Chapter 14. (Supplementary) Bayesian Filtering for State Estimation of Dynamic Systems

***Neural Networks and Learning Machines* (Haykin)**

Lecture Notes on

Self-learning Neural Algorithms

Byoung-Tak Zhang
School of Computer Science and Engineering
Seoul National University

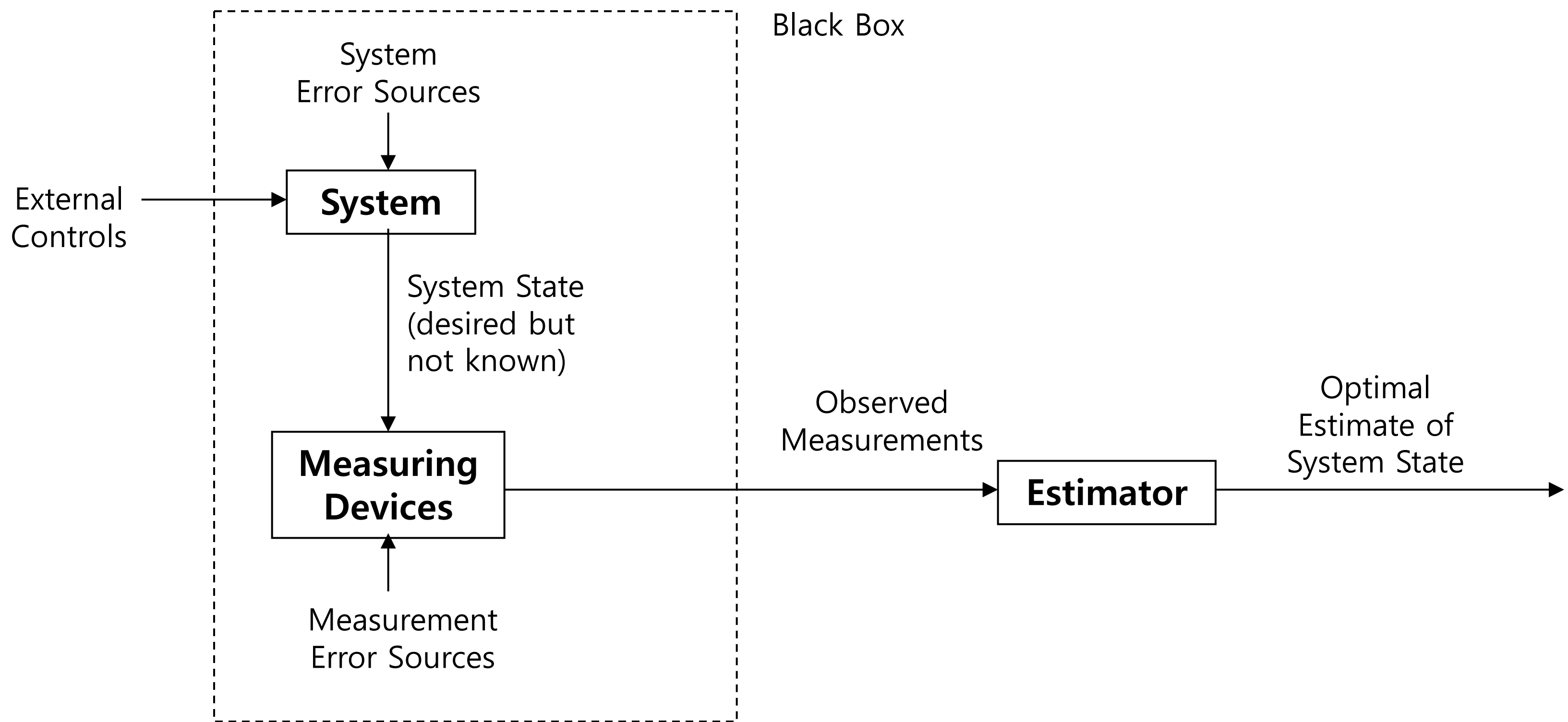
Version 20171115

Supplementary Material to Ch 14

Overview

- The Problem – Why do we need Kalman Filters?
- What is a Kalman Filter?
- Conceptual Overview
- The Theory of Kalman Filter
- Simple Example

The Problem



- System state cannot be measured directly
- Need to estimate "optimally" from measurements

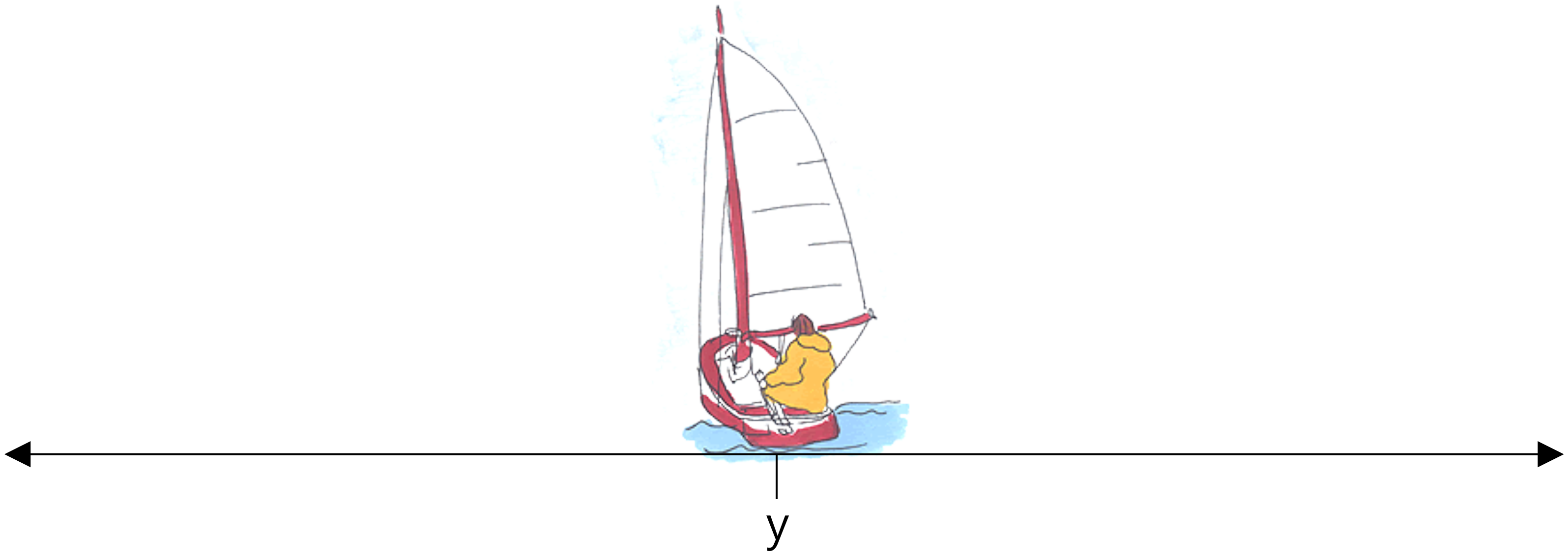
What is a Kalman Filter?

- Recursive data processing algorithm
- Generates optimal estimate of desired quantities given the set of measurements
- Optimal?
  - For linear system and white Gaussian errors, Kalman filter is "best" estimate based on all previous measurements
  - For non-linear system optimality is 'qualified'
- Recursive?
  - Doesn't need to store all previous measurements and reprocess all data each time step

Conceptual Overview

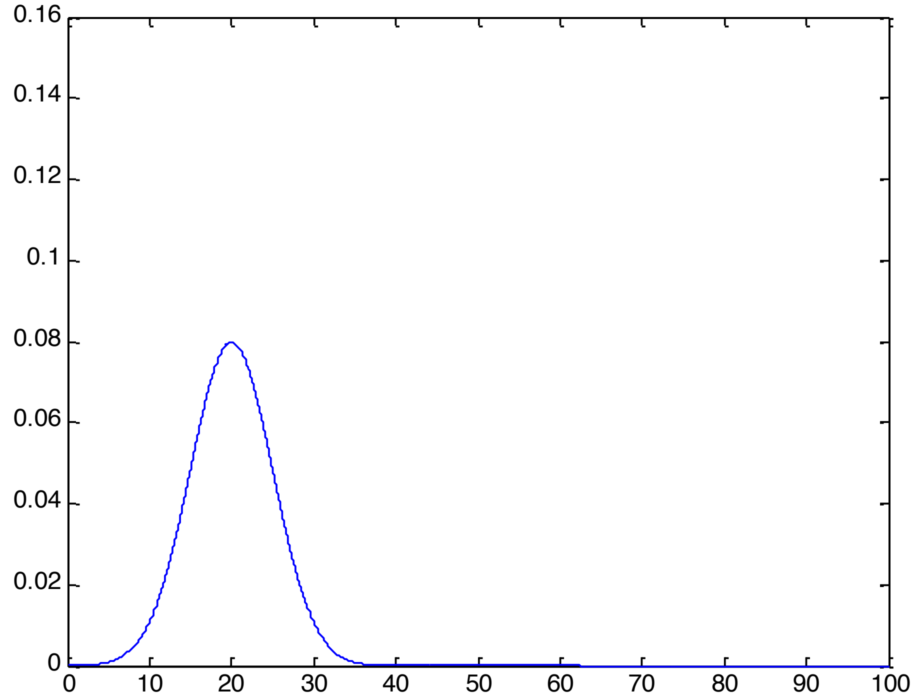
- Simple example to motivate the workings of the Kalman Filter
- Theoretical Justification to come later – for now just focus on the concept
- Important: Prediction and Correction

Conceptual Overview



- Lost on the 1-dimensional line
- Position – y(t)
- Assume Gaussian distributed measurements

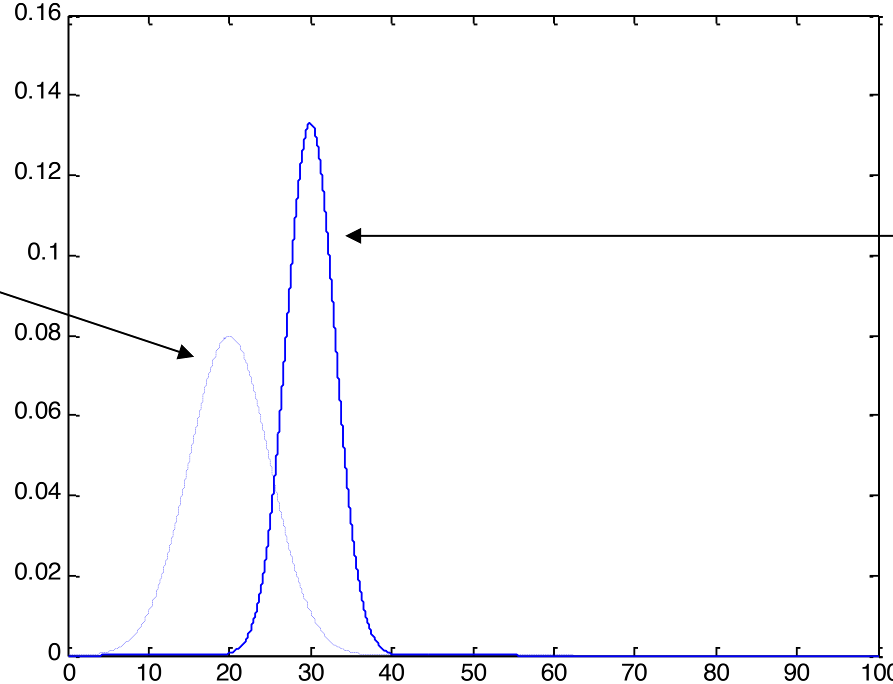
Conceptual Overview



- Sextant Measurement at t_1: Mean = z_1 and Variance = σ_{z1}
- Optimal estimate of position is: $\hat{y}(t_1) = z_1$
- Variance of error in estimate: $\sigma^2_x (t_1) = \sigma^2_{z1}$
- Boat in same position at time t_2 - Predicted position is z_1

Conceptual Overview

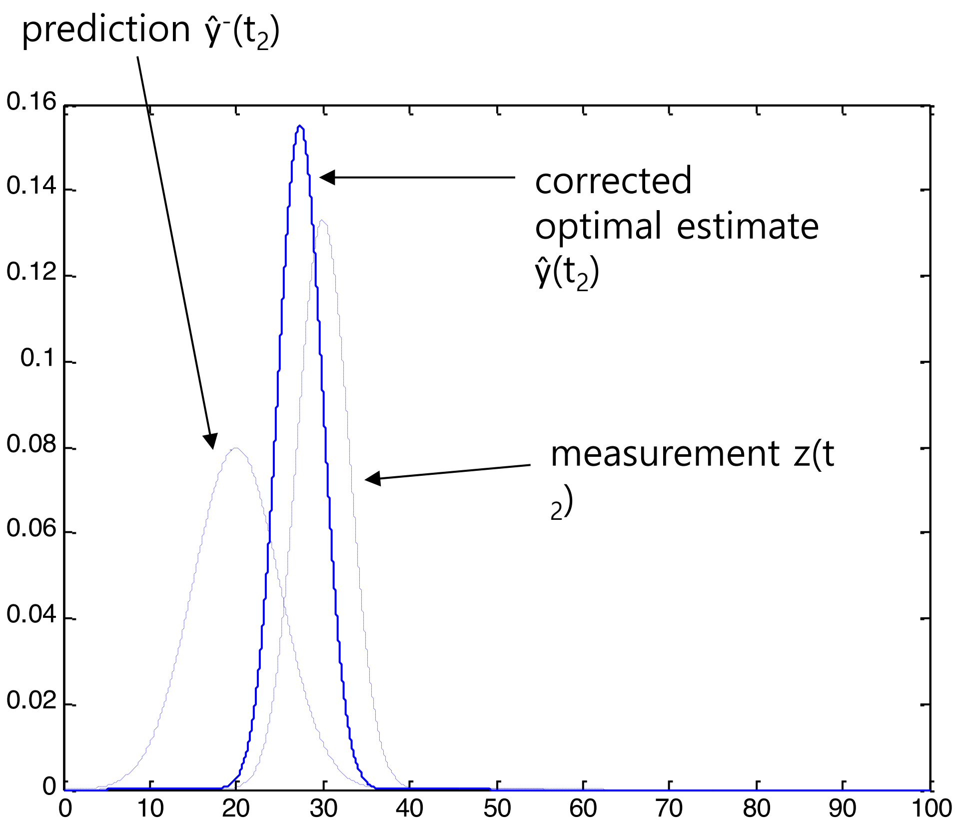
prediction $\hat{y}^{-}(t_2)$



measurement $z(t_2)$

- So we have the prediction $\hat{y}^{-}(t_2)$
- GPS Measurement at t_2: Mean = z_2 and Variance = σ_{z2}
- Need to <u>correct</u> the prediction due to measurement to get $\hat{y}(t_2)$
- Closer to more trusted measurement – linear interpolation?

Conceptual Overview



- Corrected mean is the new optimal estimate of position
- New variance is smaller than either of the previous two variances

Conceptual Overview

- Lessons so far:

Make prediction based on previous data - $\hat{y}^-$, σ^-

↓

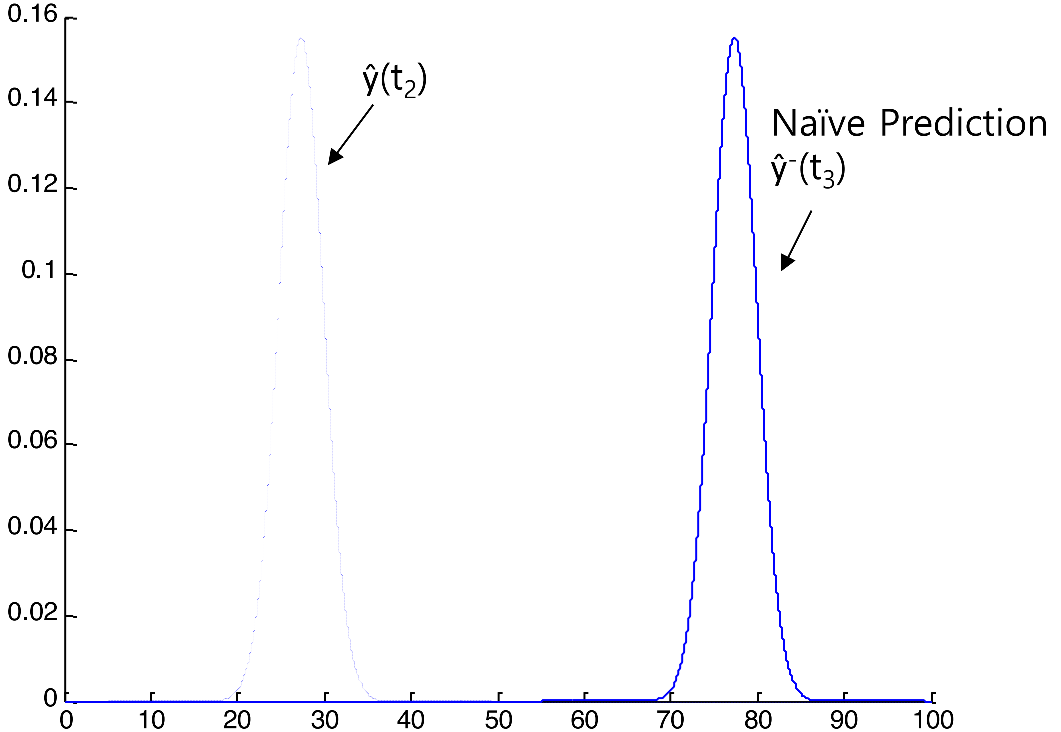
Take measurement – z_k, σ_z

↓

Optimal estimate ($\hat{y}$) = Prediction + (Kalman Gain) * (Measurement - Prediction)

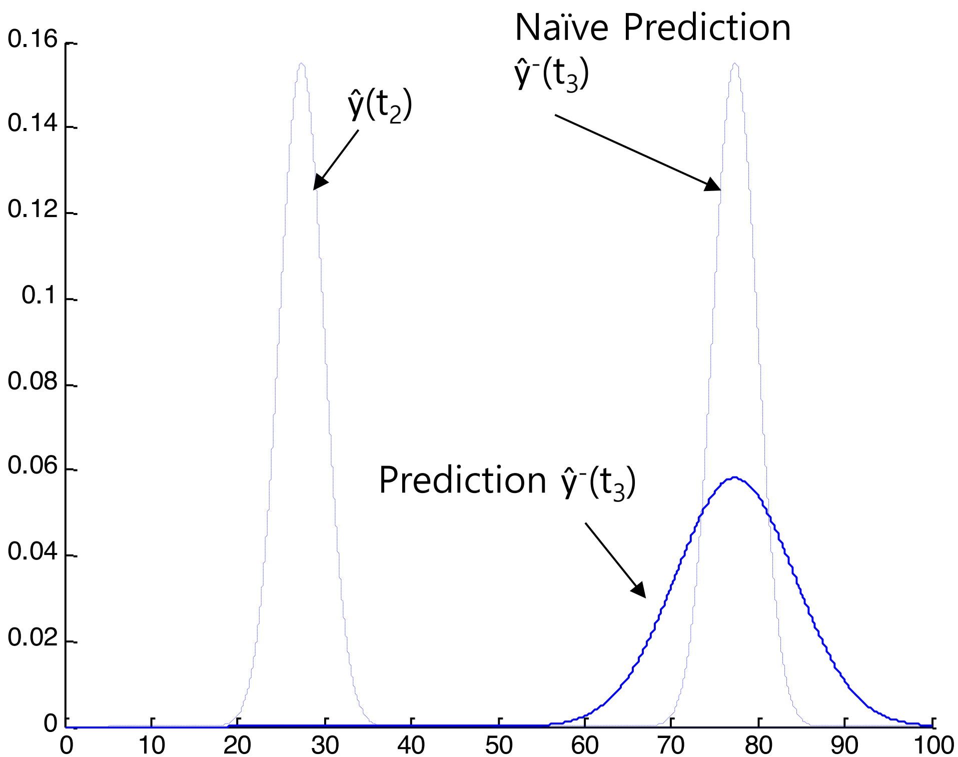
Variance of estimate = Variance of prediction * (1 – Kalman Gain)

Conceptual Overview



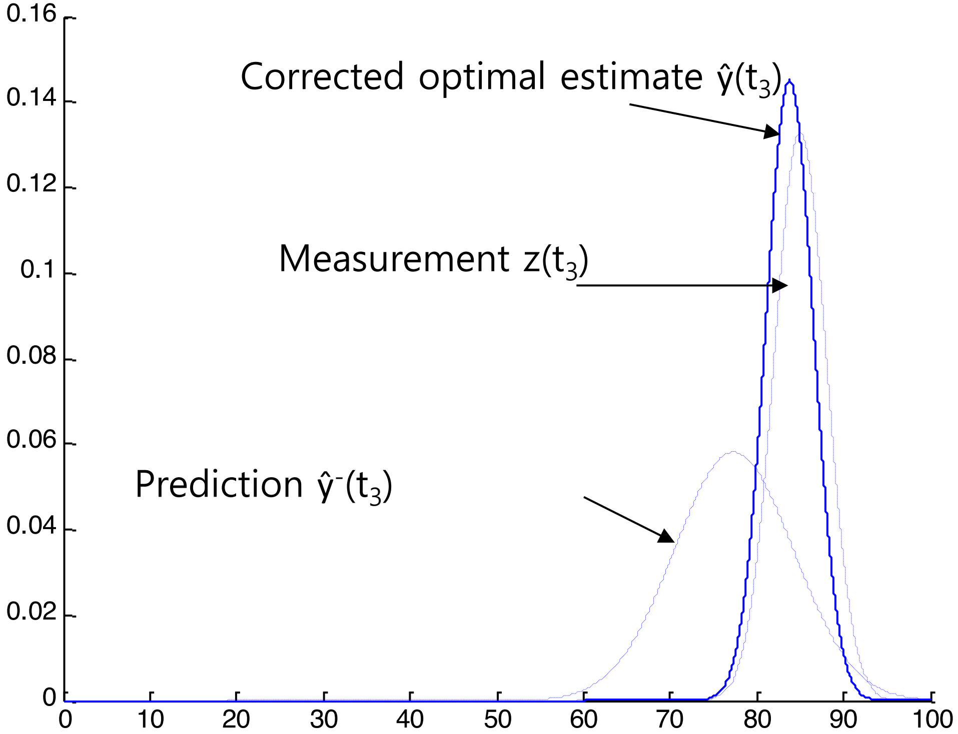
- At time t_3, boat moves with velocity dy/dt=u
- Naïve approach: Shift probability to the right to predict
- This would work if we knew the velocity exactly (perfect model)

Conceptual Overview



- Better to assume imperfect model by adding Gaussian noise
- dy/dt = u + w
- Distribution for prediction moves and spreads out

Conceptual Overview



- Now we take a measurement at t_3
- Need to once again correct the prediction
- Same as before

Conceptual Overview

- Lessons learnt from conceptual overview:
  - Initial conditions ($\hat{y}_{k-1}$ and σ_{k-1})
  - Prediction ($\hat{y}^-_k$, σ^-_k)
    - Use initial conditions and model (eg. constant velocity) to make prediction
  - Measurement (z_k)
    - Take measurement
  - Correction ($\hat{y}_k$, σ_k)
    - Use measurement to correct prediction by ‘blending’ prediction and residual – always a case of merging only two Gaussians
    - Optimal estimate with smaller variance

Theoretical Basis

- Process to be estimated:

$y_k = Ay_{k-1} + Bu_k + w_{k-1}$ Process Noise (w) with covariance Q

$z_k = Hy_k + v_k$ Measurement Noise (v) with covariance R

- Kalman Filter

Predicted: $\hat{y}^-_k$ is estimate based on measurements at previous time-steps

$$\hat{y}^-_k = Ay_{k-1} + Bu_k$$

$$P^-_k = AP_{k-1}A^T + Q$$

Corrected: $\hat{y}_k$ has additional information – the measurement at time k

$$\hat{y}_k = \hat{y}^-_k + K(z_k - H\,\hat{y}^-_k)$$

$$K = P^-_k H^T (HP^-_k H^T + R)^{-1}$$

$$P_k = (I - KH)P^-_k$$

Blending Factor

- If we are sure about measurements:
  - Measurement error covariance (R) decreases to zero
  - K decreases and weights residual more heavily than prediction
- If we are sure about prediction
  - Prediction error covariance P^{-}_{k} decreases to zero
  - K increases and weights prediction more heavily than residual

Theoretical Basis

Prediction (Time Update)

(1) Project the state ahead

$$\hat{y}^-_k = Ay_{k-1} + Bu_k$$

(2) Project the error covariance ahead

$$P^-_k = AP_{k-1}A^T + Q$$

Correction (Measurement Update)

(1) Compute the Kalman Gain

$$K = P^-_k H^T (HP^-_k H^T + R)^{-1}$$

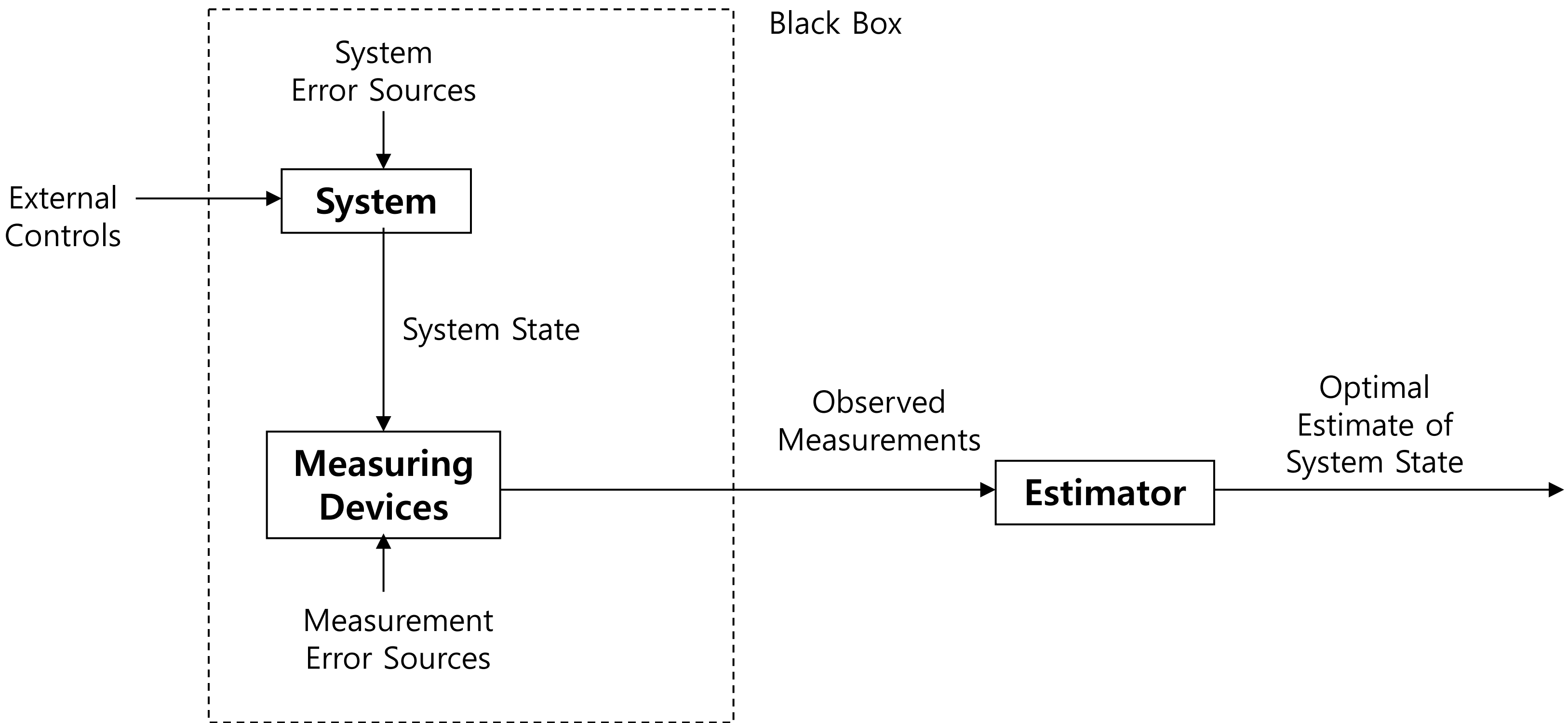
(2) Update estimate with measurement z_k

$$\hat{y}_k = \hat{y}^-_k + K(z_k - H\,\hat{y}^-_k)$$

(3) Update Error Covariance

$$P_k = (I - KH)P^-_k$$

Quick Example – Constant Model



Quick Example – Constant Model

Prediction

$$\hat{y}^-_k = y_{k-1}$$

$$P^-_k = P_{k-1}$$

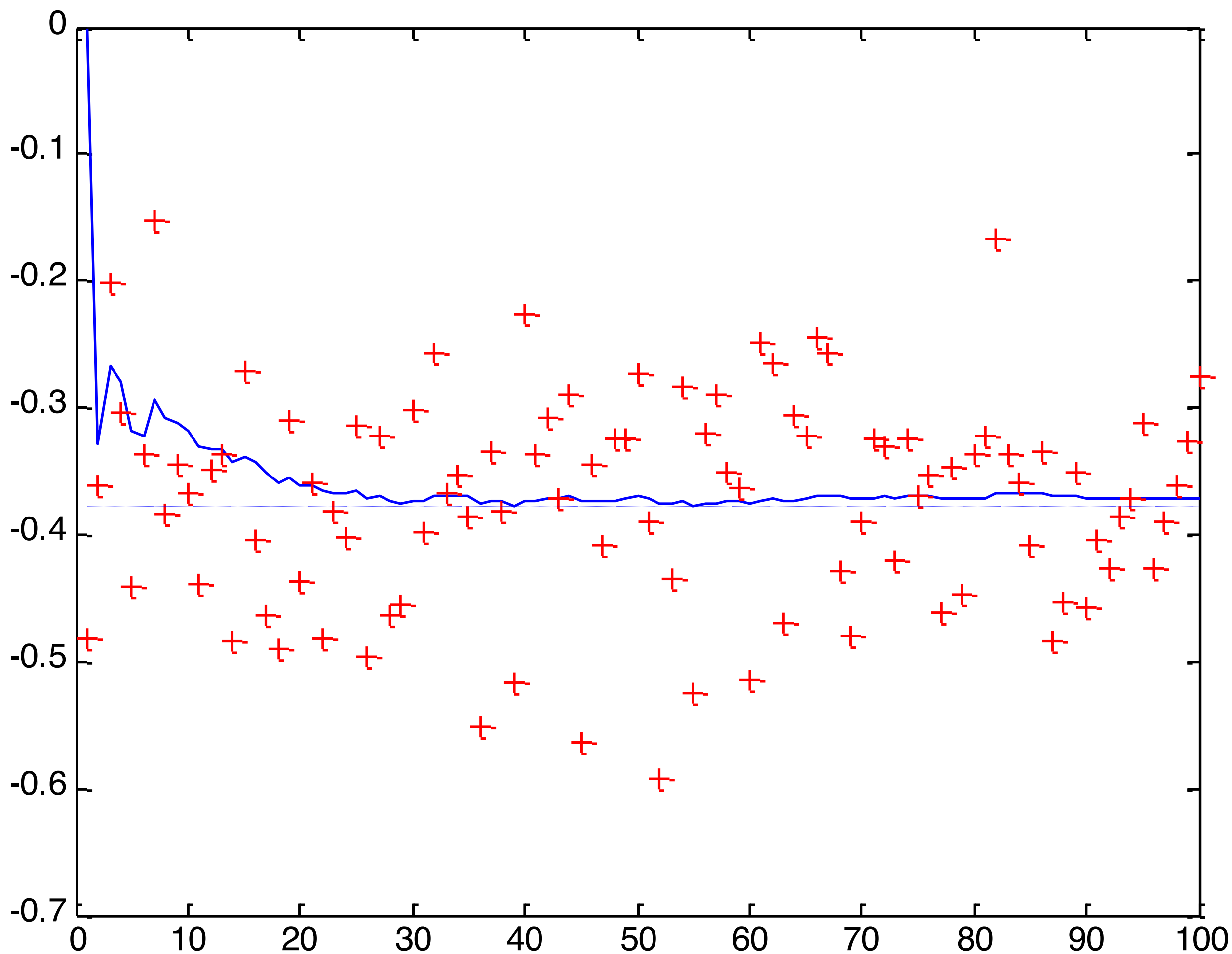
Correction

$$K = P^-_k(P^-_k + R)^{-1}$$

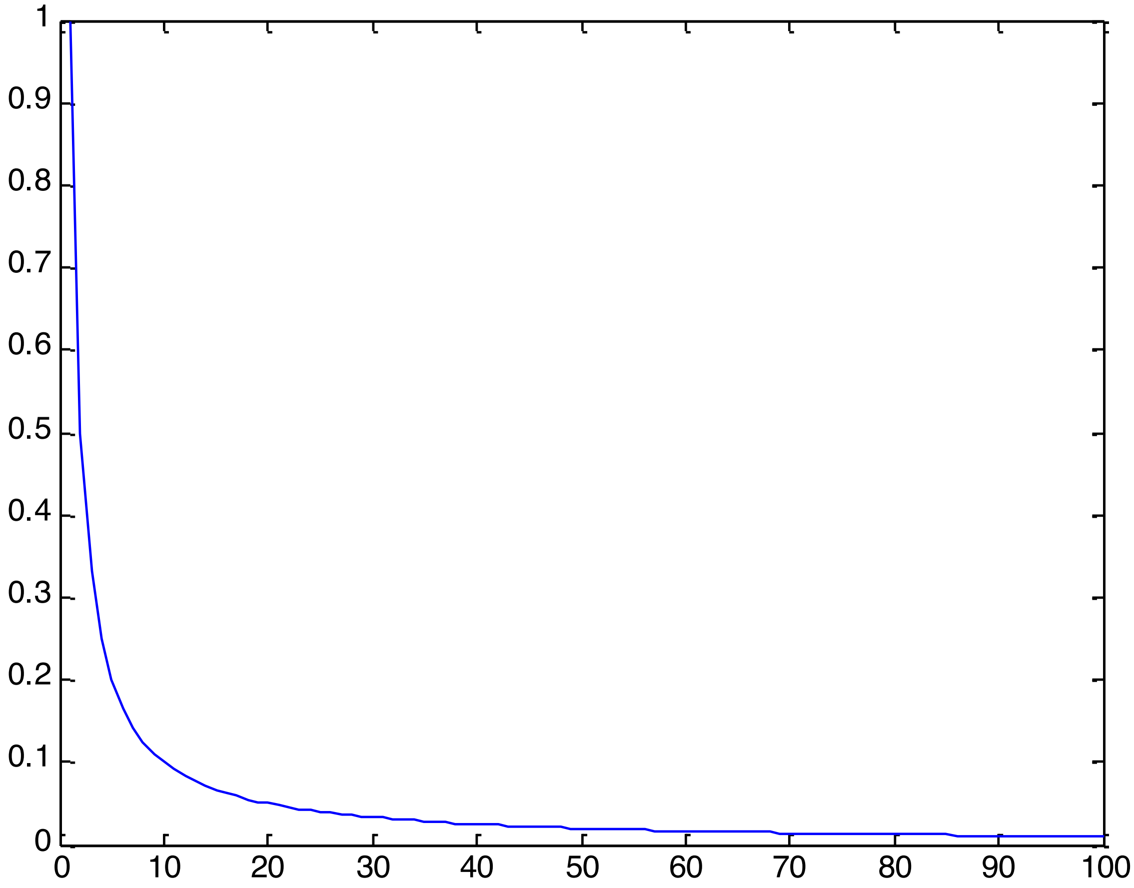
$$\hat{y}_k = \hat{y}^-_k + K(z_k - H\,\hat{y}^-_k)$$

$$P_k = (I - K)P^-_k$$

Quick Example – Constant Model

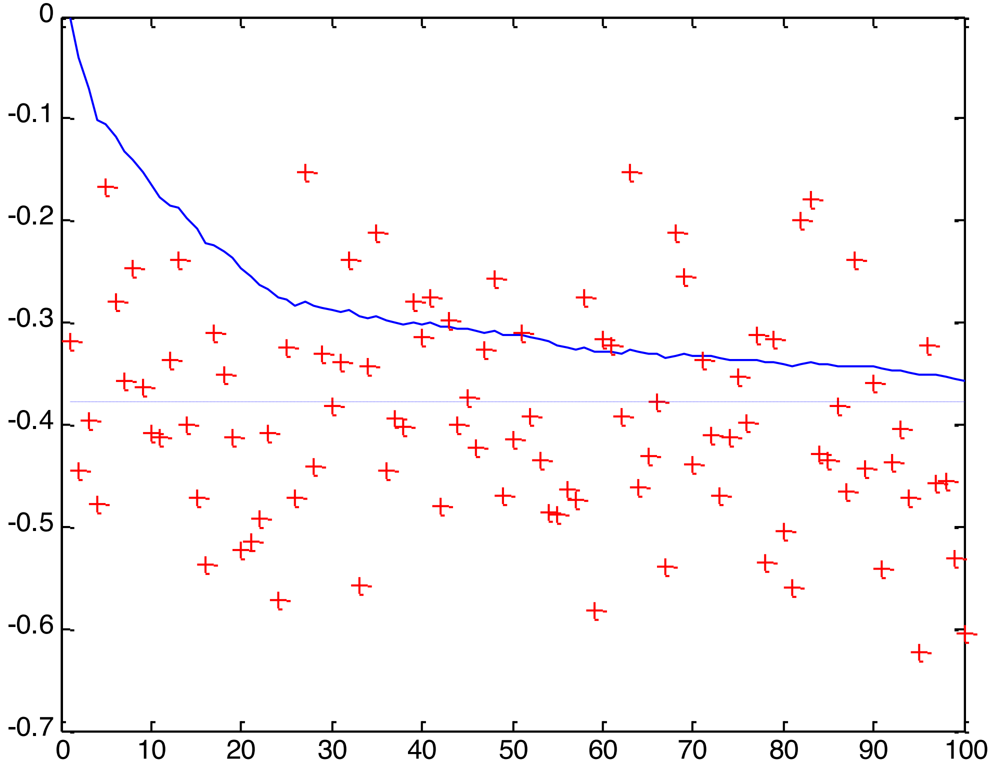


Quick Example – Constant Model



Convergence of Error Covariance - P_k

Quick Example – Constant Model



Larger value of R – the measurement error covariance (indicates poorer quality of measurements)

Filter slower to 'believe' measurements – slower convergence

References

1. Kalman, R. E. 1960. “A New Approach to Linear Filtering and Prediction Problems”, Transaction of the ASME--Journal of Basic Engineering, pp. 35-45 (March 1960).
2. Maybeck, P. S. 1979. “Stochastic Models, Estimation, and Control, Volume 1”, Academic Press, Inc.
3. Welch, G and Bishop, G. 2001. “An introduction to the Kalman Filter”, http://www.cs.unc.edu/~welch/kalman/

Sequential Monte Carlo

Monte Carlo (MC) Approximation

$$E_p[f(x)] = \int p(x) f(x) dx$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} f(x^{(i)}), \qquad x^{(i)} \sim p(x) = N(0, \sigma^2)$$

- Monte Carlo approach

1. Simulate N random variables from p(x), e.g. Normal distribution

$$x^{(i)} \sim p(x) = N(0, \sigma^2)$$

2. Compute the average

$$E_p[f(x)] = \frac{1}{N}\sum_{i=1}^{N} f(x^{(i)}),$$

MC with Importance Sampling

$$E_p[f(x)] = \int_x p(x) f(x) dx$$

$$= \int_x \frac{p(x)}{q(x)} q(x) f(x) dx$$

$$\approx \sum_{i=1}^{N} w_i f(x^{(i)})$$

$x^{(i)} \sim q(x)$ $q(x)$: proposal distribution

$w_i = \dfrac{p(x^{(i)})}{q(x^{(i)})}$ w_i: importance weight

Note: $q(x)$ is easier to sample from than $p(x)$.

Importance Sampling (IS)

$$E[f(x_{0:t})] = \int f(x_{0:t}) p(x_{0:t} \mid y_{1:t}) dx_{0:t}$$

$$\approx \sum_{i=1}^{N} w_i f(x_{0:t}^{(i)})$$

$x_{0:t}^{(i)} \sim q(x_{0:t} \mid y_{1:t})$ $\qquad$ $q(x)$: proposal distribution

$w_i = \dfrac{p(x_{0:t}^{(i)} \mid y_{1:t})}{q(x_{0:t}^{(i)} \mid y_{1:t})}$ $\qquad$ w_i: importance weight

Importance Sampling: Procedure

1. Draw N samples $x_{0:t}^{(i)}$ from proposal distribution q(.).

$$x_{0:t}^{(i)} \sim q(x_{0:t} \mid y_{1:t})$$

2. Compute importance weight

$$w(x_{0:t}^{(i)}) = \frac{p(x_{0:t}^{(i)} \mid y_{1:t})}{q(x_{0:t}^{(i)} \mid y_{1:t})}$$

3. Estimate an arbitrary function f(.):

$$E[f(x_{0:t} \mid y_{1:t})] \approx \sum_{i=1}^{N} f(x_{0:t}^{(i)})\tilde{w}_t^{(i)}, \qquad \tilde{w}_t^{(i)} = \frac{w(x_{0:t}^{(i)})}{\sum_{j=1}^{N} w(x_{0:t}^{(j)})}$$

Sequential Importance Sampling (SIS): Recursive Estimation

Augmenting the samples

$$q(x_{0:t} \mid y_{1:t}) = q(x_{0:t-1} \mid y_{1:t-1}) q(x_t \mid x_{0:t-1}, y_{1:t})$$
$$= q(x_{0:t-1} \mid y_{1:t-1}) q(x_t \mid x_{t-1}, y_t)$$

$$x_t^{(i)} \sim q(x_t \mid x_{t-1}, y_t)$$

(cf. non-sequential IS: $x_t^{(i)} \sim q(x_{0:t} \mid y_{1:t})$)

Weight update

$$w_t^{(i)} \propto w_{t-1}^{(i)} \frac{p(y_t \mid x_t^{(i)}) p(x_t^{(i)} \mid x_{t-1}^{(i)})}{q(x_t^{(i)} \mid x_{t-1}^{(i)}, y_t)}$$

Sequential Importance Sampling: Idea

- Update filtering density using Bayesian filtering
- Compute integrals using importance sampling

- The filtering density $p(x_t \mid y_{1:t})$ is represented using particles and their weights

$$\{x_t^{(i)}, w_t^{(i)}\}_{i=1}^{N}$$

- Compute weights using:

$$w_t^{(i)} = \frac{p(x_t^{(i)}, y_{1:t})}{q(x_t^{(i)}, y_{1:t})}$$

Sequential Importance Sampling: Procedure

1. Particle generation $x_t^{(i)} \sim q(x_t \mid x_{t-1}^{(i)}, y_t) = p(x_t \mid x_{t-1}^{(i)})$

2. Weight computation $w_t^{(i)} = w_{t-1}^{(i)} p(y_t \mid x_t^{(i)})$

   Weight normalization $\tilde{w}_t^{(i)} = \dfrac{w_t^{(i)}}{\sum_{j=1}^{N} w_t^{(j)}}$

3. Estimation computation $E[f(x_t \mid y_{1:t})] = \sum_{i=1}^{N} f(x_t^{(i)}) \tilde{w}_t^{(i)}$

Note: Step 1 above assumes the proposal density to be the prior. This does not use the information from observations. Alternatively, the proposal density could be

$$x_t^{(i)} \sim q(x_t \mid x_{t-1}^{(i)}, y_t) = p(x_t \mid x_{t-1}^{(i)}, y_t)$$

that minizes the variance of w_t (Doucet el al., 1999).

Resampling

- SIS suffers from degeneracy problems, i.e. a small number of particles have big weights and the rest have extremely small values.
- Remedy: SIR introduces a selection (resampling) step to eliminate samples with low importance ratios (weights) and multiply samples with high importance ratios.
- Resampling maps the weighted random measure on to the equally weighted random measure by sampling uniformly with replacement from $\{x_{0:t}^{(i)}\}_{i=1}^{N}$ with probabilities $\{w_t^{(i)}\}_{i=1}^{N}$:

$$\{\tilde{x}_{0:t}^{(i)}, N^{-1}\}_{i=1}^{N} \sim \{x_{0:t}^{(i)}, w_t^{(i)}(x_{0:t}^{(i)})\}_{i=1}^{N}$$

Sampling Importance Resampling (SIR) = Sequential Monte Carlo = Particle Filter

1. Initialize $t \leftarrow 0$

- For $i = 1, \ldots, N$: sample $x_t^{(i)} \sim p(x_0), \quad t \leftarrow 1.$

2. Importance sampling

- For $i = 1, \ldots, N$: sample $x_t^{(i)} \sim q(x_t \mid x_{t-1}^{(i)}, y_t) = p(x_t \mid x_{t-1}^{(i)})$

  Let $x_{0:t}^{(i)} \triangleq (x_{0:t-1}^{(i)}, x_t^{(i)})$

- For $i = 1, \ldots, N$: compute weights $\quad w_t^{(i)} = p(y_t \mid x_t^{(i)})$

- Normalize the weights: $\tilde{w}_t^{(i)} = w_t^{(i)} \Big/ \sum_{j=1}^{N} w_t^{(j)}$

3. Resampling

- Resample with replacement N particles $x_{0:t}^{(i)}$ according to the importance weights $w_t^{(i)}$, resulting in $\{\tilde{x}_{0:t}^{(i)}, N^{-1}\}_{i=1}^{N}$.

- New particle population $\{x_{0:t}^{(i)}\}_{i=1}^{N} \leftarrow \{\tilde{x}_{0:t}^{(i)}\}_{i=1}^{N}$.

- Set $t \leftarrow t+1$ and go to step 2.

Particle Filters

Motivating Applications

- Hand tracking using particle filters: http://www.youtube.com/watch?v=J3ioMxRI174
- Robotics - SLAM and localization with a stereo camera: http://www.youtube.com/watch?v=m3L8OfbTXH0&feature=related
- Kalman filter result on real aircraft: http://www.youtube.com/watch?v=0GSIKwfkFCA&feature=related

Problem Statement

- Tracking the state of a system as it evolves over time
- We have: Sequentially arriving (noisy or ambiguous) observations
- We want to know: Best possible estimate of the hidden variables

Bayesian Filtering / Tracking Problem

- Unknown state vector $x_{0:t} = (x_0, \ldots, x_t)$
- Observation vector $z_{1:t}$
- Find PDF $p(x_{0:t} \mid z_{1:t})$ … *posterior distribution*
- or $p(x_t \mid z_{1:t})$ … *filtering distribution*

- Prior information given:
  - $p(x_0)$ … prior on state distribution
  - $p(z_t \mid x_t)$ … sensor model
  - $p(x_t \mid x_{t-1})$ … Markovian state-space model

Sequential Update

- Storing all incoming measurements is inconvenient
- Recursive filtering:
  - Predict next state pdf from current estimate
  - Update the prediction using sequentially arriving new measurements
- Optimal Bayesian solution: recursively calculating exact posterior density

Bayesian Update and Prediction

- Prediction

$$p(x_t \mid z_{1:t-1}) = \int p(x_t \mid x_{t-1})\, p(x_{t-1} \mid z_{1:t-1})\, dx_{t-1}$$

- Update

$$p(x_t \mid z_{1:t}) = \frac{p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1})}{p(z_t \mid z_{1:t-1})}$$

$$p(z_t \mid z_{1:t-1}) = \int p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1})\, dx_t$$

Kalman Filter

- Optimal solution for linear-Gaussian case
- Assumptions:
  - State model is known linear function of last state and Gaussian noise signal
  - Sensory model is known linear function of state and Gaussian noise signal
  - Posterior density is Gaussian

Kalman Filter: Update Equations

$$x_t = F_t x_{t-1} + v_{t-1} \quad v_{t-1} \sim N(0, Q_{t-1})$$

$$z_t = H_t x_t + n_t \quad n_t \sim N(0, R_t)$$

$$F_t, H_t \text{ :known matrices}$$

$$p(x_{t-1} \mid z_{1:t-1}) = N(x_{t-1} \mid m_{t-1|t-1}, P_{t-1|t-1})$$

$$p(x_t \mid z_{1:t-1}) = N(x_t \mid m_{t|t-1}, P_{t|t-1})$$

$$p(x_t \mid z_{1:t}) = N(x_t \mid m_{t|t}, P_{t|t})$$

$$m_{t|t-1} = F_t m_{t-1|t-1}$$

$$P_{t|t-1} = Q_{t-1} + F_t P_{t-1|t-1} F_t^T$$

$$m_{t|t} = m_{t|t-1} + K_t (z_t - H_t m_{t|t-1})$$

$$P_{t|t} = P_{t|t-1} - K_t H_t P_{t|t-1}$$

$$S_t = H_t P_{t|t-1} H_t^T + R_t$$

$$K_t = P_{t|t-1} H_t^T S_t^{-1}$$

Limitations of Kalman Filtering

- Assumptions are too strong. We often find:
  - Non-linear models
  - Non-Gaussian noise or posterior
  - Multi-modal distributions
  - Skewed distributions

- Extended Kalman Filter:
  - Local linearization of non-linear models
  - Still limited to Gaussian posterior

Grid-based Methods

- Optimal for discrete and finite state space
- Keep and update an estimate of posterior pdf for every single state
- No constraints on posterior (discrete) density

Limitations of Grid-based Methods

- Computationally expensive
- Only for finite state sets
- Approximate grid-based filter
  - Divide continuous state space into finite number of cells
  - Hidden Markov model filter
  - Dimensionality increases computational costs dramatically

Many different names…

Particle Filters

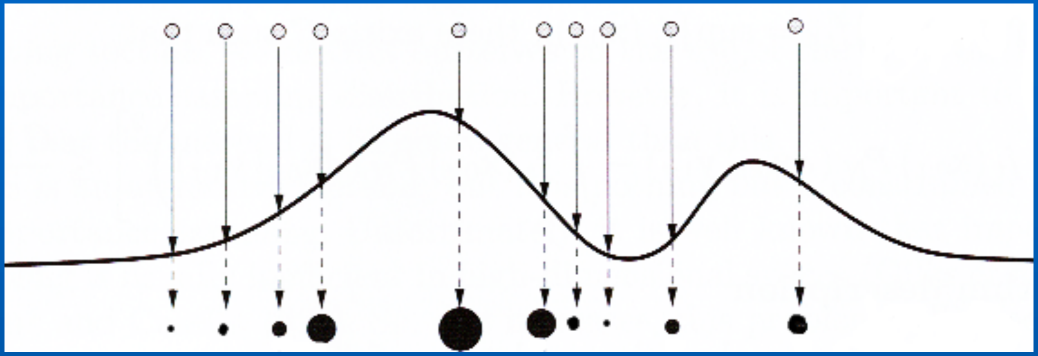
- (Sequential) Monte Carlo filters
- Bootstrap filters
- Condensation
- Interacting particle approximations
- Survival of the fittest
- …

Sample-Based PDF Representation

- Monte Carlo characterization of pdf:
  - Represent posterior density by a set of random i.i.d. samples (**particles**) from the pdf $p(x_{0:t}|z_{1:t})$
  - For larger number N of particles equivalent to functional description of pdf
  - For $N \rightarrow \infty$ approaches optimal Bayesian estimate

Sample-based PDF Representation

- Regions of high density
  - Many particles
  - Large weight of particles
- Uneven partitioning



- Discrete approximation for continuous pdf

$$P_N(x_{0:t} \mid z_{1:t}) = \sum_{i=1}^{N} w_t^i \, \delta(x_{0:t} - x_{0:t}^i)$$

Importance Sampling

- Draw N samples $x_{0:t}^{(i)}$ from importance sampling distribution $\pi(x_{0:t}|z_{1:t})$
- Importance weight

$$w(x_{0:t}) = \frac{p(x_{0:t} \mid z_{1:t})}{\pi(x_{0:t} \mid z_{1:t})}$$

- Estimation of arbitrary functions f_t:

$$\hat{I}_N(f_t) = \sum_{i=1}^{N} f_t(x_{0:t}^{(i)})\,\tilde{w}_t^{(i)}, \quad \tilde{w}_t^{(i)} = \frac{w(x_{0:t}^{(i)})}{\sum_{j=1}^{N} w(x_{0:t}^{(j)})}$$

$$\hat{I}_N(f_t) \xrightarrow[N\to\infty]{a.s.} I(f_t) = \int f_t(x_{0:t})\,p(x_{0:t} \mid y_{1:t})\,dx_{0:t}$$

Sequential Importance Sampling (SIS)

- Augmenting the samples

$$\pi(x_{0:t} \mid z_{1:t}) \quad = \pi(x_{0:t-1} \mid z_{1:t-1})\,\pi(x_t \mid x_{0:t-1}, z_{1:t}) =$$
$$= \pi(x_{0:t-1} \mid z_{1:t-1})\,\pi(x_t \mid x_{t-1}, z_t)$$
$$x_t^{(i)} \sim \pi(x_t \mid x_{t-1}^{(i)}, z_t)$$

- Weight update

$$w_t^{(i)} \propto w_{t-1}^{(i)} \frac{p(z_t \mid x_t^{(i)})\, p(x_t^{(i)} \mid x_{t-1}^{(i)})}{\pi(x_t^{(i)} \mid x_{t-1}^{(i)}, z_t)}$$

Degeneracy Problem

- After a few iterations, all but one particle will have negligible weight
- Measure for degeneracy: *effective sample size*

$$N_{eff} = \frac{N}{1 + Var(w_t^{*i})}$$

w_t^{*} ... true weights at time t

- Small N_{eff} indicates severe degeneracy
- Brute force solution: Use very large N

Choosing Importance Density

- Choose π to minimize variance of weights
- Optimal solution:

$$\pi_{opt}(x_t \mid x_{t-1}^{(i)}, z_t) = p(x_t \mid x_{t-1}^{(i)}, z_t)$$
$$\Rightarrow w_t^{(i)} \propto w_{t-1}^{(i)}\, p(z_t \mid x_{t-1}^{(i)})$$

- Practical solution
  - Importance density = prior

$$\pi(x_t \mid x_{t-1}^{(i)}, z_t) = p(x_t \mid x_{t-1}^{(i)})$$
$$\Rightarrow w_t^{(i)} \propto w_{t-1}^{(i)}\, p(z_t \mid x_t^{(i)})$$

Resampling

- Eliminate particles with small importance weights
- Concentrate on particles with large weights

- Sample N times *with replacement* from the set of particles $x_t^{(i)}$ according to importance weights $w_t^{(i)}$
- „*Survival of the fittest*“

Sampling Importance Resample Filter: Basic Algorithm

- 1. INIT, t=0
  - for i=1,..., N: sample $x_0^{(i)} \sim p(x_0)$; t:=1;
- 2. IMPORTANCE SAMPLING
  - for i=1,..., N: sample $x_t^{(i)} \sim p(x_t|x_{t-1}^{(i)})$
    - $x_{0:t}^{(i)} := (x_{0:t-1}^{(i)}, x_t^{(i)})$
  - for i=1,..., N: evaluate importance weights $w_t^{(i)}=p(z_t|x_t^{(i)})$
  - Normalize the importance weights
- 3. SELECTION / RESAMPLING
  - resample with replacement N particles $x_{0:t}^{(i)}$ according to the importance weights
  - Set t:=t+1 and go to step 2

Variations

- Auxiliary Particle Filter:
  - Resample at time t-1 with one-step lookahead (re-evaluate with new sensory information)
- Regularisation:
  - Resample from continuous approximation of posterior $p(x_t|z_{1:t})$

Visualization of Particle Filter

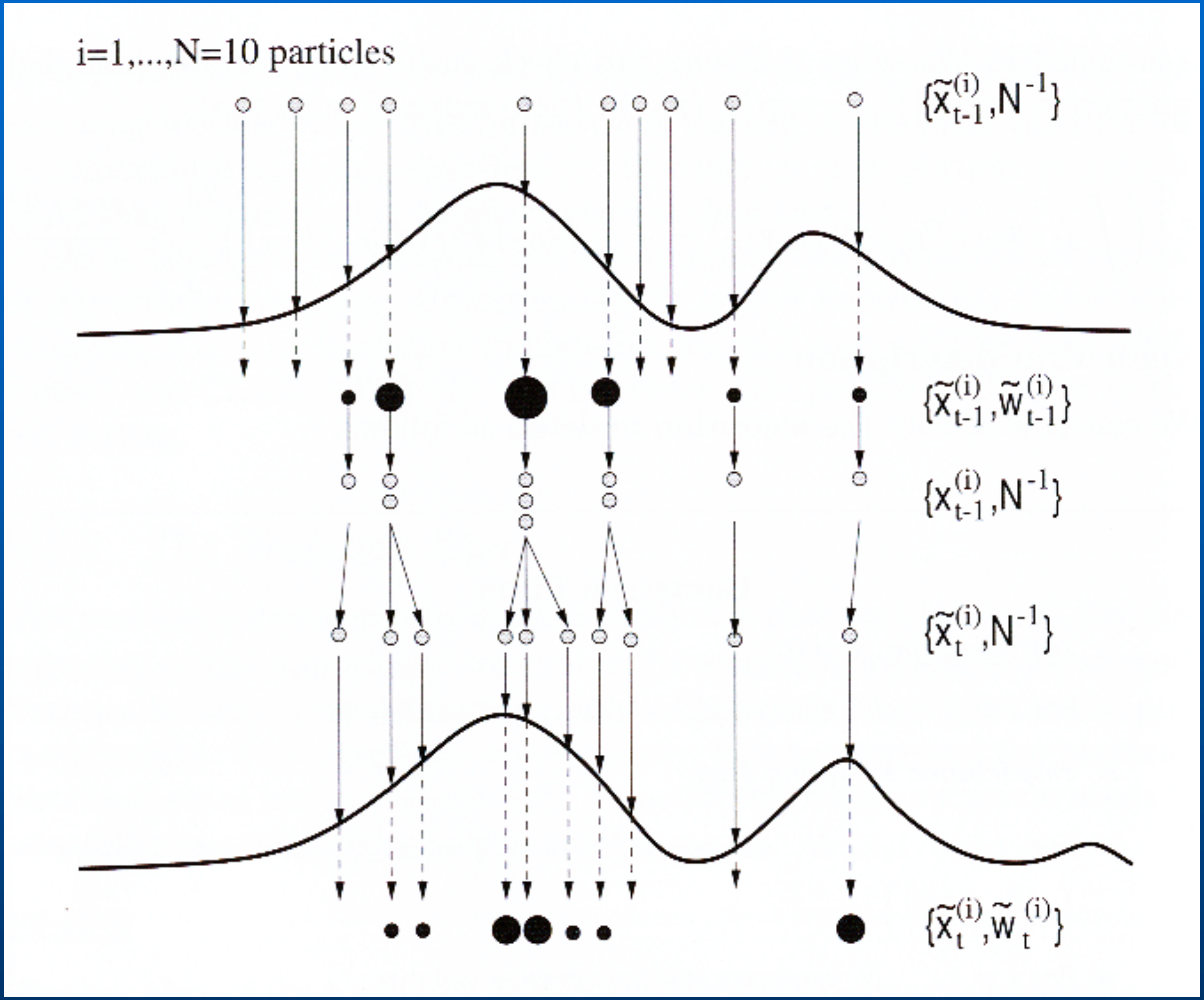
unweighted measure

compute importance weights $\Rightarrow p(x_{t-1}|z_{1:t-1})$

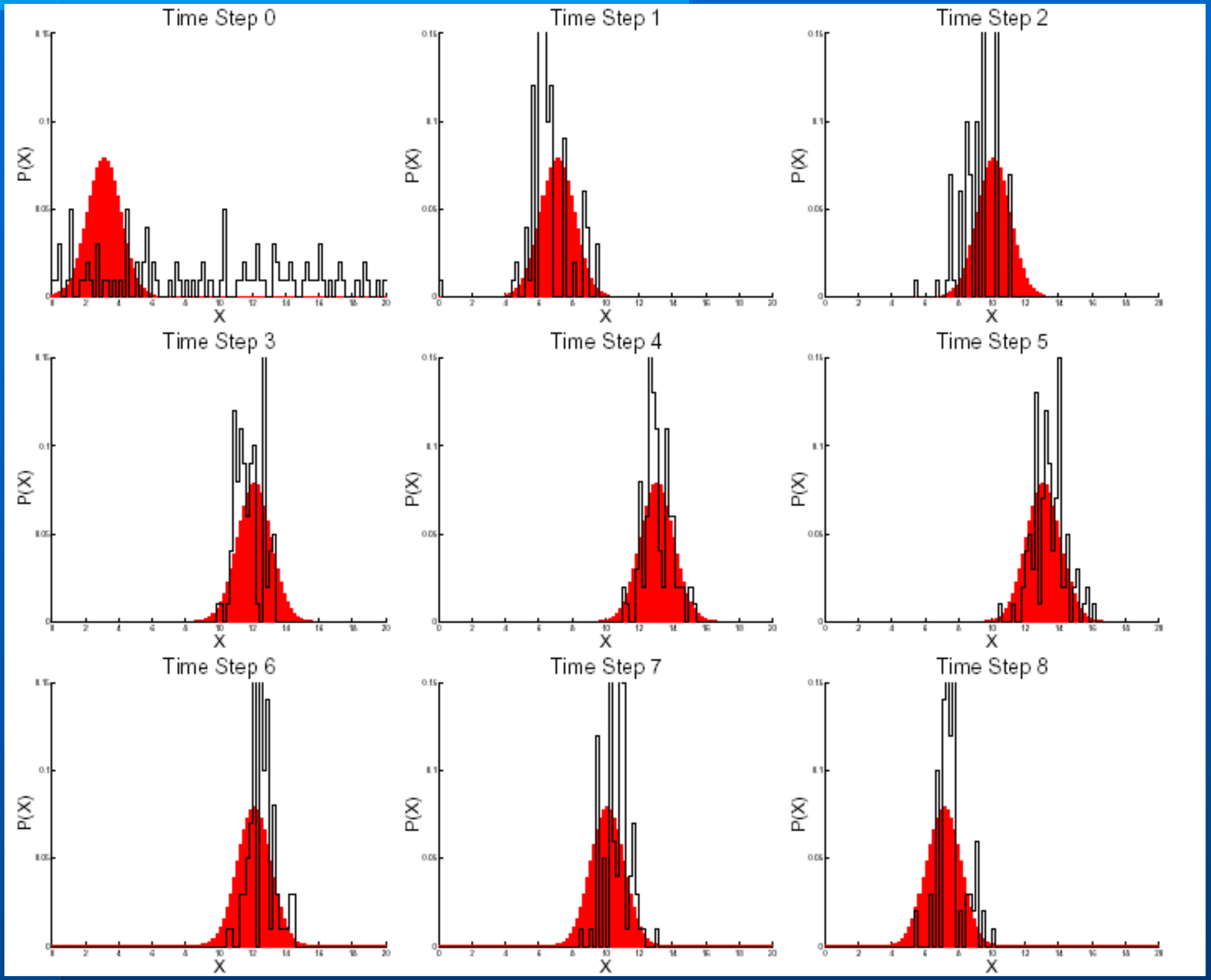
resampling

move particles

predict $p(x_t|z_{1:t-1})$

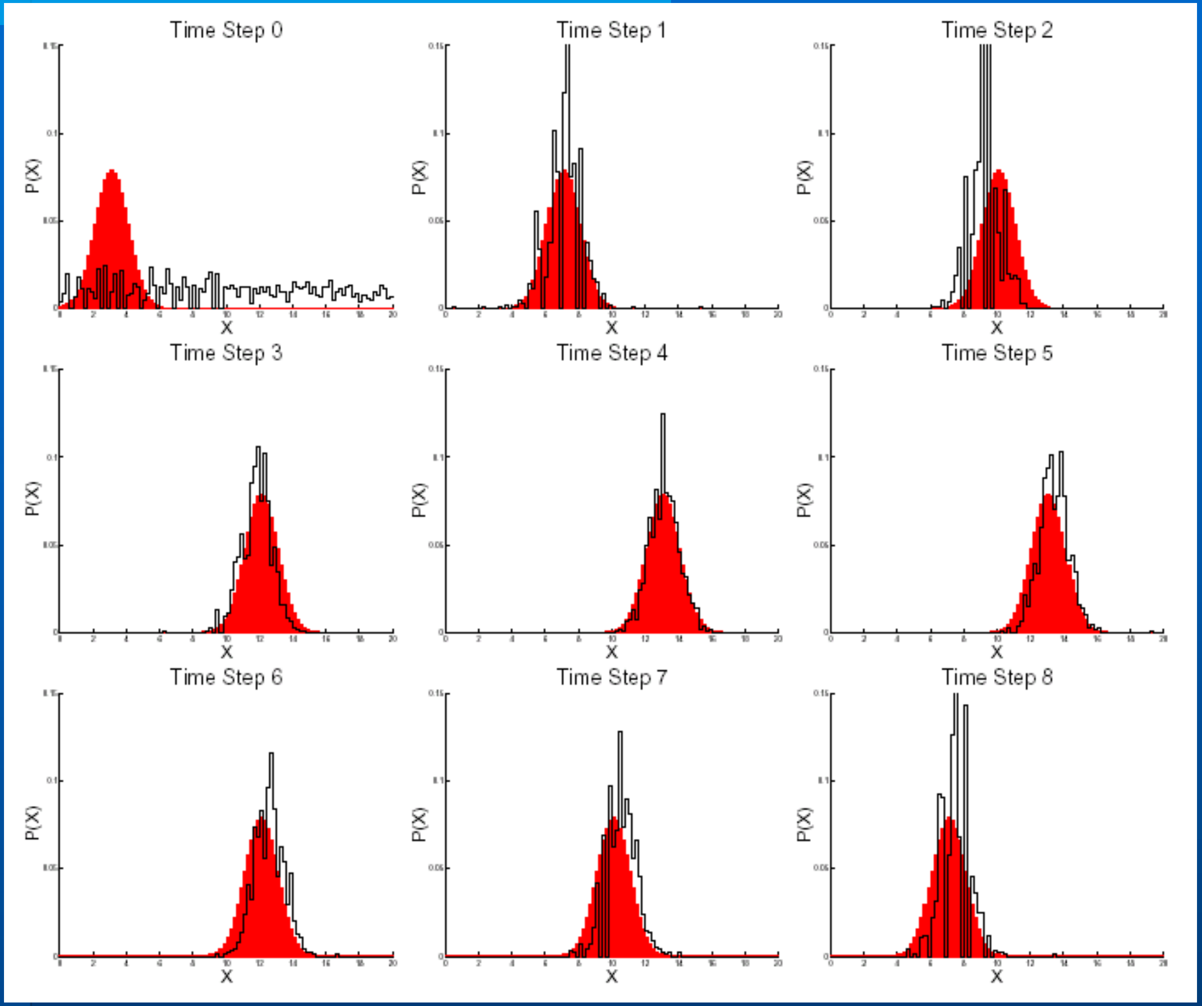


Particle Filter Demo 1



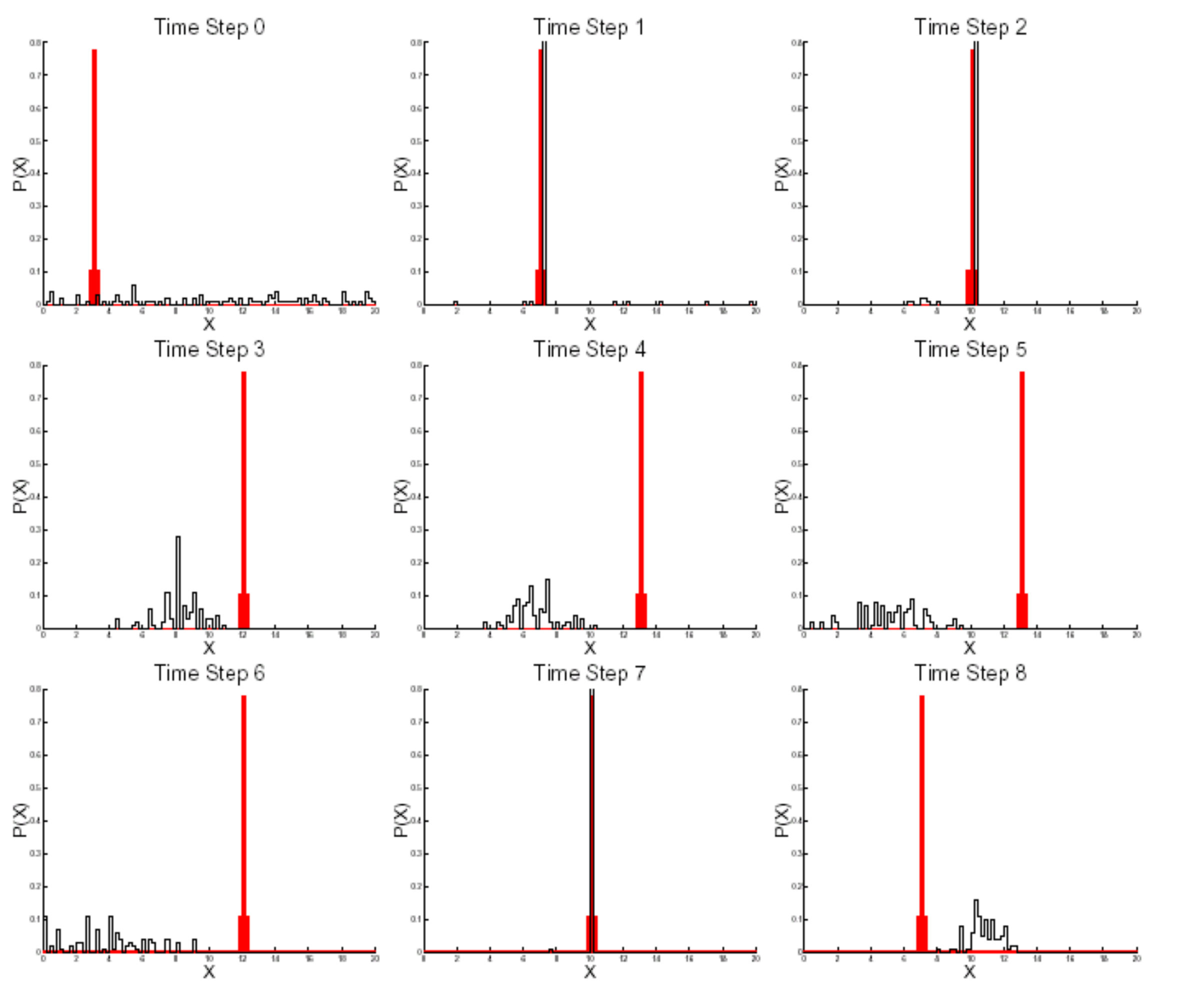
moving Gaussian + uniform, N=100 particles

Particle Filter Demo 2



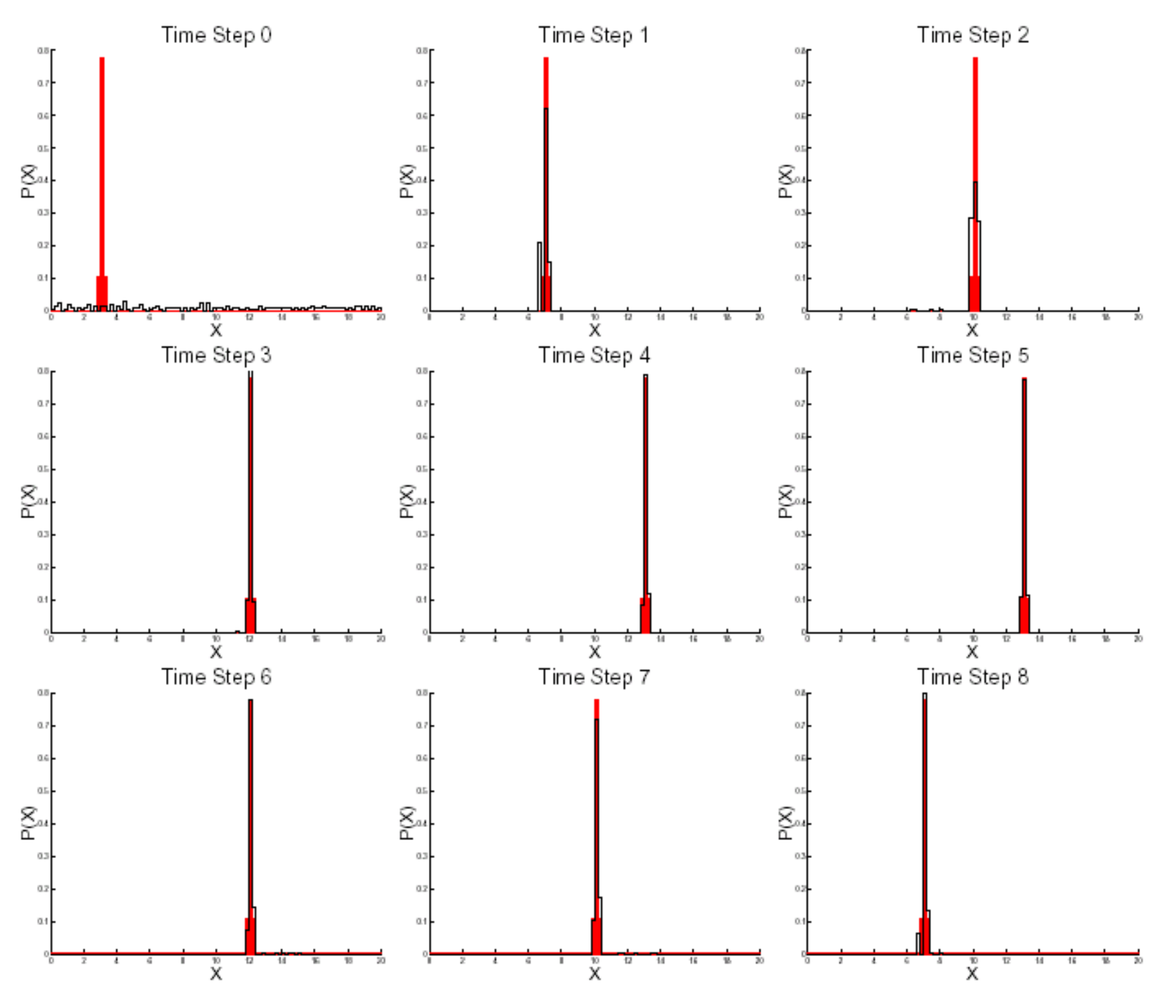
moving Gaussian + uniform, N=1000 particles

Particle Filter Demo 3



moving (sharp) Gaussian + uniform, N=100 particles

Particle Filter Demo 4

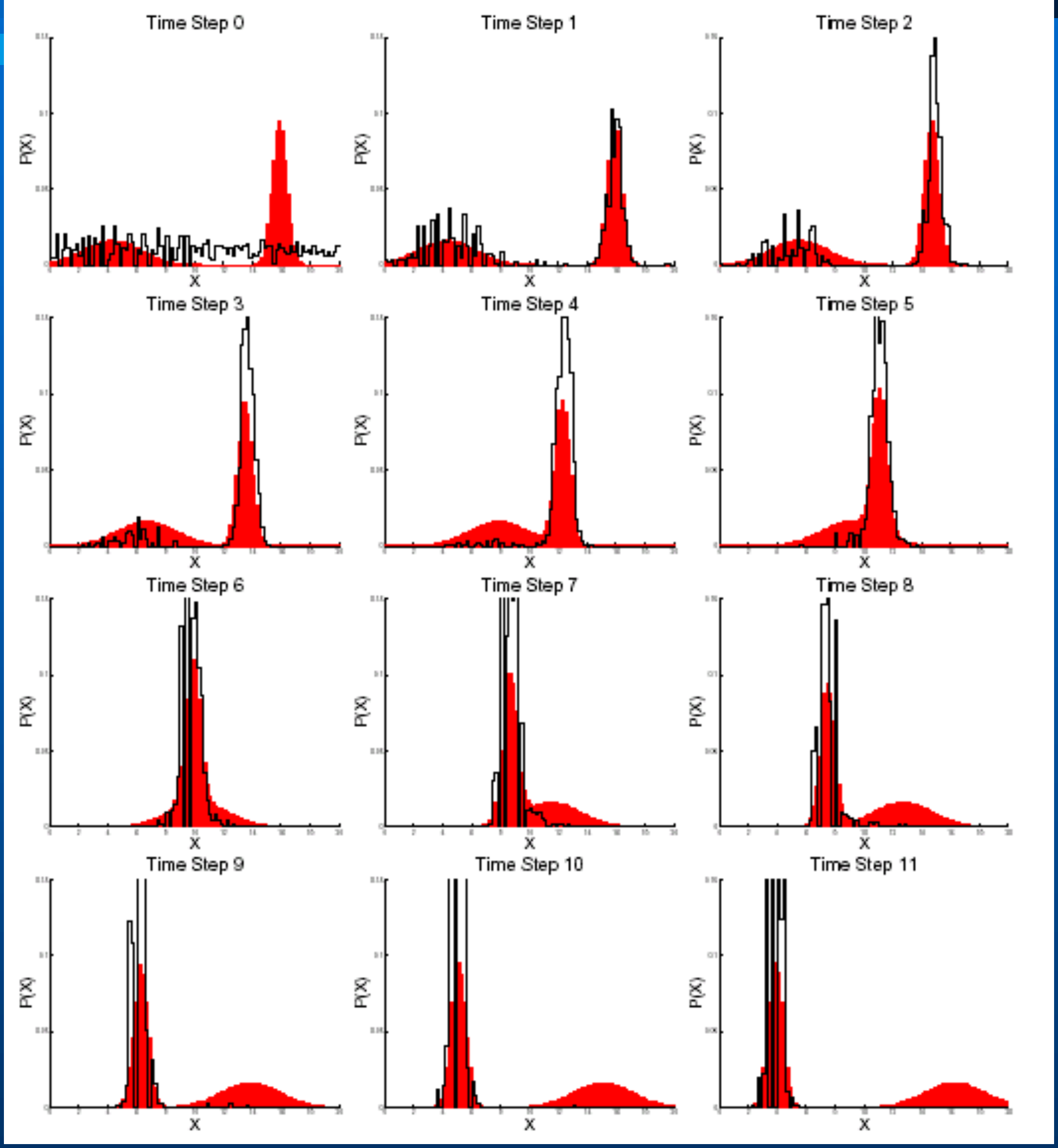


moving (sharp) Gaussian + uniform, N=1000 particles

Particle Filter Demo 5

mixture of two Gaussians,

filter loses track of smaller and less pronounced peaks



Obtaining state estimates from particles

- Any estimate of a function $f(x_t)$ can be calculated by discrete PDF-approximation

$$E\left[f(x_t)\right] = \frac{1}{N}\sum_{j=1}^{N} w_t^{(j)} f(x_t^{(j)})$$

- Mean: $E\left[x_t\right] = \frac{1}{N}\sum_{j=1}^{N} w_t^{(j)} x_t^{(j)}$
- MAP-estimate: particle with largest weight
- Robust mean: mean within window around MAP-estimate

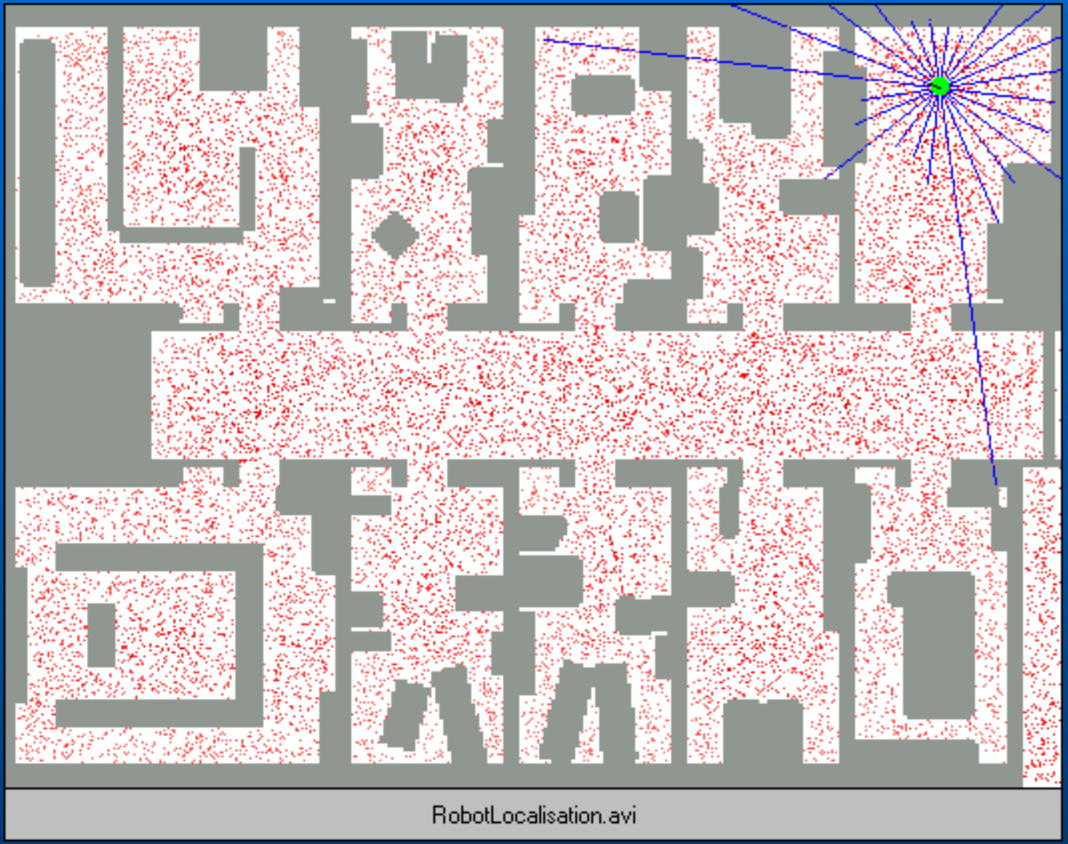
Pros and Cons of Particle Filters

- \+ Estimation of full PDFs
- \+ Non-Gaussian distributions
  - \+ e.g. multi-modal
- \+ Non-linear state and observation model
- \+ Parallelizable

- \- Degeneracy problem
- \- High number of particles needed
- \- Computationally expensive
- \- Linear-Gaussian assumption is often sufficient

Mobile Robot Localization

- Animation by Sebastian Thrun, Stanford
- http://robots.stanford.edu



Model Estimation

- Tracking with multiple motion-models
  - Discrete hidden variable indicates active model (manoever)
- Recovery of signal from noisy measurements
  - Even if signal may be absent (e.g. synaptic currents)
  - Mixture model of several hypotheses
- Neural Network model selection [de Freitas][1]
  - Estimate parameters and architecture of RBF network from input-output pairs
  - On-line classification (time-varying classes)

1: de Freitas, et.al.: Sequential Monte Carlo Methods for Neural Networks. in: Doucet, et.al.: Sequential Monte Carlo Methods in Practice, Springer Verlag, 2001

Other Applications

- Visual Tracking
  - e.g. human motion (body parts)
- Prediction of (financial) time series
  - e.g. mapping gold price → stock price
- Quality control in semiconductor industry
- Military applications
  - Target recognition from single or multiple images
  - Guidance of missiles