vibrating elastic string

- One-dimensional wave equation: $\dfrac{\partial^2 u}{\partial x^2} = c^2 \dfrac{\partial^2 u}{\partial t^2}$

☑ Initial Conditions

- Related to time (t)
- The motion of the string will depend on its initial deflection f(x) and its initial velocity g(x).

$$u(x,0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$



☑ Boundary Conditions

- Related to position (x)

$$u(0,t) = 0, \quad u(L,t) = 0, \; t > 0$$

☑ Finding a solution of PDE

Step 1. Method of Separating Variables (변수분리법)

or product method:

Setting $u(x, t) = F(x)G(t)$

Step 2. Determine solutions of ODEs that satisfy the boundary conditions

Step 3. Using Fourier series for a solution

☑ Step 1. Two ODEs from the Wave Equation

$$(1)\quad c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

$$u(x,y)=F(x)G(y) \quad\Rightarrow\quad \frac{\partial^2 u}{\partial x^2}=F''(x)G(t),\quad \frac{\partial^2 u}{\partial t^2}=F(x)\ddot{G}(t)$$

By inserting this into the wave equation and dividing by c^2FG

$$F(x)\ddot{G}(t)=c^2F''(x)G(t) \quad\Rightarrow\quad \frac{\ddot{G}(t)}{c^2G(t)}=\frac{F''(x)}{F(x)}=k$$

(Both sides should be constant!)

(depends on t?) (depends on x?)

The variables are now separated.

- The left side depending only on *t* and the right side only on *x*

$$\therefore F''(x)-kF(x)=0 \text{ and } \ddot{G}(t)-c^2kG(t)=0$$

Now we get two ODEs.

☑ Step 2. Satisfying the Boundary Conditions

Boundary conditions: $u(0, t) = F(0)G(t) = 0$ and $u(L, t) = F(L)G(t) = 0$

⇨ $F''(x) - kF(x) = 0, \quad F(0) = F(L) = 0$

$$F''(x) - kF(x) = 0$$
$$\ddot{G}(t) - c^2kG(t) = 0$$

Case 1. $\boldsymbol{k = p^2 > 0}$

$$F''(x) - p^2F(x) = 0 \quad \Rightarrow \quad F(x) = Ae^{px} + Be^{-px}$$

$$F(0) = A + B = 0 \text{ and } F(L) = Ae^{pL} + Be^{-pL} = 0$$

$$\Rightarrow B = -A \text{ and } A\left(e^{2pL} - 1\right) = 0 \Rightarrow A = 0$$

$$\therefore \quad F = 0 \quad \Rightarrow \quad u \equiv 0 \left(\text{No interest}\right)$$

Case 2. $\boldsymbol{k = 0}$

$$F''(x) = 0 \quad \Rightarrow \quad F(x) = Ax + B$$

$$F(0) = B = 0, \quad F(L) = AL + B = 0 \quad \Rightarrow \quad A = 0 \left(L \neq 0\right)$$

$$\therefore \quad F = 0 \quad \Rightarrow \quad u \equiv 0 \left(\text{No interest}\right)$$

12.3 Solution by Separating Variables. Use of Fourier Series

$$u(x,y)=F(x)G(y)$$
$$F''(x)-kF(x)=0$$
$$\ddot{G}(t)-c^2kG(t)=0$$

Case 3. $k = -p^2 < 0$

$$F''(x)+p^2F(x)=0 \quad \Rightarrow \quad F(x)=A\cos px+B\sin px$$

$$F(0)=A=0, \quad F(L)=A\cos pL+B\sin pL=0 \quad \Rightarrow \quad B\sin pL=0$$

$$B=0 \quad \Rightarrow \quad F=0 \quad \Rightarrow \quad u\equiv 0(\text{No interest})$$

$$\sin pL=0 \quad \Rightarrow \quad p=\frac{n\pi}{L}(n:\text{integer})$$

Setting $B=1$,

$$\therefore \quad F(x)=F_n(x)=\sin\frac{n\pi}{L}x \quad (n=1,\ 2,\ 3,\ \cdots)$$

Solve $\ddot{G}(t)-c^2kG(t)=0$ **with** $k=-p^2=-\left(\frac{n\pi}{L}\right)^2$

$$\ddot{G}(t)+\lambda_n{}^2G(t)=0, \quad \lambda_n=cp=\frac{cn\pi}{L} \quad \Rightarrow \quad G_n(t)=B_n\cos\lambda_n t+B_n^*\sin\lambda_n t$$

Solutions : $u(x,y)=F(x)G(y)$

$$u_n(x,t)=\left(B_n\cos\lambda_n t+B_n^*\sin\lambda_n t\right)\sin\frac{n\pi}{L}x \quad (n=1,\ 2,\ 3,\ \cdots)$$

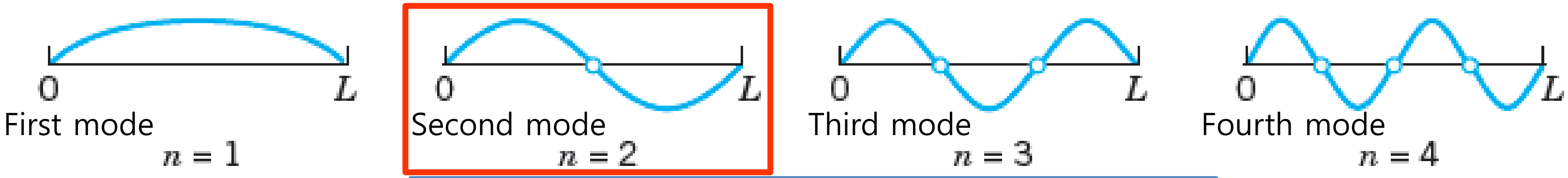
12.3 Solution by Separating Variables. Use of Fourier Series

☑ Discussion of Eigenfunctions (고유함수)

$$u_n(x,t) = \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right)\sin\frac{n\pi}{L}x \quad (n = 1,\ 2,\ 3,\ \cdots)$$

- Eigenfunctions or Characteristic Functions (고유 또는 특성함수): $u_n(x, t)$
- Eigenvalues or Characteristic Values: $\lambda_n = \dfrac{cn\pi}{L}$
- Spectrum: $\{\lambda_1,\ \lambda_2,\ \cdots\}$
- u_n represents a harmonic motion (nth normal motion) having the frequency $\dfrac{\lambda_n}{2\pi} = \dfrac{cn}{2L}$ cycles per unit time.
- **Nodes** (마디점): Points of the string that do not move.

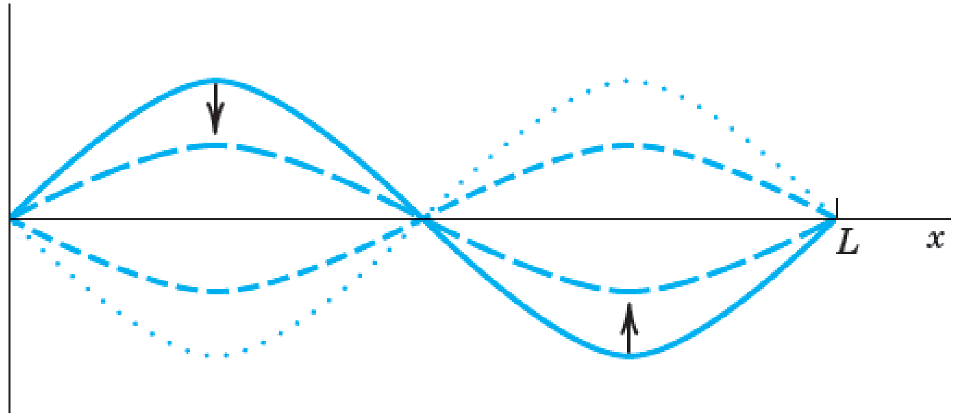
$$\sin\frac{n\pi x}{L} = 0 \quad \text{at} \;\; x = \frac{L}{n},\ \ \frac{2L}{n},\ \ \cdots,\ \ \frac{n-1}{n}L$$



Normal modes of the vibrating string

12.3 Solution by Separating Variables. Use of Fourier Series

$$u_n(x,t) = \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) \sin \frac{n\pi}{L} x \quad (n = 1,\ 2,\ 3,\ \cdots)$$



Second normal mode for various values of t

☑ Tuning (조율)

$$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}, \quad c^2 = \frac{T}{\rho} \Rightarrow \lambda_n = \frac{cn\pi}{L} \Rightarrow u_n(x,t) = \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) \sin \frac{n\pi}{L} x$$

T: Tension of the string

Q: What happens if T is increased?

If T increases, frequency (= $\frac{\lambda_n}{2\pi}$ or $\frac{cn}{2L}$) also increases.

☑ Step 3. Solution of the Entire Problem. Fourier Series

- Consider the infinite series

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\left(B_n\cos\lambda_n t+B_n^*\sin\lambda_n t\right)\sin\frac{n\pi}{L}x$$

- Satisfying Initial Condition (Given Initial Displacement)

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{L}x \quad (f \text{ odd})$$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx, \quad n=1,\ 2,\ \cdots$$

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi}{L}x=f(x)$$

Fourier sine series of ***f*** **(*x*)**:

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

- Satisfying Initial Condition (Given Initial Velocity)

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=\left[\sum_{n=1}^{\infty}\left(-B_n\lambda_n\sin\lambda_n t+B_n^*\lambda_n\cos\lambda_n t\right)\sin\frac{n\pi}{L}x\right]_{t=0}=\sum_{n=1}^{\infty}B_n^*\lambda_n\sin\frac{n\pi}{L}x=g(x)$$

Fourier sine series of ***f*****(*x*)**:

$$B_n^*\lambda_n=\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx \quad\Rightarrow\quad B_n^*=\frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

where, $\lambda_n=\frac{cn\pi}{L}$

12.3 Solution by Separating Variables. Use of Fourier Series

$$u(x,t)=\sum_{n=1}^{\infty}\left(B_n\cos\lambda_n t+B_n^*\sin\lambda_n t\right)\sin\frac{n\pi}{L}x$$

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$B_n^*=\frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

☑ Solution Established

- For the sake of simplicity, the initial velocity $g(x)$ is identically zero.

$$g(x)=0 \quad\Rightarrow\quad B_n{}^*=0$$

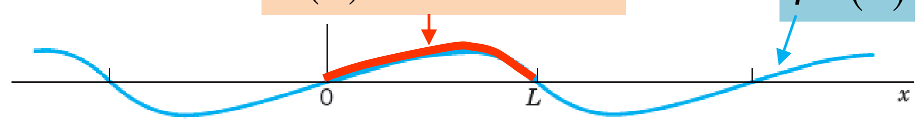
$$u(x,t)=\sum_{n=1}^{\infty}B_n\cos\lambda_n t\sin\frac{n\pi}{L}x\,,\quad \lambda_n=\frac{cn\pi}{L}\quad\Rightarrow\quad u(x,t)=\sum_{n=1}^{\infty}B_n\cos\frac{cn\pi}{L}t\sin\frac{n\pi}{L}x\,,$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}B_n\left[\sin\left\{\frac{n\pi}{L}(x-ct)\right\}+\sin\left\{\frac{n\pi}{L}(x+ct)\right\}\right]=\frac{1}{2}\left[f^*(x-ct)+f^*(x+ct)\right]$$

- ***f*(x)*: Odd periodic extension (기주기 확장) of *f*(*x*)**

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi}{L}x=f(x)$$

Let $x = x - ct,\ x = x + ct$



Odd periodic extension of *f*(x)

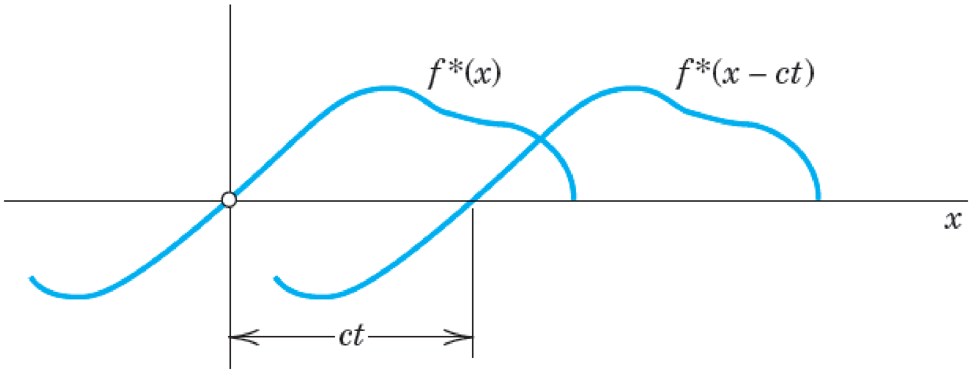
12.3 Solution by Separating Variables. Use of Fourier Series

☑ Physical interpretation of $u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct) \right]$

$f^*(x-ct)$: by shifting $f(x)$ ct units to the right $\Rightarrow$ traveling to the right

$f^*(x+ct)$: by shifting $f(x)$ ct units to the left $\Rightarrow$ traveling to the left

$u(t)$: superpositon of $f^*(x-ct)$ and $f^*(x+ct)$



12.3 Solution by Separating Variables. Use of Fourier Series

$$u(x,t)=\frac{1}{2}\left[f^*(x-ct)+f^*(x+ct)\right]$$

☑ Ex. 1 Vibrating String if the Initial Deflection is Triangular

Find the solution of the wave equation corresponding to the triangular initial deflection

$$f(x)=\begin{cases}\dfrac{2k}{L}x & \left(0<x<\dfrac{L}{2}\right)\\ \dfrac{2k}{L}(L-x) & \left(\dfrac{L}{2}<x<L\right)\end{cases}$$

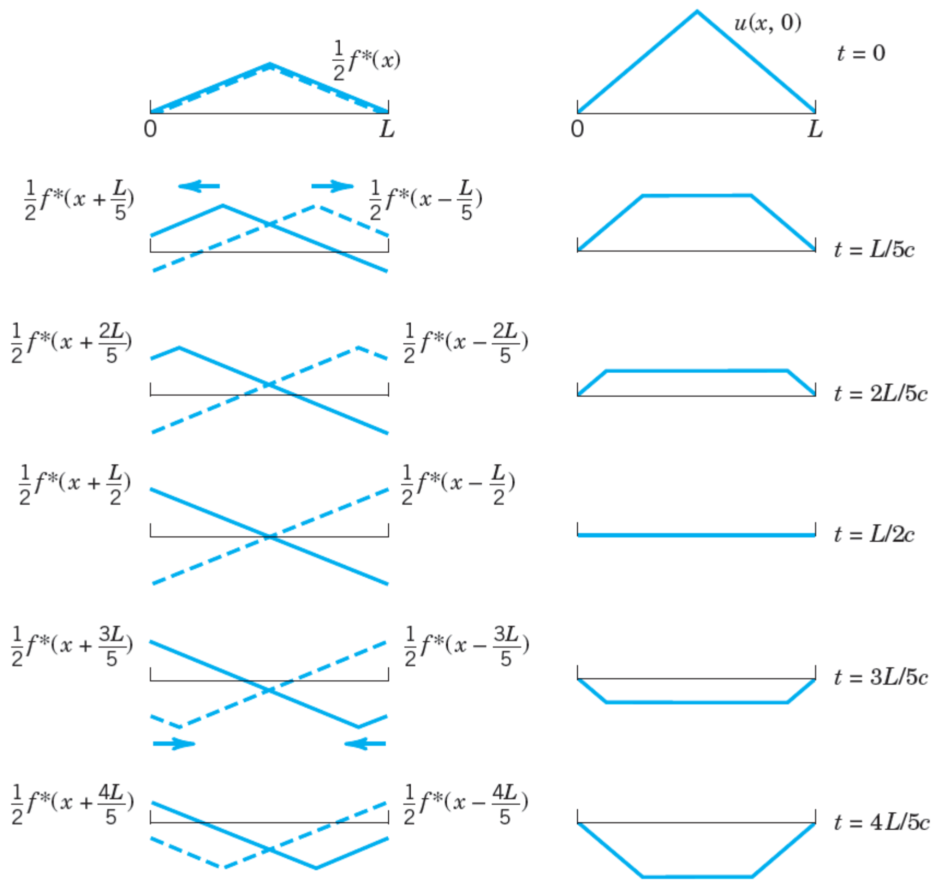
and initial velocity zero, $g(x)=0$.

Sol)

$$u(x,t)=\sum_{n=1}^{\infty}B_n\cos\lambda_n t\sin\frac{n\pi}{L}x$$

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$u(x,t)=\frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t-\frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi c}{L}t+\cdots\right]$$



Solution for various values of t obtained as the superposition of a wave traveling to the right and a wave traveling to the left

12.4 D'Alembert Solution of the Wave Equation. Characteristics

☑ Wave equation: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \qquad c^2 = \dfrac{T}{\rho}$

- Use the new independent variables $v = x + ct$ and $w = x - ct$
- u becomes a function of v and w. $v_x = \dfrac{\partial v}{\partial x} = 1, \;\; w_x = \dfrac{\partial w}{\partial x} = 1$
- The derivatives can be expressed in terms of derivatives with respect to v and w by using the chain rule.

$$u_x = u_v v_x + u_w w_x = u_v + u_w \qquad v_x = 1, \;\; w_x = 1, \; u_{vw} = u_{wv}$$

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = \boxed{u_{vv} + 2u_{vw} + u_{ww}}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \;\Rightarrow\; u_{tt} = c^2 (u_{vv} + 2u_{vw} + u_{ww})$$

12.4 D'Alembert Solution of the Wave Equation. Characteristics

☑ **Wave equation:** $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \qquad c^2 = \dfrac{T}{\rho}$

- Use the new independent variables $v = x + ct$ and $w = x - ct$
- Transform $\dfrac{\partial^2 u}{\partial t^2}$ by the same procedure

$$v_t = c, \;\; w_t = -c \qquad u_t = u_v v_t + u_w w_t = c(u_v - u_w)$$

$$u_{tt} = c(u_v - u_w)_v v_t + c(u_v - u_w)_w w_t = \boxed{c^2(u_{vv} - 2u_{vw} + u_{ww})}$$

☑ D'Alembert's Solution of the Wave Equation

$$u_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww})$$

$$u_{xx} = u_{vv} + 2u_{vw} + u_{ww}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow c^2(u_{vv} - 2u_{vw} + u_{ww}) = c^2(u_{vv} + 2u_{vw} + u_{ww})$$

$$\Rightarrow u_{vw} \equiv \frac{\partial^2 u}{\partial w \partial v} = 0$$

- Integration first with respect to w and then v

$$\frac{\partial u}{\partial v} = h(v) \Rightarrow u = \int h(v)dv + \phi(w) \Rightarrow u = \varphi(v) + \phi(w)$$

- In terms of x and t. ($v = x + ct$ and $w = x - ct$)

$$u(x,t) = \varphi(x+ct) + \phi(x-ct)$$

12.4 D'Alembert Solution of the Wave Equation. Characteristics

☑ D'Alembert Solution Satisfying the Initial Conditions

$$u(x,t)=\varphi(x+ct)+\phi(x-ct) \quad \Rightarrow \quad u_t(x,t)=c\varphi'(x+ct)-c\phi'(x-ct)$$

- Given initial displacement

$$u(x,0)=f(x) \quad \Rightarrow \quad u(x,0)=\boxed{\varphi(x)+\phi(x)=f(x)}$$

- Given initial velocity

$$u_t(x,0)=g(x) \quad \Rightarrow \quad u_t(x,0)=c\varphi'(x)-c\phi'(x)=g(x)$$

$$\Rightarrow \quad \boxed{\varphi(x)-\phi(x)=k(x_0)+\frac{1}{c}\int_{x_0}^{x} g(s)ds, \quad k(x_0)=\varphi(x_0)-\phi(x_0)}$$

$$\varphi(x)=\frac{1}{2}f(x)+\frac{1}{2c}\int_{x_0}^{x} g(s)ds+\frac{1}{2}k(x_0), \quad \phi(x)=\frac{1}{2}f(x)-\frac{1}{2c}\int_{x_0}^{x} g(s)ds-\frac{1}{2}k(x_0)$$

$$\varphi(x+ct)+\phi(x-ct)=\frac{1}{2}f(x+ct)+\frac{1}{2}f(x-ct)+\frac{1}{2c}\int_{x_0}^{x+ct} g(s)ds-\frac{1}{2c}\int_{x_0}^{x-ct} g(s)ds$$

$$\therefore \quad u(x,t)=\frac{1}{2}f(x+ct)+\frac{1}{2}f(x-ct)+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

$g(x)=0$ ⇨ $$u(x,t)=\frac{1}{2}f(x+ct)+\frac{1}{2}f(x-ct)$$

☑ Characteristics. Types and Normal Forms of PDEs

Quasilinear equation: $Au_{xx}+2Bu_{xy}+Cu_{yy} = F(x, y, u, u_x, u_y)$, or $y = ct$

(준선형식)

Type	Defining Condition	Example in Sec. 12.1
Hyperbolic (쌍곡선)	$AC - B^2 < 0$	Wave equation (1)
Parabolic (포물선)	$AC - B^2 = 0$	Heat equation (2)
Elliptic (타원)	$AC - B^2 > 0$	Laplace equation (3)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Example)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

$$A = 1, B = 0, C = -1$$

$$AC - B^2 = -1 < 0$$

Hyperbolic

$$3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$$

$$A = 3, B = 0, C = 0$$

$$AC - B^2 = 0$$

Parabolic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$A = 1, B = 0, C = 1$$

$$AC - B^2 = 1 > 0$$

Elliptic

12.4 D'Alembert Solution of the Wave Equation. Characteristics

☑ Important Second-Order PDEs

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ One-dimensional wave equation

(2) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ One-dimensional heat equation

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ Two-dimensional Laplace equation

(4) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$ Two-dimensional Poisson equation

(5) $$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$ Two-dimensional wave equation

(6) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ Three-dimensional Laplace equation

Here c is a positive constant, t is time, x, y, and z are Cartesian coordinates, and dimension is the number of these coordinates in the equation.

12.4 D'Alembert Solution of the Wave Equation. Characteristics

☑ Characteristics. Types and Normal Forms of PDEs

Quasilinear equation: $Au_{xx}+2Bu_{xy}+Cu_{yy} = F(x, y, u, u_x, u_y)$

Type	Defining Condition	Example in Sec. 12. 1
Hyperbolic	$AC-B^2<0$	Wave equation (1)
Parabolic	$AC-B^2=0$	Heat equation (2)
Elliptic	$AC-B^2>0$	Laplace equation (3)

☑ Transformation to Normal Form

Characteristic equation: $Ay'^2-2By'+Cy = 0$

Type	New Variables	Normal Form
Hyperbolic	$v=\Phi, \quad w=\Psi$	$u_{vw}=F_1$
Parabolic	$v=x, \quad w=\Phi=\Psi$	$u_{ww}=F_2$
Elliptic	$v=\frac{1}{2}(\Phi+\Psi), \quad w=\frac{1}{2i}(\Phi-\Psi)$	$u_{vv}+u_{ww}=F_3$

12.4 D'Alembert Solution of the Wave Equation. Characteristics

☑ Ex. 1 D'Alembert's Solution Obtained Systematically

The theory of characteristics (특성) gives d'Alembert's solution in a systematic fashion. Consider the wave equation $u_{tt} - c^2 u_{xx} = 0$ (Hyperbola, 쌍곡선)

Setting $y = ct \rightarrow u_t = u_y y_t = c\, u_y \rightarrow u_{tt} = c^2 u_{yy} \rightarrow u_{xx} - u_{yy} = 0$

Sol) $Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$ $Ay'^2 - 2By' + Cy = 0$ (A = 1, B = 0, C = -1)

Characteristic equation: $y'^2 - 1 = (y' + 1)(y' - 1) = 0$

Families of solutions:

$y' + 1 = 0 \Rightarrow \Phi(x, y) = y + x = \text{const.}$

$y' - 1 = 0 \Rightarrow \psi(x, y) = y - x = \text{const.}$

Type	New Variables	Normal Form
Hyperbolic	$v = \Phi, \quad w = \Psi$	$u_{vw} = F_1$

$v = y + x,\ w = y - x, \Rightarrow v_x = 1,\ w_x = -1,\ v_y = 1,\ w_y = 1$

$u_x = u_v v_x + u_w w_x = u_v - u_w$

$\Rightarrow u_{xx} = u_{vv} v_x + u_{vw} w_x - u_{wv} v_x - u_{ww} w_x = u_{vv} - 2u_{vw} + u_{ww}$

$u_y = u_v v_y + u_w w_y = u_v + u_w$

$\Rightarrow u_{yy} = u_{vv} v_y + u_{vw} w_y + u_{wv} v_y + u_{ww} w_y = u_{vv} + 2u_{vw} + u_{ww}$

☑ Ex. 1 D'Alembert's Solution Obtained Systematically

The theory of characteristics gives d'Alembert's solution in a systematic fashion. Consider the wave equation $u_{tt} - c^2 u_{xx} = 0$ (Hyperbola, 쌍곡선)

Setting $y = ct \rightarrow u_t = u_y y_t = c\, u_y \rightarrow u_{tt} = c^2 u_{yy} \rightarrow u_{xx} - u_{yy} = 0$

Families of solutions:

$$y' + 1 = 0 \;\Rightarrow\; \Phi(x, y) = v = y + x = \text{const.}$$

$$y' - 1 = 0 \;\Rightarrow\; \psi(x, y) = w = y - x = \text{const.}$$

Type	New Variables	Normal Form
Hyperbolic	$v = \Phi, \quad w = \Psi$	$u_{vw} = F_1$

$$u_{xx} = u_{vv} - 2u_{vw} + u_{ww}$$
$$u_{yy} = u_{vv} + 2u_{vw} + u_{ww}$$

⇨ $u_{xx} - u_{yy} = 0$ ⇨ $u_{vw} = 0$

$$\frac{\partial^2 u}{\partial w \partial v} = 0 \Rightarrow \frac{\partial u}{\partial v} = h(v) \Rightarrow u = \int h(v)dv + f_2(w) = f_1(v) + f_2(w)$$

$(v = x + ct \text{ and } w = x - ct)$

D'Alembert's solution: $u = f_1(x + ct) + f_2(x - ct)$

That is, we can get d'Alembert's solution from Quasilinear equation (The theory of characteristics).

12.4 D'Alembert Solution of the Wave Equation. Characteristics

☑ **Example:** Find the type, transform to normal form, and solve. Show your work in detail. $u_{xx} + 5u_{xy} + 4u_{yy} = 0$

Sol) $Au_{xx}+2Bu_{xy}+Cu_{yy} = F(x, y, u, u_x, u_y)$ $Ay'^2 - 2By'+Cy = 0$

$A = 1,\ B = -5/2,\ C = 4 \Rightarrow AC - B^2 = -9/4 < 0$ **Hyperbolic**

Characteristic equation: $y'^2 - 5y' + 4 = (y' - 4)(y' - 1) = 0$

$$y' - 4 = 0 \ \Rightarrow\ \Phi(x, y) = y - 4x = \text{const.}$$
$$y' - 1 = 0 \ \Rightarrow\ \psi(x, y) = y - x = \text{const.}$$
$$v = y - 4x,\ w = y - x,\ \Rightarrow\ v_x = -4,\ w_x = -1,\ v_y = 1,\ w_y = 1$$
$$u_x = u_v v_x + u_w w_x = -4u_v - u_w,$$
$$\Rightarrow u_{xx} = -4u_{vv}v_x - 4u_{vw}w_x - u_{wv}v_x - u_{ww}w_x = 16u_{vv} + 8u_{vw} + u_{ww}$$
$$\Rightarrow u_{xy} = -4u_{vv}v_y - 4u_{vw}w_y - u_{wv}v_y - u_{ww}w_y = -4u_{vv} - 5u_{vw} - u_{ww}$$
$$u_y = u_v v_y + u_w w_y = u_v + u_w$$
$$\Rightarrow u_{yy} = u_{vv}v_y + u_{vw}w_y + u_{wv}v_y + u_{ww}w_y = u_{vv} + 2u_{vw} + u_{ww}$$

12.4 D'Alembert Solution of the Wave Equation. Characteristics

☑ **Example:** Find the type, transform to normal form, and solve. Show your work in detail. $u_{xx} + 5u_{xy} + 4u_{yy} = 0$

$$u_{xx} = 16u_{vv} + 8u_{vw} + u_{ww}$$
$$u_{xy} = -4u_{vv} - 5u_{vw} - u_{ww}$$
$$u_{yy} = u_{vv} + 2u_{vw} + u_{ww}$$

⇨ $u_{xx} + 5u_{xy} + 4u_{yy} = 0$ ⇨ $-9u_{vw} = 0$

$$\frac{\partial^2 u}{\partial w \partial v} = 0 \Rightarrow \frac{\partial u}{\partial v} = h(v) \Rightarrow u = \int h(v)dv + f_2(w) = f_1(v) + f_2(w)$$

12.5 Modeling: Heat Flow from a Body in Space. Heat Equation

☑ Derive the equation modeling temperature distribution under the following

- **Physical Assumptions**

1. The specific heat (비열) σ and the density ρ of the material of the body are constant. No heat is produced or disappears in the body.

2. Experiments show that, in a body, heat flows in the direction of decreasing temperature (열은 온도가 높은 곳에서 낮은 곳으로 흐르고), and the rate of flow is proportional to the gradient of the temperature (열전도율은 온도의 기울기에 비례);
   the velocity $\mathbf{v}$ of the heat flow in the body:

   $$v = -K\,\text{grad}(u)$$

   where **$u(x, y, z, t)$: the temperature at a point (x, y, z) and time t.**

3. The thermal conductivity K (열전도율) is constant, as is the case for homogeneous material and nonextreme temperatures.

12.5 Modeling: Heat Flow from a Body in Space. Heat Equation

☑ Derivation of the PDE of the Model

$v = -K\,\mathrm{grad}(u)$

- T: a region in the body bounded by a surface S
- Total amount of heat that flows across S from T: $\iint_S \mathbf{v}\cdot\mathbf{n}\,dA$
- Using Gauss's theorem of divergence

$$\iiint_T \mathrm{div}\,\mathbf{F}\,dV = \iint_S \mathbf{F}\bullet\mathbf{n}\,dA$$

$$\mathrm{div}(\mathrm{grad}\,u) = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\iint_S \mathbf{v}\cdot\mathbf{n}\,dA = -K\iint_S (\mathrm{grad}\,u)\cdot\mathbf{n}\,dA = -K\iiint_T \mathrm{div}(\mathrm{grad}\,u)\,dxdydz = -K\iiint_T \nabla^2 u\,dxdydz$$

- Total amount of heat in T: $H = \iiint_T \sigma\rho u\,dxdydz$
- Time rate of decrease of H: $-\frac{dH}{dt} = -\iiint_T \sigma\rho\frac{\partial u}{\partial t}\,dxdydz$

It's same. Since No heat is produced or disappears in the body

$$\Rightarrow\quad -\iiint_T \sigma\rho\frac{\partial u}{\partial t}\,dxdydz = -K\iiint_T \nabla^2 u\,dxdydz \quad\Rightarrow\quad \frac{\partial u}{\partial t} = c^2\nabla^2 u,\quad c^2 = \frac{K}{\sigma\rho}$$

12.6 Heat Equation: Solution by Fourier Series. Steady Two-Dimensional Heat Problems. Dirichlet Problem

☑ Heat Equation (열전도 방정식): $\dfrac{\partial u}{\partial t} = c^2 \nabla^2 u, \quad c^2 = \dfrac{K}{\sigma\rho}$

- c^2 : Thermal diffusivity (열확산 계수)
- K : Thermal conductivity (열전도도), kcal/m·sec·°C
- σ : Specific heat (비열), kcal/kg·°C
- P : Density (밀도), kg/m^3
- $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$
  : Laplacian of u with respect to Cartesian coordinates *x*, *y*, *z*

☑ One-dimensional heat equation: $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

- Boundary Condition: $u(0, t) = u\,(L, t) = 0$ for all t
- Initial Condition: $u(x,0) = f\,(x)$, ($f\,(x)$ given)

* 열전도도: 열의 이동이 정상 상태에서 면적 1m^2, 두께 1m의 물질 내를 온도차 1℃로 하고 1초에 흐르는 열량을 kcal로 나타낸 것

12.6 Heat Equation: Solution by Fourier Series. Steady Two-Dimensional Heat Problems. Dirichlet Problem

Step 1. Two ODEs from the heat equation

Substitution of a product $u(x, t) = F(x)G(t)$ into the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad F\dot{G} = c^2 F''G \qquad \frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} \left(= -p^2\right)$$

(depends on t?) (depends on x?) (Both sides should be constant!)

$$\therefore F''(x) + p^2 F(x) = 0, \quad \dot{G}(t) + c^2 p^2 G(t) = 0$$

Step 2. Satisfying the boundary conditions

Boundary condition

$u(0, t) = F(0)G(t) = 0$ and $u(L, t) = F(L)G(t) = 0 \Rightarrow F(0) = F(L) = 0$

General solution of $F''(x) + p^2 F(x) = 0$: $F(x) = A \cos px + B \sin px$

$$\therefore F(x) = F_n(x) = \sin \frac{n\pi}{L} x, \quad p = \frac{n\pi}{L} \left(n = 1, \ 2, \ 3, \ \cdots\right)$$

General solution of $\dot{G}(t) + c^2 p^2 G(t) = 0 \Rightarrow G_n(t) = B_n e^{-\lambda_n^2 t}, \ \lambda_n = \dfrac{cn\pi}{L}$

12.6 Heat Equation: Solution by Fourier Series. Steady Two-Dimensional Heat Problems. Dirichlet Problem

Solutions of the heat equation: $u_n(x,t) = B_n \sin \frac{n\pi x}{L} e^{-\lambda_n{}^2 t} \quad (n = 1,\ 2,\ 3,\ \cdots)$

Step 3. Solution of the entire problem. Fourier series.

Consider a series of these eigenfunctions

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n{}^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L} \right)$$

Initial condition

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x)$$

Fourier sine series

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

12.6 Heat Equation: Solution by Fourier Series. Steady Two-Dimensional Heat Problems. Dirichlet Problem

☑ Ex. 1 Sinusoidal Initial Temperature

Find the temperature $u(x, t)$ in a laterally insulated (측면이 절연된) copper bar 80 cm long if the initial temperature is $100\sin(\pi x/80)$°C and the ends are kept at 0°C. How long will it take for the maximum temperature in the bar to drop to 50°C? First guess, then calculate.

- Physical data for copper:
  density (ρ): 8.92g/cm^3,
  specific heat (σ): 0.092cal/g·°C,
  thermal conductivity (K): 0.095cal/cm·sec·°C

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

$$\lambda_n=\frac{cn\pi}{L}, \quad c^2=\frac{K}{\sigma\rho}$$

c^2 : Thermal diffusivity (열확산 계수)
K : Thermal conductivity (열전도도), kcal/m·sec·°C
σ : Specific heat (비열), kcal/kg·°C

Sol) Initial condition:

$$u(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{80}=f(x)=100\sin\frac{\pi x}{80} \Rightarrow B_1=100,\ B_2=B_3=\cdots=0$$

$$c^2=\frac{K}{\sigma\rho}=\frac{0.95}{0.092\cdot 8.92}=1.158\left[\mathrm{cm}^2/\mathrm{sec}\right] \Rightarrow \lambda_1^2=\frac{c^2\pi^2}{L^2}=1.158\cdot\frac{9.870}{80^2}=0.001785\left[\mathrm{sec}^{-1}\right]$$

Solution:

$$u(x,t)=100\sin\frac{\pi x}{80}e^{-0.001785t}$$

$$100e^{-0.001785t}=50 \Rightarrow t=\frac{\ln 0.5}{-0.001785}=388[\mathrm{sec}]\approx 6.5[\mathrm{min}]$$

(maximum temperature)

12.6 Heat Equation: Solution by Fourier Series. Steady Two-Dimensional Heat Problems. Dirichlet Problem

☑ Ex. 4 Bar with Insulated Ends. Eigenvalue 0

Find a solution formula of one-dimensional heat equation replaced by the condition that both ends of the bar are insulated (단열된).

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Boundary Condition: $u_x(0, t), u_x(L, t) = 0$ for all t
- Initial Condition: $u(x,0) = f(x)$, ($f(x)$ given)

Sol) Physical experiments:

The rate of heat flow is proportional to the gradient of the temperature.
The ends of the bar are insulated. ⇨ No heat can flow through the ends.

Boundary condition: $u_x(0, t), u_x(L, t) = 0$ for all t ⇨ $F'(0)G(t) = F'(L)G(t) = 0$

⇨ $F'(0) = F'(L) = 0$

$$F(x) = A\cos px + B\sin px \quad \Rightarrow \quad F'(x) = -Ap\sin px + Bp\cos px$$

$$F'(0) = Bp = 0, \quad F'(L) = -Ap\sin pL = 0,$$

$$\Rightarrow \quad F_n(x) = \cos\frac{n\pi x}{L} \Rightarrow p = p_n = \frac{n\pi}{L}$$

☑ Ex. 4 Bar with Insulated Ends. Eigenvalue 0

Find a solution formula of one-dimensional heat equation replaced by the condition that both ends of the bar are insulated (단열된).

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Boundary Condition: $u_x(0, t), u_x(L, t) = 0$ for all t
- Initial Condition: $u(x,0) = f(x)$, ($f(x)$ given)

Sol) Eigenfunctions: $F_n(x) = \cos\frac{n\pi x}{L}$, $G_n(t) = B_n e^{-\lambda_n^2 t}$ (use A_n instead of B_n)

$$u_n(x,t) = F_n(x)G_n(t) = A_n \cos\frac{n\pi x}{L} e^{-\lambda_n^2 t} \qquad \left(\text{eigen values } \lambda_n = \frac{cn\pi}{L}\right)$$

Fourier cosine series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-\lambda_n^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right)$$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x) \quad \Rightarrow \quad A_0 = \frac{1}{L}\int_0^L f(x)dx, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

12.6 Heat Equation: Solution by Fourier Series.

☑ Steady Two-Dimensional Heat Problems (정상상태 2차원 열전도). Laplace's Equation

- Two-dimensional heat equation : $\dfrac{\partial u}{\partial t} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$
- Steady (time-independent) ⇨ $\dfrac{\partial u}{\partial t} = 0$ ⇨ Laplace's equation: $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

☑ Boundary Value Problem (BVP)

- First BVP or Dirichlet Problem: u is prescribed on C (Dirichlet boundary condition)
- Second BVP or Neumann Problem
  : The normal derivative $u_n = \dfrac{\partial u}{\partial n}$ is prescribed on C (Neumann boundary condition)
- Third BVP, Mixed BVP, or Robin Problem
  : u is prescribed on a portion of C and u_n on the rest of C (Mixed boundary condition)

12.6 Heat Equation: Laplace's Equation
- Dirichlet Problem

* Dirichlet Problem: u is prescribed on C (Dirichlet boundary condition)

▪ Laplace's equation

(Steady Two-Dimensional Heat Problem)

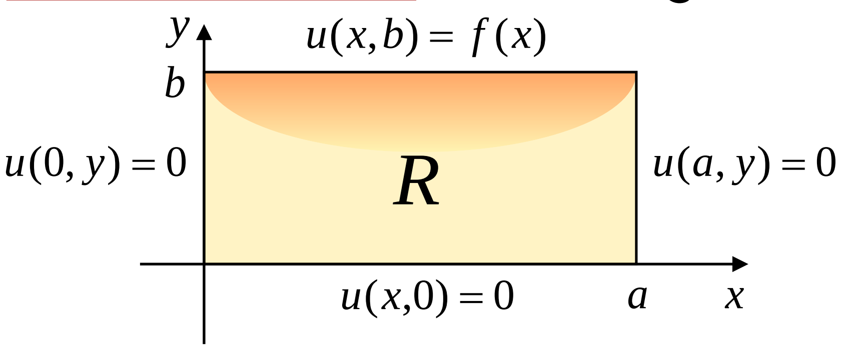
Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Steady $\Downarrow$ $\frac{\partial u}{\partial t} = 0$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Dirichlet Problem in a Rectangle ***R***



$$u(x,y) = F(x)G(y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 F}{\partial x^2}G(y) + F(x)\frac{\partial^2 G}{\partial y^2} = 0$$

Separating variable,

$$\frac{1}{F}\frac{\partial^2 F}{\partial x^2} = -\frac{1}{G}\frac{\partial^2 G}{\partial y^2} = -\lambda < 0$$

Two ODEs

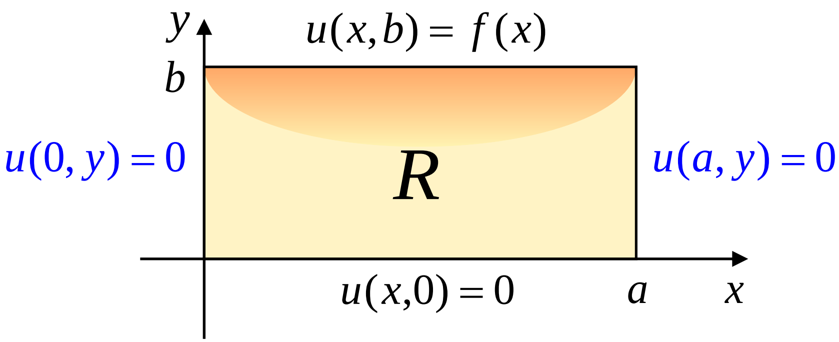
$$F'' + \lambda F = 0$$

$$G'' - \lambda G = 0$$

12.6 Heat Equation: Laplace's Equation - Dirichlet Problem

$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$

* Dirichlet Problem: u is prescribed on C (Dirichlet boundary condition)



$$F'' + \lambda F = 0$$

- **General solution**

$$F(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

- **Boundary condition**

$$u(0,y) = F(0)G(y) = 0$$

$$u(a,y) = F(a)G(y) = 0$$

$$\therefore F(0) = F(a) = 0$$

$$F(0) = A = 0$$

$$\therefore F(x) = B\sin\sqrt{\lambda}x$$

$$F(a) = B\sin\sqrt{\lambda}a = 0$$

$$\therefore \sin\sqrt{\lambda}a = 0$$

$$a\sqrt{\lambda} = n\pi, \quad \sqrt{\lambda} = \frac{n\pi}{a}, \ (n = 1, 2, \cdots)$$

$$\therefore \lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

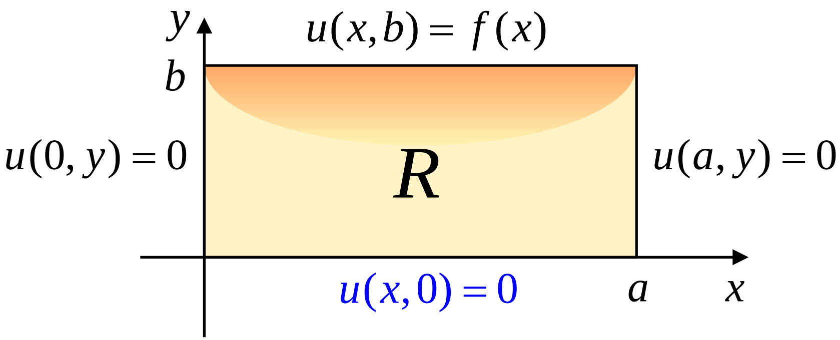
Setting $B = 1$,

$$\therefore F(x) = F_n(x)$$

$$= \sin\frac{n\pi}{a}x, \quad (n = 1, 2, \cdots)$$

12.6 Heat Equation: Laplace's Equation
- Dirichlet Problem

* Dirichlet Problem: u is prescribed on C (Dirichlet boundary condition)



$$G'' - \lambda G = 0, \qquad \lambda = \left(\frac{n\pi}{a}\right)^2, \quad (n = 1, 2, \cdots)$$

$$\therefore G'' - \frac{n^2\pi^2}{a^2} G = 0$$

- **General solution**

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$$

- **Boundary condition**

$$u(x,0) = F(x)G(0) = 0$$

$$F_n(x) = \sin\frac{n\pi}{a}x$$

$$\therefore G(0) = A_n + B_n = 0$$

$$\therefore B_n = -A_n$$

$\sinh(x) = (e^x - e^{-x})/2$

$$\therefore G_n(y) = A_n(e^{n\pi y/a} - e^{-n\pi y/a})$$

$$= 2A_n \sinh\frac{n\pi y}{a}$$

$$\therefore G_n(y) = A_n^* \sinh\frac{n\pi y}{a}$$

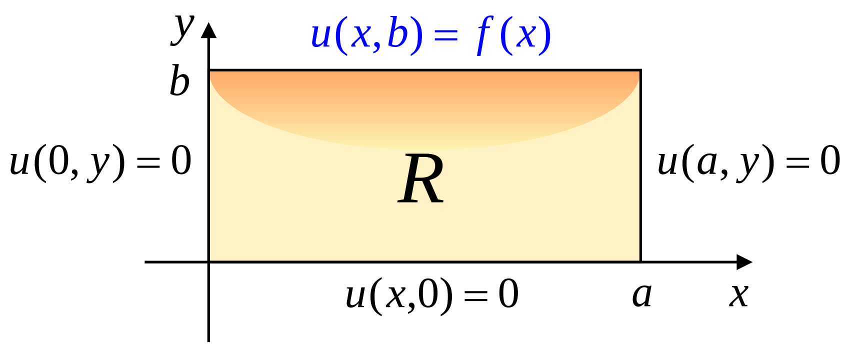
$$\text{where } A_n^* = 2A_n$$

$$\therefore u_n(x,y) = F_n(x)G_n(y)$$

$$= A_n^* \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

12.6 Heat Equation: Laplace's Equation
- Dirichlet Problem

* Dirichlet Problem: u is prescribed on C (Dirichlet boundary condition)



$$u_n(x,y) = A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

By Superposition

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$$

$$= \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

- **Boundary condition**

$$u(x,b) = F(x)G(b) = f(x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \quad (f \text{ odd})$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx, \quad n = 1, 2, \cdots$$

$$u(x,b) = f(x)$$

$$= \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

$$= \sum_{n=1}^{\infty}\left(A_n^* \sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$$

Fourier sine series of *f*(*x*)

$$\therefore A_n^* \sinh\frac{n\pi b}{a}$$

$$= \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

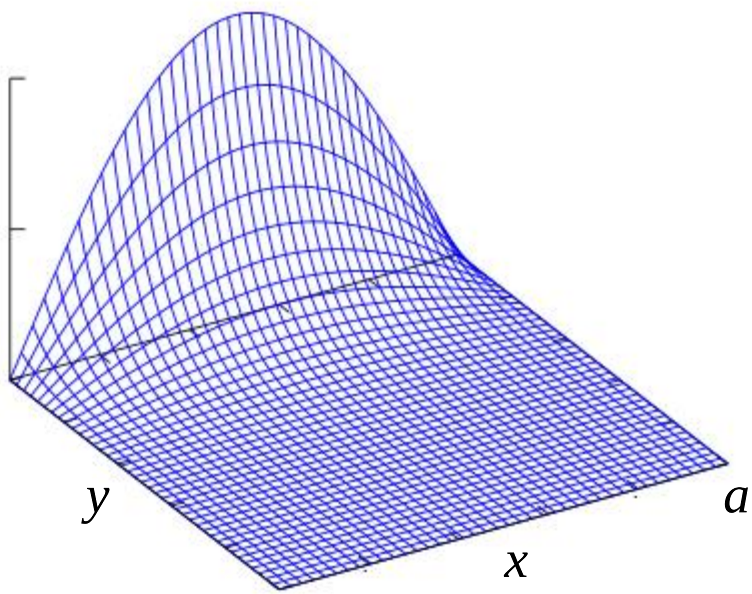
$$\therefore A_n^* = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

12.6 Heat Equation: Laplace's Equation
- Dirichlet Problem

* Dirichlet Problem: u is prescribed on C (Dirichlet boundary condition)

$$u(x,y)=\sum_{n=1}^{\infty}u_n(x,y)$$

$$=\sum_{n=1}^{\infty}A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$



$$u(x,b)=F(x)G(b)=f(x)$$

$$u(x,b)=f(x)$$

$$=\sum_{n=1}^{\infty}A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

$$\therefore A_n^*=\frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

$$u(x,y)=A_n^*\sin\frac{\pi x}{a}\sinh\frac{\pi y}{a}$$

$$A_n^*=\frac{2}{a\sinh(\pi b/a)}$$

$$\sinh\frac{\pi y}{a}=\frac{(e^{\pi y/a}-e^{-\pi y/a})}{2}$$

• Laplace's equation

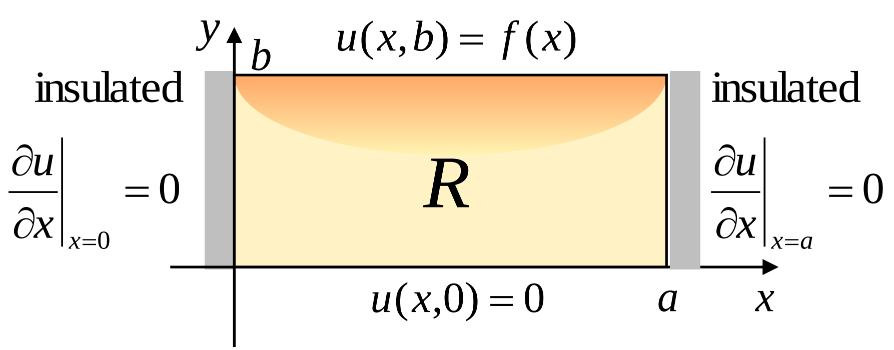
(Steady Two-Dimensional Heat Problem)

Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Steady $\Downarrow$ $\frac{\partial u}{\partial t} = 0$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



Neumann Problem $u_n = \frac{\partial u}{\partial n}$ is prescribed on C

$$u(x,y) = X(x)Y(y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 X}{\partial x^2} Y(y) + X(x) \frac{\partial^2 Y}{\partial y^2} = 0$$

Separating variable,

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -\lambda < 0$$

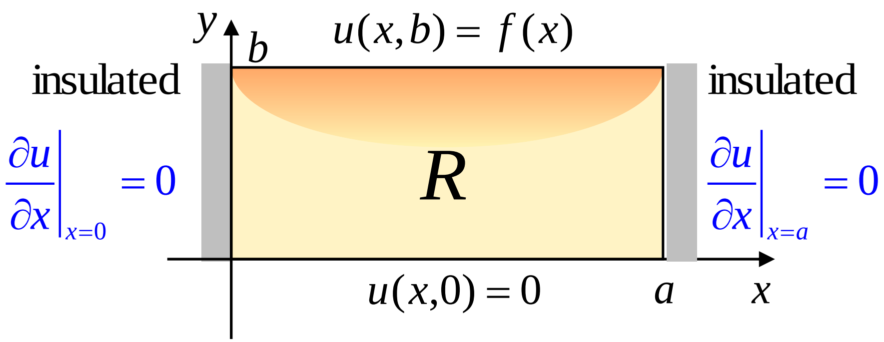
Two ODEs

$$X'' + \lambda X = 0$$

$$Y'' - \lambda Y = 0$$

$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$

* Neumann Problem: The normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on C



$$X'' + \lambda X = 0$$

1) λ = 0

$$X'' = 0 \;\Rightarrow X(x) = c_1 x + c_2$$

- **Boundary condition**

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = X'(0)Y(y) = 0 \;, \left.\frac{\partial u}{\partial x}\right|_{x=a} = X'(a)Y(y) = 0$$

$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = c_1 = 0$$

For any c_2, the second b/c $X'(a) = 0$ is satisfied

So for $c_2 \neq 0$, $X(x) = c_2$: nontrivial solution!

2) λ = -α² < 0

$$X'' - \alpha^2 X = 0$$

$$X(x) = c_3 e^{\alpha x} + c_4 e^{-\alpha x}$$

- **Boundary condition**

$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = (c_3 - c_4)\alpha = 0 \;\; \therefore c_3 = c_4$$

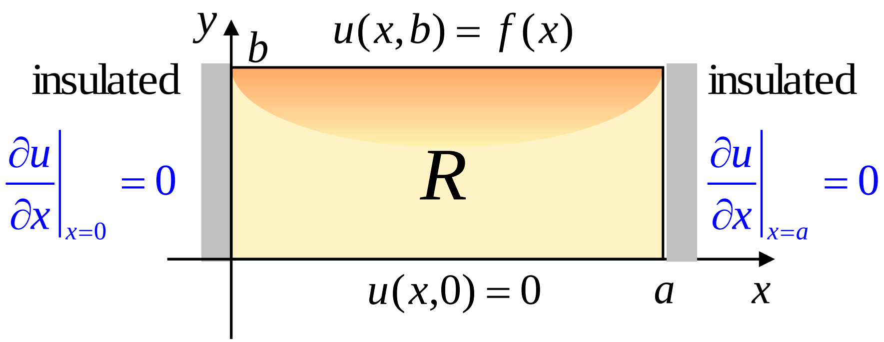
$$X'(a) = c_3 \alpha (e^{\alpha a} - e^{-\alpha a}) = 0$$

if $c_3 = 0 \rightarrow c_4 = 0$ ⇨ $X(x) = 0$

trivial solution → no interest

12.6 Heat Equation: Laplace's Equation
- Neumann Problem

* Neumann Problem: The normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on C



$$X'' + \lambda X = 0$$

3) $\lambda = \alpha^2 > 0$

$$X'' + \alpha^2 X = 0$$

$$X(x) = c_5 \cos \alpha x + c_6 \sin \alpha x$$

- **General solution**

$$X(x) = c_5 \cos \sqrt{\lambda} x + c_6 \sin \sqrt{\lambda} x$$

- **Boundary condition**

$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = c_6 \sqrt{\lambda} = 0$$

$$\therefore X(x) = c_5 \cos \sqrt{\lambda} x$$

$$X'(a) = -c_5 \sqrt{\lambda} \sin \sqrt{\lambda} a = 0$$

$$\therefore \sin \sqrt{\lambda} a = 0$$

$$a\sqrt{\lambda} = n\pi, \quad \sqrt{\lambda} = \frac{n\pi}{a}, \quad (n = 1, 2, \cdots)$$

$$\therefore \lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad (n = 1, 2, \cdots)$$

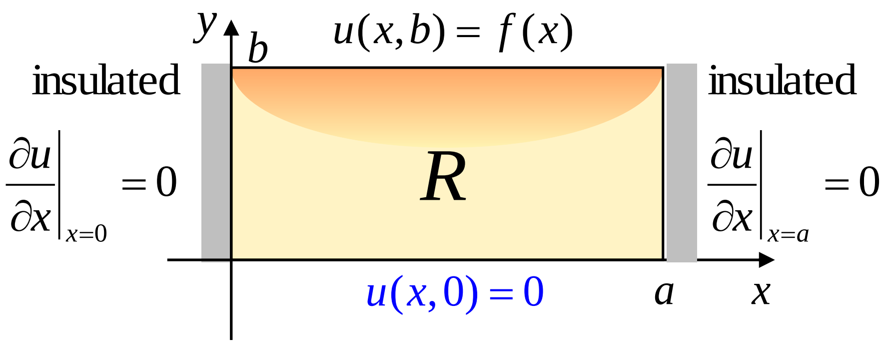
$$\therefore X(x) = X_n(x) = c_5 \cos \frac{n\pi}{a} x, \quad (n = 1, 2, \cdots)$$

by corresponding $\lambda_0 = 0$ with $n = 0$

$$\begin{cases} X(x) = c_2, \; n = 0 \\ X_n(x) = c_5 \cos \frac{n\pi}{a} x, \quad (n = 1, 2, \cdots) \end{cases}$$

12.6 Heat Equation: Laplace's Equation
- Neumann Problem

* Neumann Problem: The normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on C



$$Y'' - \lambda Y = 0$$

First, for $\lambda_0 = 0 \ (n=0)$

$$Y'' = 0 \Rightarrow Y(x) = c_7 y + c_8$$

- **Boundary condition**

$$u(x,0) = X(x)Y(0) = 0$$

$$\therefore Y(0) = 0$$

$$Y(0) = c_8 = 0$$

$$\therefore Y(x) = c_7 y$$ **: nontrivial solution!**

Second, for $\lambda_n = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$

$$Y'' - \frac{n^2\pi^2}{a^2} Y = 0$$

- **General solution**

$$Y(y) = Y_n(y) = c_9 e^{n\pi y/a} + c_{10} e^{-n\pi y/a}$$

$$Y(0) = c_9 + c_{10} = 0 \qquad \therefore c_{10} = -c_9$$

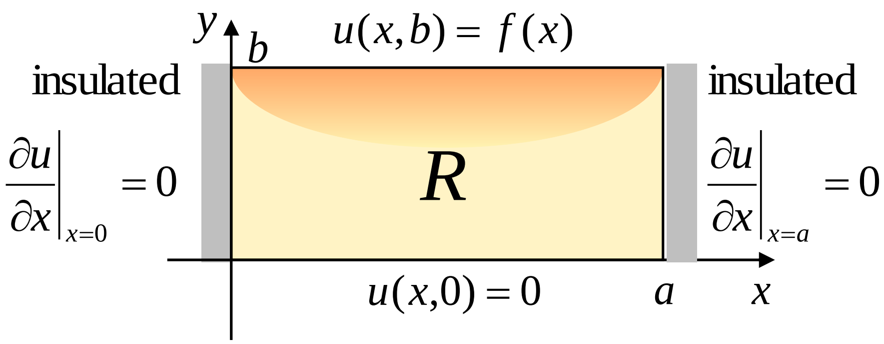
$$\therefore Y_n(y) = c_9 (e^{n\pi y/a} - e^{-n\pi y/a}) = 2c_9 \sinh \frac{n\pi y}{a}$$

$\sinh(x) = (e^x - e^{-x})/2$

$$\begin{cases} Y(y) = c_7 y, \ n = 0 \\ Y_n(y) = c_9^* \sinh \frac{n\pi y}{a}, \quad (n = 1, 2, \cdots), c_9^* = 2c_9 \end{cases}$$

12.6 Heat Equation: Laplace's Equation
- Neumann Problem

* Neumann Problem: The normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on C



$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

$$\begin{cases} X(x) = c_2, \; n = 0 \\ X_n(x) = c_5 \cos \dfrac{n\pi}{a} x, \quad (n = 1, 2, \cdots) \end{cases}$$

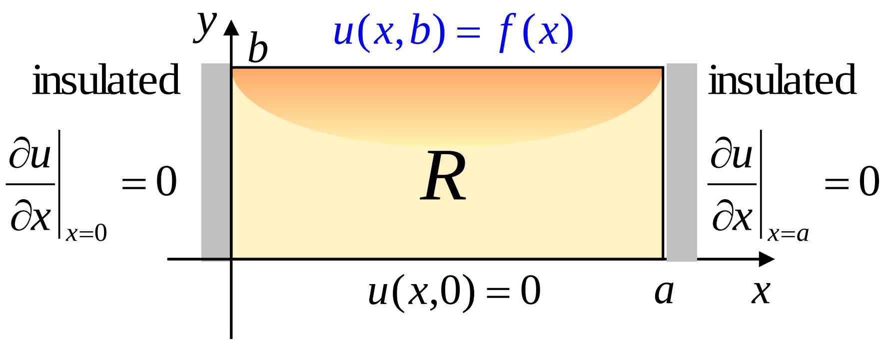
$$\begin{cases} Y(y) = c_7 y, \; n = 0 \\ Y_n(y) = c_9^* \sinh \dfrac{n\pi y}{a}, \quad (n = 1, 2, \cdots) \end{cases}$$

$$\therefore u_n(x,y) = X_n(x) Y_n(y) = \begin{cases} A_0^* y, \quad (n = 0) \\ A_n^* \sinh \dfrac{n\pi y}{a} \cos \dfrac{n\pi x}{a}, \quad (n = 1, 2, \ldots) \end{cases}$$

$$where, \; A_0^* = c_2 c_7, \; A_n^* = c_5 c_9^*$$

12.6 Heat Equation: Laplace's Equation
- Neumann Problem

* Neumann Problem: The normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on C



$$u_n(x,y) = X_n(x)Y_n(y)$$

$$= \begin{cases} A_0^* y, & (n=0) \\ A_n^* \sinh \frac{n\pi y}{a} \cos \frac{n\pi x}{a}, & (n=1,2,\ldots) \end{cases}$$

By Superposition

$$u(x,t) = A_0^* y + \sum_{n=1}^{\infty} u_n(x,y)$$

$$= A_0^* y + \sum_{n=1}^{\infty} A_n^* \sinh \frac{n\pi y}{a} \cos \frac{n\pi x}{a}$$

- **Boundary condition**

$$u(x,b) = X(x)Y(b) = f(x)$$

$$u(x,b) = f(x)$$

$$= A_0^* b + \sum_{n=1}^{\infty} A_n^* \sinh \frac{n\pi b}{a} \cos \frac{n\pi x}{a}$$

$$= A_0^* b + \sum_{n=1}^{\infty} \left(A_n^* \sinh \frac{n\pi b}{a} \right) \cos \frac{n\pi x}{a}$$

Fourier cosine series of *f*(*x*)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

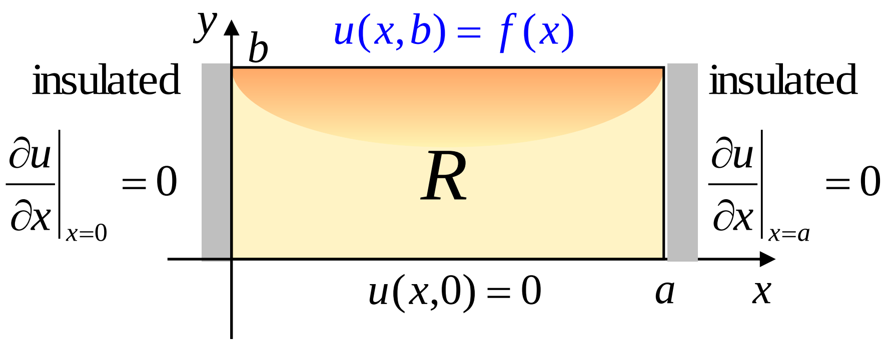
$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx$$

$$A_0^* b = \frac{1}{a}\int_0^a f(x)\,dx$$

$$A_0^* = \frac{1}{ab}\int_0^a f(x)\,dx$$

12.6 Heat Equation: Laplace's Equation
- Neumann Problem

* Neumann Problem: The normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on C



$$u_n(x,y) = X_n(x)Y_n(y)$$

$$= \begin{cases} A_0^* y, & (n=0) \\ A_n^* \sinh \dfrac{n\pi y}{a} \cos \dfrac{n\pi x}{a}, & (n=1,2,\ldots) \end{cases}$$

By Superposition

$$u(x,y) = A_0^* y + \sum_{n=1}^{\infty} u_n(x,y)$$

$$= A_0^* y + \sum_{n=1}^{\infty} A_n^* \sinh \frac{n\pi y}{a} \cos \frac{n\pi x}{a}$$

- **Boundary condition**

$$u(x,b) = X(x)Y(b) = f(x)$$

$$u(x,b) = f(x)$$

$$= A_0^* b + \sum_{n=1}^{\infty} A_n^* \sinh \frac{n\pi b}{a} \cos \frac{n\pi x}{a}$$

$$= A_0^* b + \sum_{n=1}^{\infty} \left(A_n^* \sinh \frac{n\pi b}{a} \right) \cos \frac{n\pi x}{a}$$

Fourier cosine series of *f*(*x*)

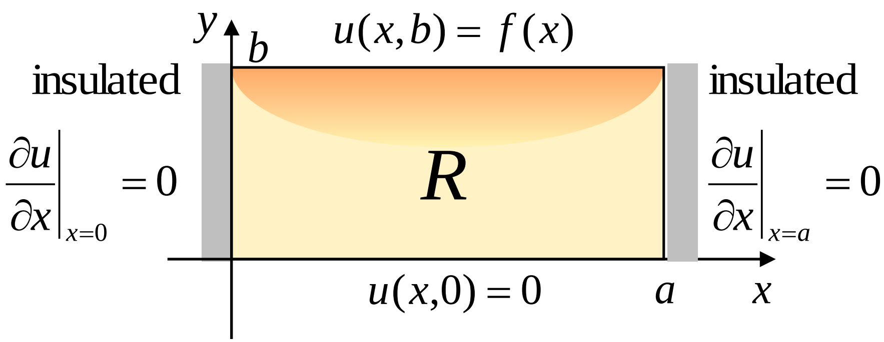
$$A_0^* b = \frac{1}{ab} \int_0^a f(x)\,dx$$

$$A_n^* \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi x}{a}\,dx$$

$$\therefore A_n^* = \frac{2}{a \sinh(n\pi b / a)} \int_0^a f(x) \cos \frac{n\pi x}{a}\,dx$$

12.6 Heat Equation: Laplace's Equation
- Neumann Problem

* Neumann Problem: The normal derivative $u_n = \frac{\partial u}{\partial n}$ is prescribed on C

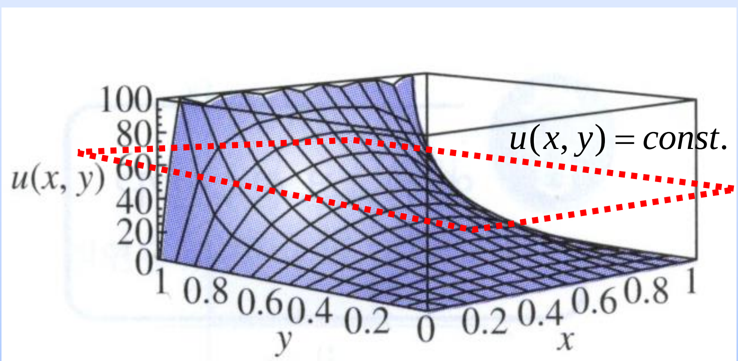


$$u(x,y) = A_0^* y + \sum_{n=1}^{\infty} A_n^* \sinh\frac{n\pi y}{a}\cos\frac{n\pi x}{a}$$

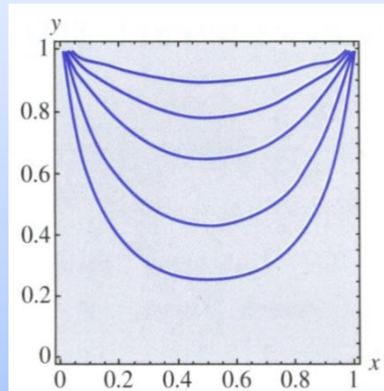
$$A_0^* b = \frac{1}{ab}\int_0^a f(x)dx$$

$$A_n^* = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\cos\frac{n\pi x}{a}dx$$

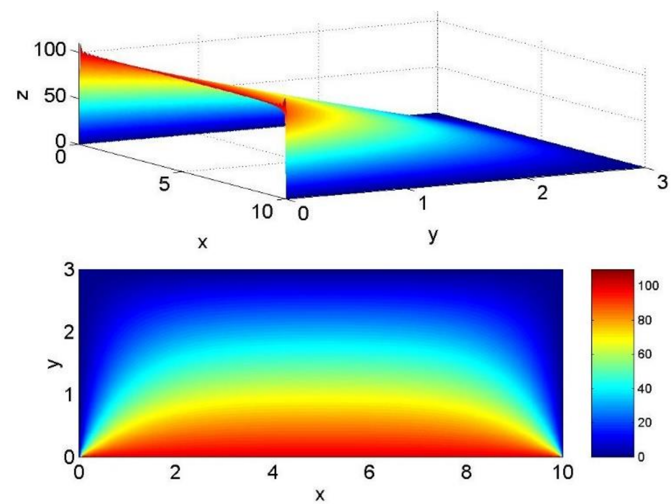
Ex) f(x)=100, a=1, b=1



(a) 곡면



(b) 등온선



12.6 Heat Equation: Laplace's Equation
- Superposition

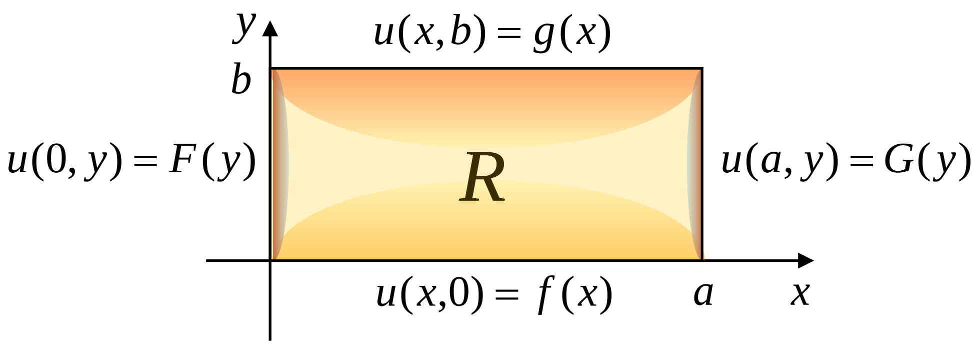
Superposition

The method of separation of variables is not applicable to a Dirichlet problem when the boundary conditions on all four sides of the rectangle are non-homogeneous.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

$$u(0, y) = F(y), \; u(a, y) = G(y), \; 0 < y < b$$

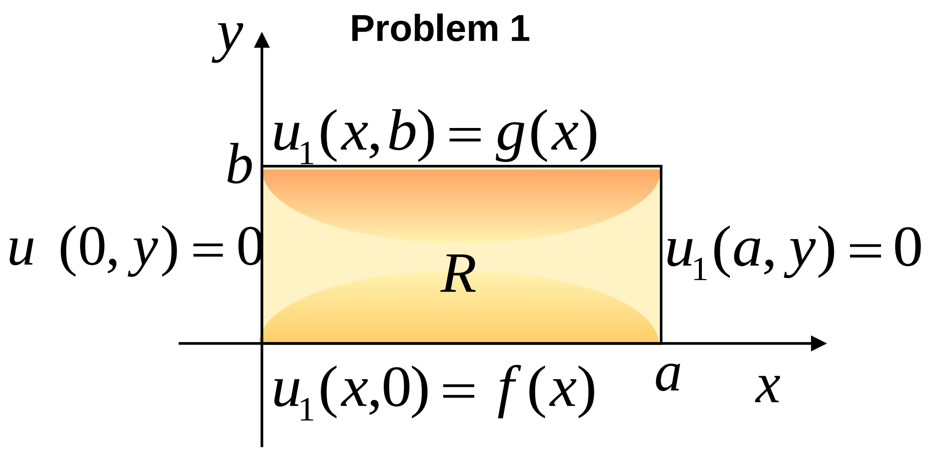
$$u(x, 0) = f(x), \; u(x, b) = g(x), \; 0 < x < a$$

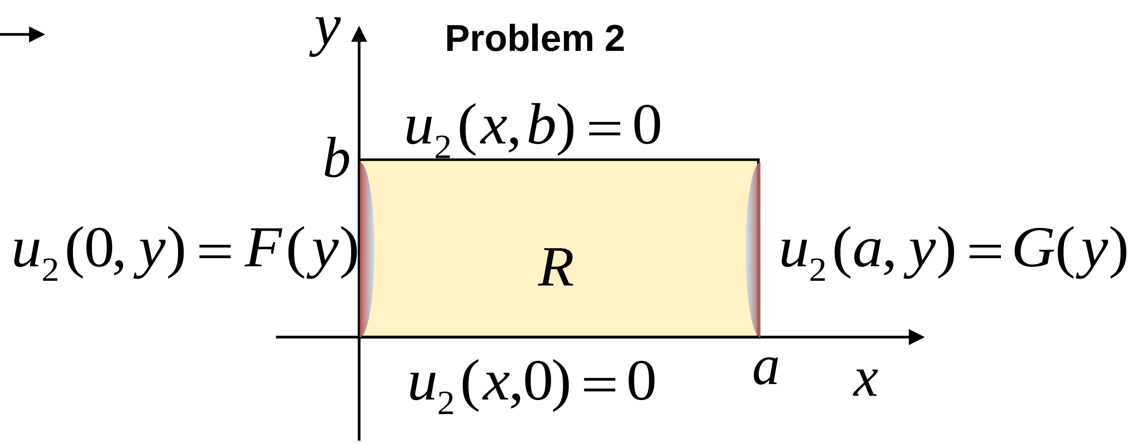


12.6 Heat Equation: Laplace's Equation
- Superposition

Superposition

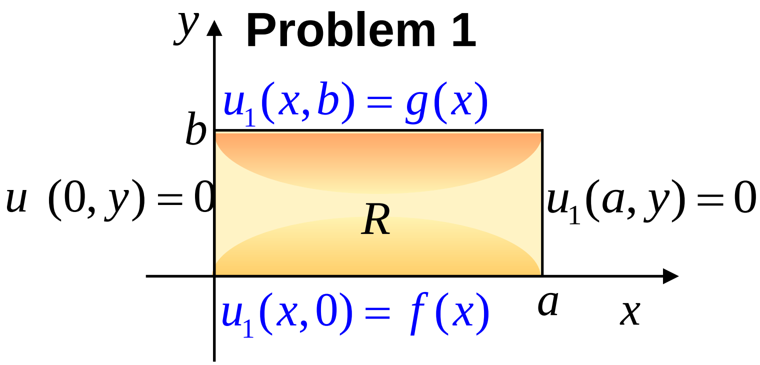
To get around this difficulty we break the problem into two problems, each of which has homogeneous boundary conditions on parallel boundaries, as shown.



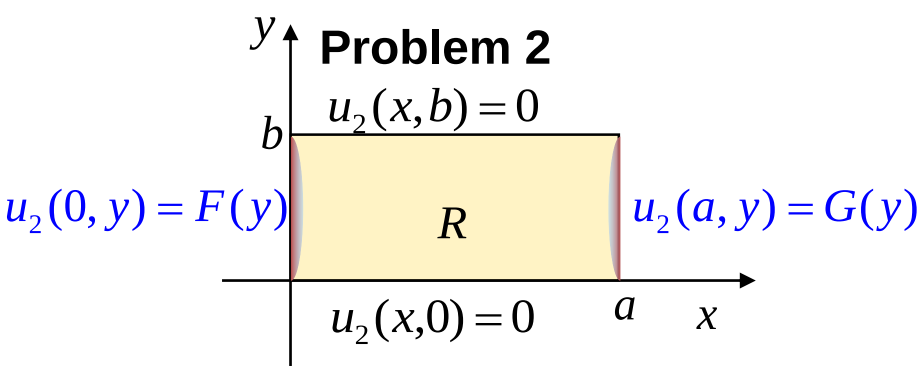


12.6 Heat Equation: Laplace's Equation - Superposition

Superposition



$$\begin{cases} \dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b \\ u_1(0,y) = 0, \qquad u_1(a,y) = 0, \quad 0 < y < b \\ u_1(x,0) = f(x), \quad u_1(x,b) = g(x), \quad 0 < x < a \end{cases}$$



$$\begin{cases} \dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\partial^2 u_2}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b \\ u_2(0,y) = F(y), \quad u_2(a,y) = G(y), \quad 0 < y < b \\ u_2(x,0) = 0, \qquad u_2(x,b) = 0, \quad 0 < x < a \end{cases}$$

12.6 Heat Equation: Laplace's Equation - Superposition

Superposition

Problem 1

$$\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}=0, \quad 0<x<a, \quad 0<y<b$$

$$u_1(0,y)=0, \qquad u_1(a,y)=0, \quad 0<y<b$$

$$u_1(x,0)=f(x), \quad u_1(x,b)=g(x), \quad 0<x<a$$

Problem 2

$$\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}=0, \quad 0<x<a, \quad 0<y<b$$

$$u_2(0,y)=F(y), \quad u_2(a,y)=G(y), \quad 0<y<b$$

$$u_2(x,0)=0, \qquad u_2(x,b)=0, \quad 0<x<a$$

Suppose u_1 and u_2 are the solutions of Problems 1 and 2, respectively. If we define $u(x, y) = u_1(x, y) + u_2(x, y)$, it is seen that u satisfies all boundary conditions in the original problem above.

$$u(0,y)=u_1(0,y)+u_2(0,y)=0+F(y)=F(y)$$

$$u(a,y)=u_1(a,y)+u_2(a,y)=0+G(y)=G(y)$$

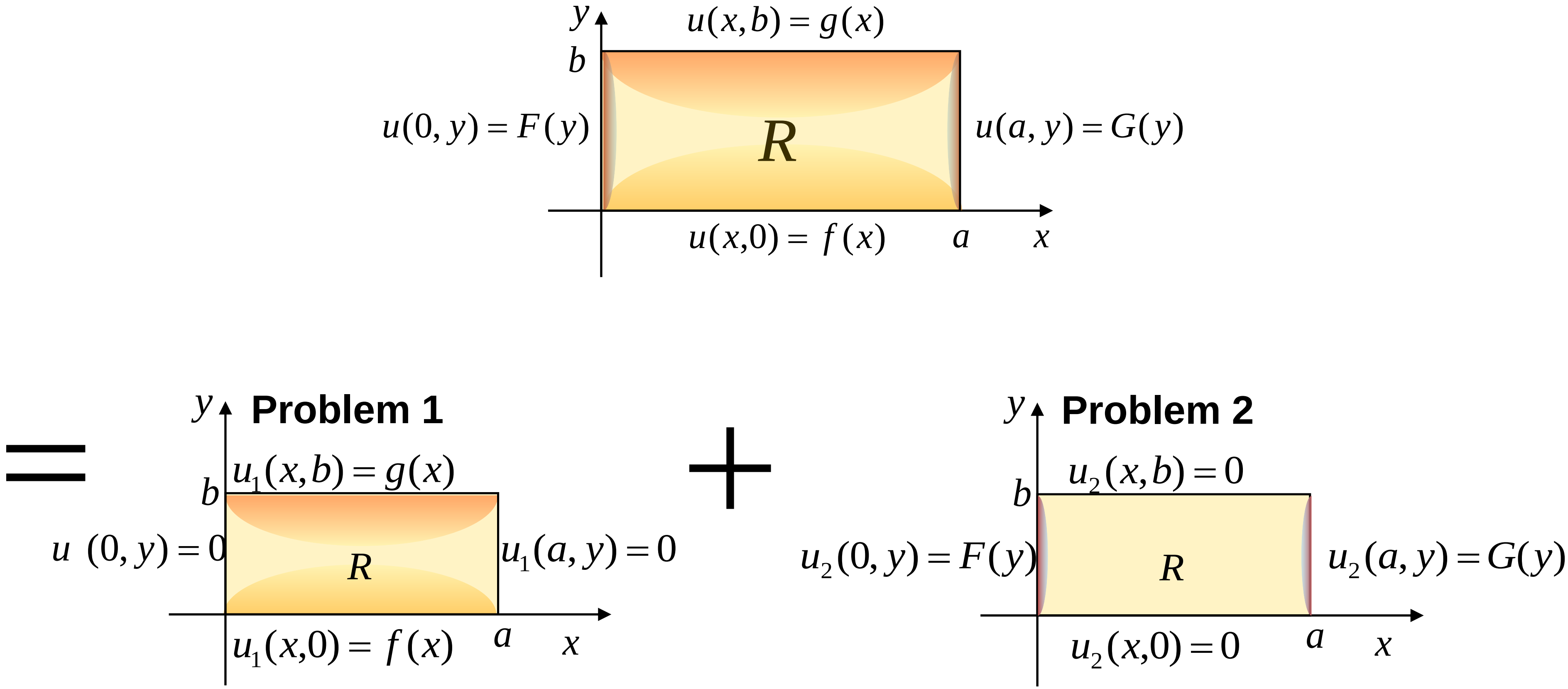
$$u(x,0)=u_1(x,0)+u_2(x,0)=f(x)+0=f(x)$$

$$u(x,b)=u_1(x,b)+u_2(x,b)=g(x)+0=g(x)$$

12.6 Heat Equation: Laplace's Equation
- Superposition

Superposition

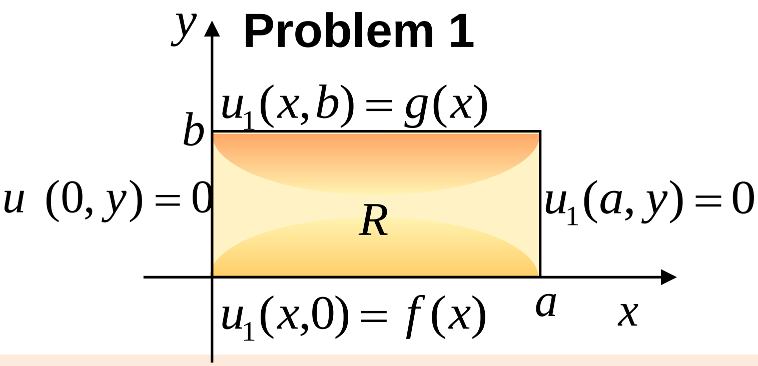
$$u(x,y) = u_1(x,y) + u_2(x,y)$$



12.6 Heat Equation: Laplace's Equation - Superposition

$$\frac{1}{F}\frac{\partial^2 F}{\partial x^2} = -\frac{1}{G}\frac{\partial^2 G}{\partial y^2} = -k < 0$$

Superposition



Problem 1

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

$$u_1(0,y) = 0, \ u_1(a,y) = 0, \ 0 < y < b,$$

$$u_1(x,0) = f(x), \ u_1(x,b) = g(x), \quad 0 < x < a$$

$$\frac{d^2F}{dx^2} + kF = 0, \quad \frac{d^2G}{dy^2} - kG = 0$$

$$k = \left(\frac{n\pi}{a}\right)^2$$

$$F_{1n}(x) = \sin\frac{n\pi}{a}x, \quad (n = 1, 2, \cdots)$$

$$\therefore \frac{d^2G}{dy^2} - \frac{n^2\pi^2}{a^2}G = 0$$

$$G_{1n}(y) = A_{1n}\cosh\frac{n\pi}{a}y + B_{1n}\sinh\frac{n\pi}{a}y$$

$$\therefore u_{1n} = F_{1n}(x)G_{1n}(y)$$

$$= \left(A_{1n}\cosh\frac{n\pi}{a}y + B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \quad (f \text{ odd})$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx, \quad n = 1, 2, \cdots$$

Superposition

$$u_{1n} = F_{1n}(x)G_{1n}(y) = \left(A_{1n}\cosh\frac{n\pi}{a}y + B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

- **General solution of the problem 1**

$$\therefore u_1(x,y) = \sum_{n=1}^{\infty} u_{1n} = \sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y + B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

- **Boundary condition to find A_n**

$$u_1(x,0) = f(x) = \sum_{n=1}^{\infty} A_{1n}\sin\frac{n\pi}{a}x$$

Fourier sine series of $f(x)$

$$\therefore A_{1n} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$$

12.6 Heat Equation: Laplace's Equation - Superposition

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \quad (f \text{ odd})$$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx, \quad n=1,\ 2,\ \cdots$$

Superposition

$$u_1(x,y)=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

$$A_{1n}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$$

- **Boundary condition to find B_n**

$$u_1(x,b)=g(x)=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi b}{a}+B_{1n}\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi}{a}x$$

Fourier sine series of $g(x)$

$$\therefore A_{1n}\cosh\frac{n\pi b}{a}+B_{1n}\sinh\frac{n\pi b}{a}=\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx$$

$$\therefore B_{1n}=\frac{1}{\sinh(n\pi b/a)}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx-A_{1n}\cosh\frac{n\pi}{a}b\right)$$

12.6 Heat Equation: Laplace's Equation - Superposition

$$\therefore u_1(x,y)=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

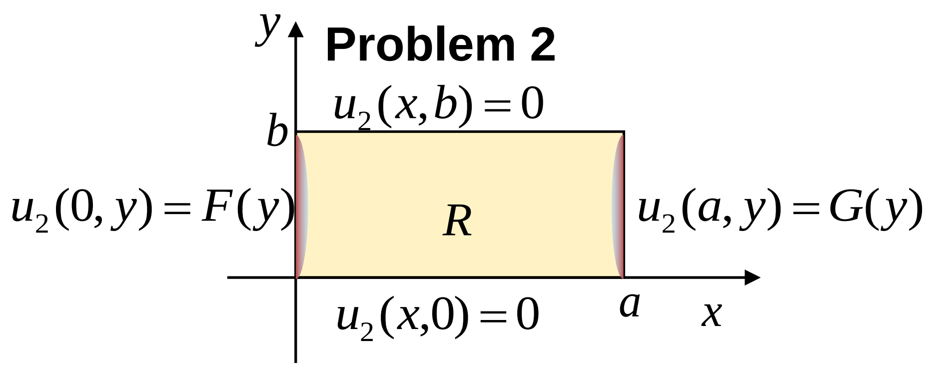
Superposition

Problem 2

$$\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

$$u_2(0,y)=F(y),\ u_2(a,y)=G(y),\ 0<y<b,$$

$$u_2(x,0)=0,\quad u_2(x,b)=0,\qquad 0<x<a$$



$$u_2(x,y)=\sum_{n=1}^{\infty}\left\{A_{2n}\cosh\frac{n\pi}{b}x+B_{2n}\sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$$

$$A_{1n}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$$

$$B_{1n}=\frac{1}{\sinh(n\pi b/a)}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx-A_{1n}\cosh\frac{n\pi}{a}b\right)$$

$$A_{2n}=\frac{2}{b}\int_0^b F(y)\sin\frac{n\pi}{b}y\,dy$$

$$B_{2n}=\frac{1}{\sinh\frac{n\pi}{b}a}\left(\frac{2}{b}\int_0^b G(y)\sin\frac{n\pi}{b}y\,dy-A_{2n}\cosh\frac{n\pi}{b}a\right)$$

12.6 Heat Equation: Laplace's Equation
- Superposition

Superposition

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

$$u_1(x,y) = \sum_{n=1}^{\infty} \left\{ A_{1n} \cosh \frac{n\pi}{a} y + B_{1n} \sinh \frac{n\pi}{a} y \right\} \sin \frac{n\pi}{a} x$$

$$A_{1n} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x dx, \qquad B_{1n} = \frac{1}{\sinh \frac{n\pi}{a} b} \left(\frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x dx - A_{1n} \cosh \frac{n\pi}{a} b \right)$$

$$u_2(x,y) = \sum_{n=1}^{\infty} \left\{ A_{2n} \cosh \frac{n\pi}{b} x + B_{2n} \sinh \frac{n\pi}{b} x \right\} \sin \frac{n\pi}{b} y$$

$$A_{2n} = \frac{2}{b} \int_0^b F(y) \sin \frac{n\pi}{b} y dy, \qquad B_{2n} = \frac{1}{\sinh \frac{n\pi}{b} a} \left(\frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y dy - A_{2n} \cosh \frac{n\pi}{b} a \right)$$

12.7 Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

☑ Model of bars of infinite length

- The role of Fourier series in the solution problem will be takend by Fourier Integrals.
- Heat equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
- Initial condition: $u(x,0) = f(x)$

$$u(x,t) = F(x)G(t) \qquad \Rightarrow \qquad F'' + p^2 F = 0, \qquad \dot{G} + c^2 p^2 G = 0$$

$$F(x) = A\cos px + B\sin px, \qquad G(t) = e^{-c^2 p^2 t}$$

$$u(x,t;p) = FG = (A\cos px + B\sin px)e^{-c^2 p^2 t}$$

- **No boundary condition** (Since the length of bar is infinite)

12.7 Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

$$f(x)=\int_0^\infty \left[A(w)\cos wx + B(w)\sin wx\right]dw$$

$$A(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv,\quad B(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

☑ Use of Fourier Integrals

Heat equation is linear and homogeneous.

Solution: $$u(x,t)=\int_0^\infty u(x,t;p)\,dp=\int_0^\infty (A\cos px + B\sin px)e^{-c^2p^2t}\,dp$$

Determination of $A(p)$ and $B(p)$ from the Initial Condition

$$u(x,0)=\int_0^\infty (A\cos px + B\sin px)\,dp = f(x)$$

Complex Form of the Fourier Integral (Sec. 11.9)

$$\Rightarrow \quad A(p)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv\,dv \quad \text{and} \quad B(p)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin pv\,dv$$

$$\therefore \quad u(x,t)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^\infty e^{-c^2p^2t}\cos pv\cos px + \sin pv\sin px\,dp\right]dv$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^\infty e^{-c^2p^2t}\cos(px-pv)\,dp\right]dv$$

$\sin(x+y)=\sin x\cos y+\cos x\sin y$

$\sin(x-y)=\sin x\cos y-\cos x\sin y$

$\cos(x+y)=\cos x\cos y-\sin x\sin y$

$\cos(x-y)=\cos x\cos y+\sin x\sin y$

12.7 Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

$$u(x,t)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^{\infty} e^{-c^2p^2t}\cos(px-pv)\,dp\right]dv$$

- **Using the formula** $\displaystyle\int_0^{\infty} e^{-s^2}\cos 2bs\,ds=\frac{\sqrt{\pi}}{2}e^{-b^2}$

we choose , $p=\dfrac{s}{c\sqrt{t}}$, $dp=\dfrac{ds}{c\sqrt{t}}$, $s=pc\sqrt{t}$

$$b=\frac{p(x-v)}{2s}=\frac{p(x-v)}{2pc\sqrt{t}}=\frac{(x-v)}{2c\sqrt{t}}$$

$$\int_0^{\infty} e^{-c^2p^2t}\cos(px-pv)\,dp=\frac{\sqrt{\pi}}{2c\sqrt{t}}\exp\left\{-\frac{(x-v)^2}{4c^2t}\right\}$$

$$z=\frac{(v-x)}{2c\sqrt{t}},\quad dz=\frac{dv}{2c\sqrt{t}}$$

$$v=x+2cz\sqrt{t}$$

$$u(x,t)=\frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\exp\left\{-\frac{(x-v)^2}{4c^2t}\right\}dv=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f\left(x+2cz\sqrt{t}\right)e^{-z^2}\,dz$$

12.7 Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

☑ Ex. 1 Temperature in an Infinite Bar

Find the temperature in the infinite bar if the initial temperature is

$$f(x) = \begin{cases} U_0 = \text{const.} & (|x| < 1) \\ 0 & (|x| > 1) \end{cases} \qquad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u(x,0) = f(x),$$

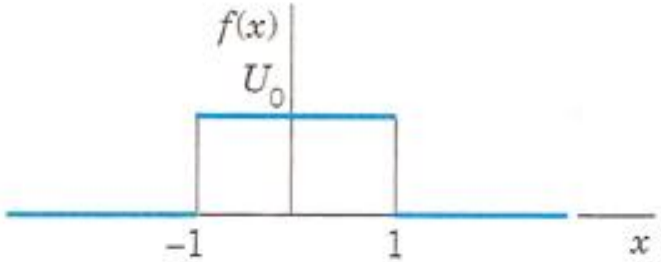
Sol)

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\} dv = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f\left(x + 2cz\sqrt{t}\right) e^{-z^2} dz$$

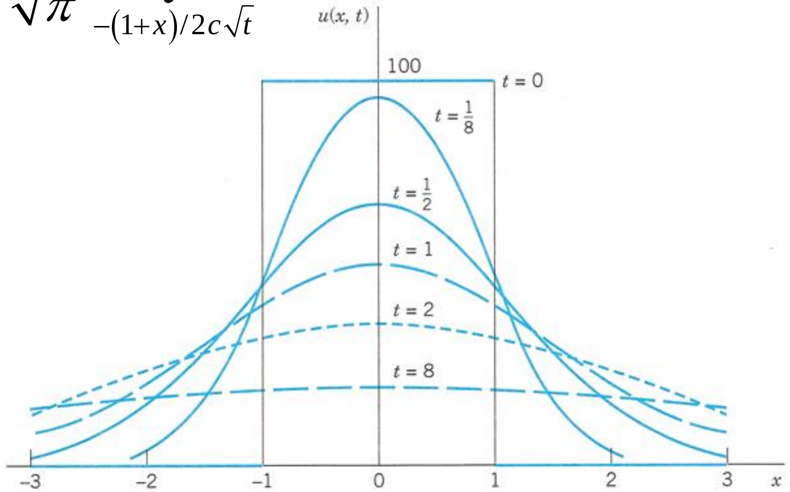
$$u(x,t) = \frac{U_0}{2c\sqrt{\pi t}} \int_{-1}^{1} \exp\left\{-\frac{(x-v)^2}{4c^2 t}\right\} dv = \frac{U_0}{\sqrt{\pi}} \int_{-(1+x)/2c\sqrt{t}}^{(1-x)/2c\sqrt{t}} e^{-z^2} dz$$

$$z = \frac{(v-x)}{2c\sqrt{t}}$$

$$dz = \frac{dv}{2c\sqrt{t}}$$



Initial temperature



Solution $u(x,t)$

☑ Ex. 2 Temperature in the Infinite Bar in Example 1

Solve Example 1 using the Fourier transform.

$$f(x)=\begin{cases} U_0 = \text{const.} & (|x|<1) \\ 0 & (|x|>1) \end{cases} \qquad \frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2} \qquad u(x,0)=f(x),$$

Sol) $\hat{u}=\mathcal{F}(u)$: Fourier transform of u, regarded as a function of *x*

$$\mathcal{F}\{f'(x)\}=iw\mathcal{F}\{f(x)\} \qquad \mathcal{F}\{f''(x)\}=-w^2\mathcal{F}\{f(x)\}$$

$$\mathcal{F}(f)=\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$$

Heat equation: $\mathcal{F}(u_t)=c^2\mathcal{F}(u_{xx})=c^2(-w^2)\mathcal{F}(u)=-c^2w^2\hat{u}$

Interchange the order of differentiation and integration

$$\mathcal{F}(u_t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t e^{-iwx}dx=\frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} ue^{-iwx}dx=\frac{\partial \hat{u}}{\partial t} \qquad \Rightarrow \qquad \frac{\partial \hat{u}}{\partial t}=-c^2w^2\hat{u}$$

☑ Ex. 2 Temperature in the Infinite Bar in Example 1

Solve Example 1 using the Fourier transform.

Sol) $\hat{u} = \mathcal{F}(u)$: Fourier transform of *u*, regarded as a function of *x*

$$\mathcal{F}(f) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

$$\frac{\partial \hat{u}}{\partial t} = -c^2 w^2 \hat{u}$$ (1st order ODE for independent variable *t*)

General solution: $\hat{u}(w,t) = C(w) e^{-c^2 w^2 t}$

Initial condition: $u(x,0) = f(x) \Rightarrow \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\}$

$$\Rightarrow \hat{u}(w,0) = \hat{f}(w) \Rightarrow \hat{u}(w,t) = \hat{f}(w) e^{-c^2 w^2 t}$$

$(\therefore \hat{f}(w) = C(w))$

☑ Ex. 2 Temperature in the Infinite Bar in Example 1

Solve Example 1 using the Fourier transform.

$$\hat{u}(w,t) = \hat{f}(w)e^{-c^2w^2t}$$

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(w)e^{iwx}dw$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{iwx}dw$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)e^{-iwv}dv$$

Sol-continued)

(Inverse Fourier Transform)

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(w)e^{iwx}dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{-c^2w^2t}e^{iwx}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}f(v)\left[\int_{-\infty}^{\infty}e^{-c^2w^2t}e^{i(wx-wv)}dw\right]dv$$

$$e^{-c^2w^2t}e^{i(wx-wv)} = e^{-c^2w^2t}\cos(wx-wv) + ie^{-c^2w^2t}\sin(wx-wv)$$

(Euler formula)

odd function of $w \rightarrow \int_{-\infty}^{\infty} f_{odd}(w)dw = 0$

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}f(v)\left[\int_{0}^{\infty}e^{-c^2w^2t}\cos(wx-wv)dw\right]dv$$

☑ Ex. 3 Solution in Example 1 by the Method of Convolution

Solve the heat problem in Example 1 by the method of convolution

Sol) $$(f*g)(x)=\int_{-\infty}^{\infty} f(p)g(x-p)dp=\int_{-\infty}^{\infty} f(x-p)g(p)dp$$

$$\mathcal{F}(f*g)=\sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g) \qquad \mathcal{F}^{-1}(f)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{iwx}dw \qquad (f*g)(x)=\int_{-\infty}^{\infty}\hat{f}(w)\hat{g}(w)e^{iwx}dw$$

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{-c^2w^2t}e^{iwx}dw, \qquad \hat{g}(w)=\frac{1}{\sqrt{2\pi}}e^{-c^2w^2t}$$

$$=\int_{-\infty}^{\infty}\hat{f}(w)\hat{g}(w)e^{iwx}dw=(f*g)(x)=\int_{-\infty}^{\infty}f(p)g(x-p)dp$$

$$\mathcal{F}\left(e^{-ax^2}\right)=\frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}$$

$$\Rightarrow \qquad a=\frac{1}{4c^2t}, \quad \mathcal{F}\left(e^{-\frac{x^2}{4c^2t}}\right)=\sqrt{2c^2t}\,e^{-c^2w^2t}=\sqrt{2c^2t}\sqrt{2\pi}\,\hat{g}(w)$$

12.7 Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

☑ Ex. 3 Solution in Example 1 by the Method of Convolution

Solve the heat problem in Example 1 by the method of convolution

Sol)

$$(f*g)(x)=\int_{-\infty}^{\infty} f(p)g(x-p)dp=\int_{-\infty}^{\infty} f(x-p)g(p)dp$$

$$\mathcal{F}(f*g)=\sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g)$$

$$\mathcal{F}^{-1}(f)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{iwx}dw$$

$$(f*g)(x)=\int_{-\infty}^{\infty}\hat{f}(w)\hat{g}(w)e^{iwx}dw$$

$$\mathcal{F}\left(e^{-\frac{x^2}{4c^2t}}\right)=\sqrt{2c^2t}\sqrt{2\pi}\hat{g}(w)$$

$$\mathcal{F}\left(\frac{e^{-\frac{x^2}{4c^2t}}}{\sqrt{2\pi}\sqrt{2c^2t}}\right)=\hat{g}(w)$$

$$\mathcal{F}^{-1}\{\hat{g}(w)\}=g(x)=\mathcal{F}^{-1}\mathcal{F}\left(\frac{e^{-\frac{x^2}{4c^2t}}}{\sqrt{2\pi}\sqrt{2c^2t}}\right)=\frac{e^{-\frac{x^2}{4c^2t}}}{\sqrt{2\pi}\sqrt{2c^2t}}$$

$$\therefore g(x)=\frac{e^{-\frac{x^2}{4c^2t}}}{\sqrt{2\pi}\sqrt{2c^2t}}$$

$$\therefore\ u(w,t)=(f*g)(x)=\frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(p)\exp\left\{-\frac{(x-p)^2}{4c^2t}\right\}dp$$

12.7 Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

☑ Ex. 4 Fourier Sine Transform Applied to the Heat Equation

- If a laterally insulated bar extends from x = 0 to infinity.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textbf{Subject to} \qquad u(x,0) = f(x), \quad u(0,t) = 0$$

$$\Rightarrow \quad u(0,0) = \boxed{f(0) = 0}$$

Sol)

$$\mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0), \quad \mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}$$

$$\mathcal{F}_c\{f''(x)\} = w\mathcal{F}_s\{f'(x)\} - \sqrt{\frac{2}{\pi}} f'(0) = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\mathcal{F}_s\{f''(x)\} = -w\mathcal{F}_c\{f'(x)\} = -w^2\mathcal{F}_s\{f(x)\} + \boxed{\sqrt{\frac{2}{\pi}} w f(0)}$$

"Fourier Sine Transform 이 적합한 이유"

$$\mathcal{F}_s\{u_t\} = \frac{\partial \hat{u}_s}{\partial t} = c^2\mathcal{F}_s\{u_{xx}\} = -c^2w^2\mathcal{F}_s\{u\} + c^2\sqrt{\frac{2}{\pi}} w f(0) = -c^2w^2\hat{u}_s(w,t)$$

$$\Rightarrow \quad \frac{\partial \hat{u}_s}{\partial t} + c^2w^2\hat{u}_s(w,t) = 0$$

Homogeneous PDE로 만들어 줌

☑ Ex. 4 Fourier Sine Transform Applied to the Heat Equation

- If a laterally insulated bar extends from x = 0 to infinity.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Subject to $u(x,0) = f(x)$, $u(0,t) = 0$ ⇨ $u(0,0) = f(0) = 0$

Sol)

$$\frac{\partial \hat{u}_s}{\partial t} + c^2 w^2 \hat{u}_s(w,t) = 0 \quad \Rightarrow \quad \hat{u}_s(w,t) = C(w) e^{-c^2 w^2 t}$$

$$u(x,0) = f(x) \quad \Rightarrow \quad \mathcal{F}_s\{u(x,0)\} = \mathcal{F}_s\{f(x)\}$$

$$\Rightarrow \hat{u}_s(w,0) = \hat{f}_s(w) \quad \Rightarrow \quad \hat{u}_s(w,t) = \hat{f}_s(w) e^{-c^2 w^2 t}$$

$$(\therefore \hat{f}_s(w) = C(w))$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(p) \sin wp \, dp$$

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_s(w) \sin wx dw$$

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}(w) e^{-c^2 w^2 t} \sin wx \, dw = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(p) \sin wp \, e^{-c^2 w^2 t} \sin wx \, dp \, dw$$

12.7 Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

☑ Ex. 5 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<\pi,\ y>0$$

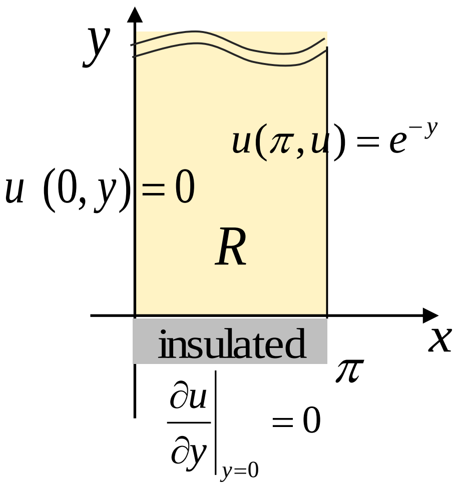
Subject to

$$u(0,y)=0,\ \ u(\pi,u)=e^{-y},\ \ y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$

"Fourier cosine Transform이 적합한 이유"

Sol)



Fourier series는 y 방향으로만 의미 있음, x는 상수 취급

y에 대한 Fourier transform 적용, $\mathcal{F}_c^{\,y}$ 로 정의

$$\mathcal{F}_c^{\,y}\{f''(x)\}=-w^2\mathcal{F}_c^{\,y}\{f(x)\}-\sqrt{\frac{2}{\pi}}f'(0)\ \Rightarrow\ \mathcal{F}_c^{\,y}\{u_{yy}\}=-w^2\mathcal{F}_c\{u\}-\sqrt{\frac{2}{\pi}}u_y(0)$$

$$\mathcal{F}_s\{f''(x)\}=-w^2\mathcal{F}_s\{f(x)\}+\sqrt{\frac{2}{\pi}}wf(0)\ \Rightarrow\ \mathcal{F}_s^{\,y}\{u_{xx}\}=-w^2\mathcal{F}_c\{u\}+\sqrt{\frac{2}{\pi}}wu(0)$$

$$\mathcal{F}_c^{\,y}(u_{xx})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_{xx}e^{-iwy}dy=\frac{1}{\sqrt{2\pi}}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}ue^{-iwy}dy=\frac{\partial^2\hat{u}}{\partial x^2}$$

$$\mathcal{F}(f)=\hat{f}(w)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-iwx}dx$$

12.7 Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

☑ Ex. 5 Using the Cosine Transform

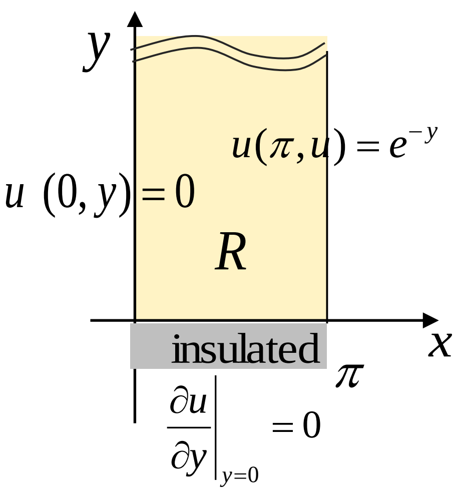
The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<\pi,\ y>0$$

Subject to

$$u(0,y)=0,\ \ u(\pi,u)=e^{-y},\ \ y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$



$$\mathrm{F}_c^{\ y}(u_{xx})=\frac{\partial^2 \hat{u}}{\partial x^2}$$

$$\mathrm{F}_c^{\ y}\{u_{yy}\}=-w^2\mathrm{F}_c\{u\}-\sqrt{\frac{2}{\pi}}u_y(0)$$

$$\mathrm{F}_c^{\ y}\left\{\frac{\partial^2 u}{\partial x^2}\right\}+\mathrm{F}_c^{\ y}\left\{\frac{\partial^2 u}{\partial y^2}\right\}=\mathrm{F}_c^{\ y}\{0\}$$

Homogeneous PDE로 만들어 줌

$$\frac{d^2\hat{u}}{dx^2}-w^2\hat{u}(x,w)-\sqrt{\frac{2}{\pi}}u_y(x,0)=0\ \Rightarrow\ \frac{d^2\hat{u}}{dx^2}-w^2\hat{u}=0$$

Fourier transform은 y에 대해서만 적용하였으므로, x는 그대로 유지되고, y만 w로 변경됨

☑ Ex. 5 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<\pi,\ y>0$$

Subject to

$$u(0,y)=0,\ \ u(\pi,u)=e^{-y},\ \ y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$

$$\frac{d^2\hat{u}}{dx^2}-w^2\hat{u}=0 \ \Rightarrow\ \hat{u}(x,w)=c_1\cosh wx+c_2\sinh wx$$

Boundary condition

$$\mathcal{F}_c^{\,y}\{u(0,y)\}=\hat{u}(0,w)=\mathcal{F}_c^{\,y}\{0\}$$

$$\therefore \hat{u}(0,w)=0$$

$$\mathcal{F}_c^{\,y}\{u(\pi,y)\}=\hat{u}(\pi,w)=\mathcal{F}_c^{\,y}\{e^{-y}\}$$

$$\therefore \hat{u}(\pi,w)=\frac{1}{1+w^2}$$

$$\int_0^\infty e^{-x}\cos wx\,dx=\left[e^{-x}\frac{\sin wx}{w}\right]_0^\infty-\int_0^\infty(-e^{-x})\frac{\sin wx}{w}dx$$

$$=\left[e^{-x}\frac{(-\cos wx)}{w^2}\right]_0^\infty-\int_0^\infty(-e^{-x})\frac{(-\cos wx)}{w^2}dx$$

$$=\frac{1}{w^2}-\frac{1}{w^2}\int_0^\infty e^{-x}\cos wx dx$$

$$\therefore \int_0^\infty e^{-x}\cos wx dx=\frac{1}{1+w^2}$$

☑ Ex. 5 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<\pi,\ y>0$$

Subject to

$$u(0,y)=0,\ \ u(\pi,u)=e^{-y},\ \ y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$

$$\frac{d^2\hat{u}}{dx^2}-w^2\hat{u}=0 \ \Rightarrow\ \hat{u}(x,w)=c_1\cosh wx+c_2\sinh wx$$

Boundary condition

$$\hat{u}(0,w)=0 \qquad \hat{u}(\pi,w)=\frac{1}{1+w^2}$$

$$\hat{u}(0,w)=c_1=0$$

$$\hat{u}(\pi,w)=c_2\sinh w\pi=\frac{1}{1+w^2} \qquad \therefore c_2=\frac{1}{(1+w^2)\sinh w\pi}$$

$$\therefore \hat{u}(x,w)=\frac{\sinh wx}{(1+w^2)\sinh w\pi}$$

☑ Ex. 5 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<\pi,\ y>0$$

Subject to

$$u(0,y)=0,\ \ u(\pi,u)=e^{-y},\ \ y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$

$$\hat{u}(x,w)=\frac{\sinh wx}{(1+w^2)\sinh w\pi}$$

$$\mathcal{F}_c(f)=\hat{f}_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx\,dw$$

Recall, definition

$$\mathcal{F}_c^{\,y}\{u(x,y)\}=\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,y)\cos wy\,dy=\hat{u}(x,w)$$

$$u(x,y)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}(x,w)\cos wy\,dw$$

$$\therefore u(x,y)=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sinh wx}{(1+w^2)\sinh w\pi}\cos wy\,dw$$