Ch. 15 Power Series, Taylor Series

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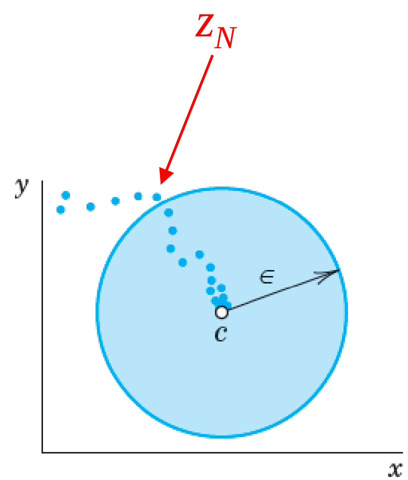
2017.12

※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

15.1 Sequences (수열), Series (급수), Convergence Tests (수렴판정)

☑ Sequences: Obtained by assigning to each positive integer n a number z_n

- Term: z_n $\quad z_1,\ z_2,\ \cdots \quad$ or $\quad \{z_1,\ z_2,\ \cdots\}$ $\quad$ or briefly $\quad \{z_n\}$
- Real sequence (실수열): Sequence whose terms are real



Convergent complex sequence

☑ Convergence

- Convergent sequence (수렴수열): Sequence that has a limit c

$$\lim_{n\to\infty} z_n = c \quad \text{or simply} \quad z_n \to c$$

- For every $\varepsilon > 0$, we can find N such that

$$|z_n - c| < \varepsilon \quad \text{for all } n > N$$

→ all terms z_n with $n > N$ lie in the open disk of radius ε and center c.

- Divergent sequence (발산수열): Sequence that does not converge.

15.1 Sequences, Series, Convergence Tests

☑ Convergence

- Convergent sequence: Sequence that has a limit c

$$\lim_{n\to\infty} z_n = c \quad \text{or simply} \quad z_n \to c$$

☑ Ex. 1 Convergent and Divergent Sequences

Sequence $\left\{\frac{i^n}{n}\right\} = \left\{i,\ -\frac{1}{2},\ -\frac{i}{3},\ \frac{1}{4},\ \cdots\right\}$ is convergent with limit 0.

Sequence $\{i^n\} = \{i,\ -1,\ -i,\ 1,\ \cdots\}$ is divergent.

Sequence $\{z_n\}$ with $z_n = (1 + i)^n$ is divergent.

15.1 Sequences, Series, Convergence Tests

☑ Theorem 1 Sequences of the Real and the Imaginary Parts

- A sequence $z_1, z_2, z_3, \ldots$ of complex numbers $z_n = x_n + iy_n$ converges to $c = a + ib$
- if and only if the sequence of the real parts $x_1, x_2, \ldots$ converges to $\boldsymbol{a}$
- and the sequence of the imaginary parts $y_1, y_2, \ldots$ converges to $\boldsymbol{b}$.

☑ Ex. 2 Sequences of the Real and the Imaginary Parts

Sequence $\{z_n\}$ with $z_n = x_n + iy_n = 1 - \frac{1}{n^2} + i\left(2 + \frac{4}{n}\right)$ converges to $c = 1+2i$.

$x_n = 1 - \frac{1}{n^2}$ has the limit $1 = \operatorname{Re} c$ and $y_n = 2 + \frac{4}{n}$ has the limit $2 = \operatorname{Im} c$.

15.1 Sequences, Series, Convergence Tests

☑ **Series (급수):** $\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$

- Nth partial sum: $s_n = z_1 + z_2 + \cdots + z_n$
- Term of the series: $z_1, \ z_2, \ \cdots$
- Convergent series (수렴급수): Series whose sequence of partial sums converges

$$\lim_{n\to\infty} s_n = s \quad \text{Then we write} \quad s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

- Sum or Value: s
- Divergent series (발산급수): Series that is not convergent
- Remainder: $R_n = z_{n+1} + z_{n+2} + z_{n+3} + \cdots$

☑ **Theorem 2 Real and the Imaginary Parts**

A series $\sum_{m=1}^{\infty} z_m$ with $z_m = x_m + iy_m$ converges and has the sum $s = u + iv$ if and only if $x_1 + x_2 + \ldots$ converges and has the sum u and $y_1 + y_2 + \ldots$ converges and has the sum v.

15.1 Sequences, Series, Convergence Tests

☑ Tests for Convergence and Divergence of Series

☑ Theorem 3 Divergence

If a series $z_1 + z_2 + \ldots$ converges, then $\lim_{m\to\infty} z_m = 0$.
Hence if this does not hold, the series diverges.

Proof) If a series $z_1 + z_2 + \ldots$ converges, with the sum s,

$$z_m = s_m - s_{m-1} \quad \Rightarrow \quad \lim_{m\to\infty} z_m = s_m - s_{m-1} = s - s = 0$$

- $z_m \to 0$ is necessary for convergence of series but not sufficient.

☑ **Ex)** The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which satisfies this condition but diverges.

- The practical difficulty in proving convergence is that, in most cases, the sum of a series is unknown.

- Cauchy overcame this by showing that a series converges if and only if its partial sums eventually get close to each other.

15.1 Sequences, Series, Convergence Tests

☑ **Ex)** The harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges

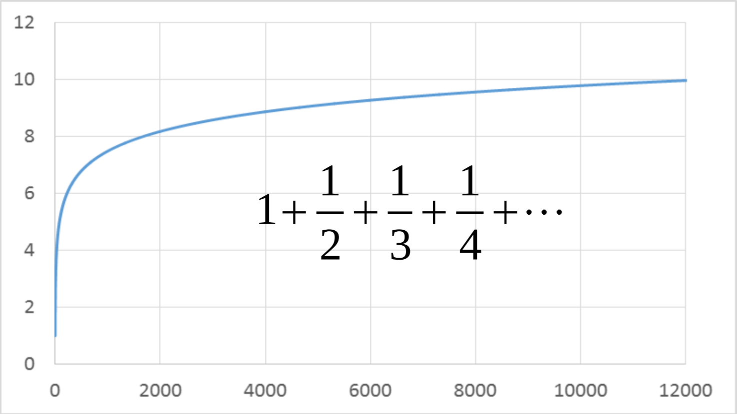
Proof)

$$S=1+\frac{1}{2}\quad+\frac{1}{3}+\frac{1}{4}\quad+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\quad+\frac{1}{9}\cdots+\frac{1}{16}+\cdots$$

$$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)\cdots$$

$$\boxed{=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots}$$

diverge!

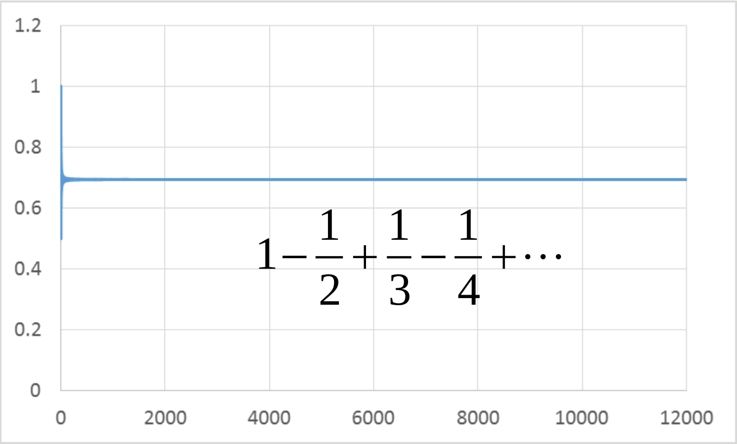


☑ **Ex)** The harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges

$$S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\cdots$$

$$<1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{4}-\frac{1}{6}+\frac{1}{6}-\frac{1}{8}+\frac{1}{8}+\cdots=1$$

converge!



15.1 Sequences, Series, Convergence Tests

☑ Theorem 4 Cauchy's Convergence Principle for Series

A series $z_1 + z_2 + \ldots$ is convergent if and only if for every given $\varepsilon > 0$ (no matter how small) we can find an N (*which depends on* ε *in general)* such that

$$\left|z_{n+1} + z_{n+2} + \cdots + z_{n+p}\right| < \varepsilon \qquad \text{for every } n > N \text{ and } p = 1,\ 2,\ \cdots$$

☑ Absolute Convergence (절대 수렴)

- Absolute convergent: Series of the absolute value of the terms

  $\sum_{m=1}^{\infty} |z_m| = |z_1| + |z_2| + \cdots$ is convergent.

- Conditionally convergent (조건 수렴): $z_1 + z_2 + \ldots$ converges but $|z_1| + |z_2| + \ldots$ diverges.

☑ **Ex. 3** The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges but $|z_1| + |z_2| + \cdots$ diverges, then the series $z_1 + z_2 + \ldots$ is called conditionally convergent.

15.1 Sequences, Series, Convergence Tests

☑ Theorem 5 Comparison Test (비교 판정법)

If a series $z_1 + z_2 + \ldots$ is given and we can find a convergent series $b_1 + b_2 + \ldots$ with nonnegative real terms such that $|z_1| < b_1$, $|z_2| < b_2, \ldots,$ then the given series converges, even absolutely.

Proof) by Cauchy's principle,

$$b_{n+1} + b_{n+2} + \cdots + b_{n+p} < \varepsilon \qquad \text{for every } n > N \text{ and } p = 1,\ 2,\ \cdots$$

- From this and $|z_1| < b_1$, $|z_2| < b_2, \ldots,$

$$|z_{n+1}| \cdots + |z_{n+p}| \leq b_{n+1} + \cdots + b_{n+p} < \varepsilon$$

- $|z_1| + |z_2| + \ldots$ converges, so that $z_1 + z_2 + \ldots$ is absolutely convergent.

15.1 Sequences, Series, Convergence Tests

☑ **Theorem 6 Geometric Series (기하 급수)**

The geometric series $\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \cdots$.

converges with the sum $\frac{1}{1-q}$ if $|q| < 1$ and diverges if $|q| \geq 1$.

Proof) $s_n = 1 + q + q^2 + \cdots + q^n$

$$qs_n = q + q^2 + \cdots + q^{n+1}$$

$$s_n - qs_n = (1-q)s_n = 1 - q^{n+1}$$

$$s_n = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q} \quad \text{since } |q|<1, \ n \to \infty \Rightarrow \frac{q^{n+1}}{1-q} \to 0$$

$$\therefore s_n \to \frac{1}{1-q}$$

15.1 Sequences, Series, Convergence Tests

☑ Theorem 7 Ratio Test (비 판정법)

If a series $z_1 + z_2 + \dots$ with $z_n \neq 0 \ (n = 1,\ 2,\ \cdots)$ has the property that for every n greater than some N,

$$\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1 \qquad (n > N)$$

(where $q < 1$ is fixed), this series converges absolutely.

If for every $n > N$ $\left|\frac{z_{n+1}}{z_n}\right| \geq 1 \qquad (n > N)$, the series diverges.

Proof) i) $\left|\frac{z_{n+1}}{z_n}\right| \geq 1 \ \Rightarrow |z_{n+1}| \geq |z_n| \Rightarrow z_1 + z_2 + \cdots$ diverges

ii) $|z_{n+1}| \leq |z_n| q \quad \text{for } n > N \Rightarrow |z_{N+2}| \leq |z_{N+1}| q \ \Rightarrow \ |z_{N+3}| \leq |z_{N+2}| q \leq |z_{N+1}| q^2$

$$|z_{N+p}| \leq |z_{N+1}| q^{p-1}$$

$$|z_{N+1}| + |z_{N+2}| + |z_{N+3}| \cdots \leq |z_{N+1}| (1 + q + q^2 + \cdots) \leq |z_{N+1}| \frac{1}{1-q}$$

Absolutely convergence follows from Theorem 5 Comparison Test

15.1 Sequences, Series, Convergence Tests

☑ Theorem 8 Ratio Test

If a series $z_1 + z_2 + \ldots$ with $z_n \neq 0 \ (n = 1,\ 2,\ \cdots)$ is such that $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$, then:

a. If L < 1, the series converges absolutely.

b. If L > 1, the series diverges.

c. If L = 1, the series may converge or diverge, so that the test fails and permits no conclusion.

Proof) (a) $k_n = |z_{n+1} / z_n|$, let $L = 1 - b < 1$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L \Rightarrow k_n \to 1 - b \Rightarrow \text{say } k_n \le q = 1 - \frac{1}{2}b < 1 \quad \textit{for } n > N$$

Theorem 7 Ratio Test $\Rightarrow$ $z_1 + z_2 + \ldots$ converges

Theorem 7 Ratio Test

$$\left|\frac{z_{n+1}}{z_n}\right| \le q < 1 \quad (n > N)$$

$\Rightarrow$ the series converges

(b) $k_n = |z_{n+1} / z_n|$, let $L = 1 + c > 1$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L \Rightarrow k_n \to 1 + c \Rightarrow \text{say } k_n \ge 1 + \tfrac{1}{2}c > 1 \quad \textit{for } n > N$$

Theorem 7 Ratio Test $\Rightarrow$ $z_1 + z_2 + \ldots$ diverges

$$\left|\frac{z_{n+1}}{z_n}\right| \ge 1 \quad (n > N)$$

$\Rightarrow$ the series diverge

☑ Theorem 8 Ratio Test

If a series $z_1 + z_2 + \ldots$ with $z_n \neq 0$ $(n = 1,\ 2,\ \cdots)$ is such that $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$, then:

a. If L < 1, the series converges absolutely.

b. If L > 1, the series diverges.

c. If L = 1, the series may converge or diverge, so that the test fails and permits no conclusion.

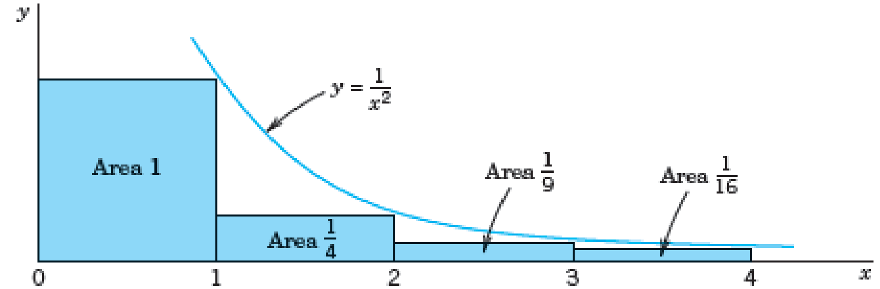
Proof) (c) harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{n}{n+1},\ as\ n\to\infty,\ \left|\frac{z_{n+1}}{z_n}\right| \to 1,\ L=1$$ diverge!

another series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$ $\left|\frac{z_{n+1}}{z_n}\right| = \frac{n^2}{(n+1)^2},\ as\ n\to\infty,\ \left|\frac{z_{n+1}}{z_n}\right| \to 1,\ L=1$

$$s_n = 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} \le 1+\int_1^n \frac{dx}{x^2} = 2-\frac{1}{n}$$

converge!



15.1 Sequences, Series, Convergence Tests

☑ Theorem 8 Ratio Test

If a series $z_1 + z_2 + \ldots$ with $z_n \neq 0 \ (n = 1, 2, \cdots)$ is such that $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$, then:

a. If L < 1, the series converges absolutely.

b. If L > 1, the series diverges.

c. If L = 1, the series may converge or diverge, so that the test fails and permits no conclusion.

☑ Ex. 4 Ratio Test

Is the following series convergent or divergent?

$$\sum_{n=0}^{\infty}\frac{(100+75i)^n}{n!} = 1 + (100+75i) + \frac{1}{2!}(100+75i)^2 + \cdots$$

Sol) The series is convergent, since

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{\frac{(100+75i)^{n+1}}{(n+1)!}}{\frac{(100+75i)^n}{n!}}\right| = \frac{|100+75i|}{n+1} = \frac{125}{n+1} \to 0 = L$$

15.1 Sequences, Series, Convergence Tests

☑ **Theorem 9 Root Test (근 판정법)**

If a series $z_1 + z_2 + \ldots$ is such that for every n greater than some N

$$\sqrt[n]{|z_n|} \le q < 1 \qquad (n > N)$$

(where **$q < 1$ is fixed**), this series converges absolutely.

If for infinitely many n $\sqrt[n]{|z_n|} \ge 1$, the series diverges.

Proof) (a) $\sqrt[n]{|z_n|} \le q < 1 \Rightarrow |z_n| \le q^n < 1$ for all $n > N$

$$|z_1| + |z_2| + |z_3| \cdots \le (1 + q + q^2 + \cdots) \le \frac{1}{1-q}$$

☑ **Theorem 5 Comparison Test (비교 판정법)**

If a series $z_1 + z_2 + \ldots$ is given and we can find a convergent series $b_1 + b_2 + \ldots$ with nonnegative real terms such that $|z_1| < b_1$, $|z_2| < b_2, \ldots$, then the given series converges, even absolutely.

Absolutely convergence follows from Theorem 5 Comparison Test

(b) $\sqrt[n]{|z_n|} \ge 1 \Rightarrow |z_n| \ge 1 \Rightarrow z_1 + z_2 + \ldots$ diverges.

This diverges from Theorem 3 Divergence.

☑ **Theorem 3 Divergence**

If a series $z_1 + z_2 + \ldots$ converges, then $\lim_{m \to \infty} z_m = 0$.
Hence if this does not hold, the series diverges.

15.1 Sequences, Series, Convergence Tests

☑ **Caution!**

harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

$\sqrt[n]{|z_n|}=\sqrt[n]{1/n}<1$ but diverge! because $q < 1$ is not fixed.

☑ **Theorem 10 Root Test**

If a series $z_1 + z_2 + \dots$ is such that $\lim_{n\to\infty}\sqrt[n]{|z_n|}=L$, then:

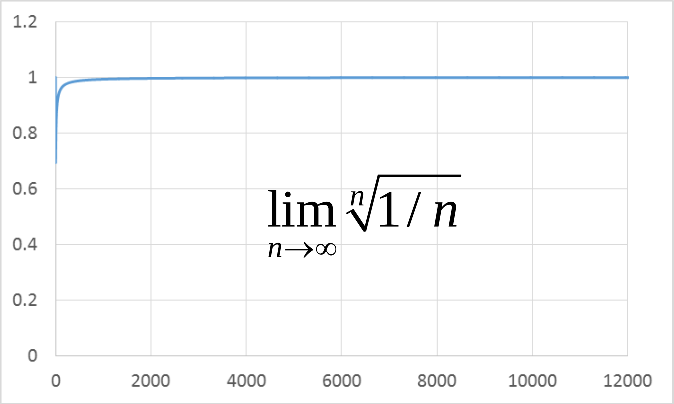
a. The series converges absolutely if L < 1.

b. The series diverges if L > 1.

c. If L = 1, the test fails; that is, no conclusion is possible.

Ex) harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

$\lim_{n\to\infty}\sqrt[n]{|z_n|}=\lim_{n\to\infty}\sqrt[n]{1/n}=1$ the test fails.



15.2 Power Series

☑ Power Series (거듭제곱급수)

- Power series in powers of $z - z_0$: $\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$

  Coefficients: Complex (or real) constants a_0, a_1, ...

  Center: Complex (or real) constant z_0

- A power series in powers of z: $\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$

☑ Convergence Behavior of Power Series

☑ Ex. 1 Convergence in a Disk. Geometric Series

The geometric series $\sum_{n=0}^{\infty} z_n = 1 + z + z^2 + \cdots$

converges absolutely if $|z| < 1$

diverges if $|z| \geq 1$

☑ Theorem 6 Geometric Series (기하 급수)

The geometric series $\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \cdots$.

converges with the sum $\frac{1}{1-q}$ if $|q| < 1$ and diverges if $|q| \geq 1$.

15.2 Power Series

☑ Convergence of Power Series

- Power series play an important role in complex analysis.
- The sums are analytic functions (Theorem 5, Sec. 15.3) ⇒ The sum should be convergent.
- Every analytic function $f(z)$ can be represented by power series at z_0 (Theorem 1, Sec. 15.4) ⇒ The sum should converge to $f(z_0)$.

- Ex) Maclaurin series of e^z

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

- For specific $z = z_0$, the left side and the right side have the same value.

  ⇒ The right sum should converge for the specific value.

  ⇒ This doesn't mean convergence of e^z as $z \to \infty$.

15.2 Power Series

☑ Ex. 2 Convergence for Every z

The power series (which is the Maclaurin series of e^z) $\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

is absolutely convergent for every z

By the ratio test, for any fixed z,

$$\left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right| = \frac{|z|}{n+1} \to 0 \qquad \text{as} \qquad n \to \infty$$

☑ Ex. 3 Convergence Only at the Center (Useless Series)

The following series converges only at $z = 0$, but diverges for every $z \neq 0$

$$\sum_{n=0}^{\infty} n! z^n = 1 + z + 2z^2 + 6z^3 + \cdots$$

$$\left| \frac{(n+1)! z^{n+1}}{n! z^n} \right| = (n+1)|z| \to \infty \qquad \text{as} \qquad n \to \infty \qquad (z \text{ fixed and} \neq 0)$$

15.2 Power Series

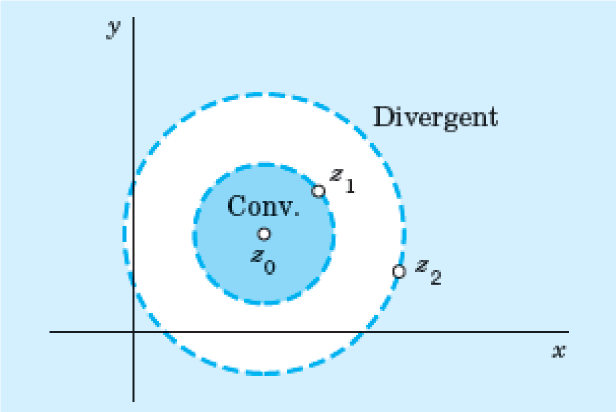
☑ Theorem 1 Convergence of a Power Series

a. Every power series converges at the center z_0.

b. If a power series converges at a point $z = z_1 \neq z_0$, it converges absolutely for every z closer to z_0 than z_1, that is, $|z - z_0| < |z_1 - z_0|$.

c. If a power series diverges at a $z = z_2$, it diverges for every z farther away from z_0 than z_2.

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$



15.2 Power Series

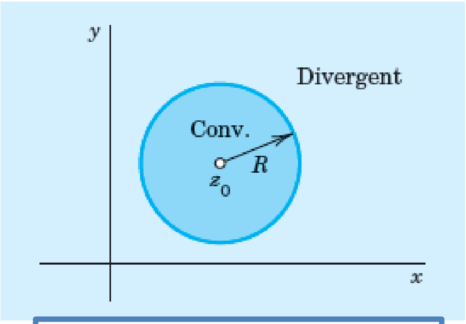
☑ Radius of Convergence (수렴 반지름) of a Power

- Circle of convergence (수렴원): The smallest circle with center z_0 that includes all the points at which a given power series converges.
- Radius of convergence (수렴반경): Radius of the circle of convergence.

  $|z - z_0| = R$ is the circle of convergence and its radius R the radius of convergence.

  ⟺ Convergence everywhere within that circle, that is, for all z for which $|z - z_0| = R$

  Diverges for all z for which $|z - z_0| > R$.



Circle of convergence

- Notations $R = \infty$ and $R = 0$.

  $R = \infty$: the series converges for all z.

  $R = 0$: the series converges only at the center.

- Real power series: In which powers, coefficients, and center are real.
- Convergence interval (수렴구간): Interval $|x\text{-}x_0| < R$ of length $2R$ on the real line.

15.2 Power Series

☑ **Theorem 2 Radius of Convergence R**

Suppose that the sequence $\left|a_{n+1}/a_n\right|, \quad n=1,\ 2,\ \cdots$, converges with limit L^*. If $L^* = 0$, then $R = \infty$; that is, the power series converges for all z. If $L^* \neq 0$ (hence $L^* > 0$), then

$$R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad (\textbf{Cauchy-Hadamard formula})$$

If $\left|a_{n+1}/a_n\right| \to \infty$, then $R = 0$ (convergence only at the center z_0)

Proof) The ratio of the terms in the ratio test is

$$\left|\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n}\right| = \left|\frac{a_{n+1}}{a_n}\right|\left|z-z_0\right| \qquad L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\left|z-z_0\right| = L^*\left|z-z_0\right| \qquad L = L^*\left|z-z_0\right|$$

i) Let $L^* \neq 0$, thus $L^* > 0$

The series converges if $L^* = L^*|z - z_0| < 1$, $|z - z_0| < 1/L^*$ and diverge if $|z - z_0| > 1/L^*$.

$\Rightarrow$ $1/L^*$ is radius of convergence.

ii) If $L^* = 0$, then $L = 0$ for every z. $\Rightarrow$ convergence for all z by the ratio test.

iii) If $\left|a_{n+1}/a_n\right| \to \infty$, then $\left|\frac{a_{n+1}}{a_n}\right|\left|z-z_0\right| > 1$ $\Rightarrow$ diverge for any $z \neq z_0$ and all sufficiently large n.

15.2 Power Series

☑ **Theorem 2 Radius of Convergence R**

Suppose that the sequence $\left|a_{n+1}/a_n\right|$, $n=1, 2, \cdots$, converges with limit L^*. If $L^*=0$, then $R=\infty$; that is, the power series converges for all z. If $L^*\neq 0$ (hence $L^*>0$), then

$$R=\frac{1}{L^*}=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad (\textbf{Cauchy - Hadamard formula})$$

If $\left|a_{n+1}/a_n\right|\to\infty$, then $R=0$ (convergence only at the center z_0)

☑ **Ex. 5 Radius of Convergence**

Radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}(z-3i)^n$

Sol) $$R=\lim_{n\to\infty}\left[\frac{\frac{(2n)!}{(n!)^2}}{\frac{(2n+2)!}{((n+1)!)^2}}\right]=\lim_{n\to\infty}\left[\frac{(2n)!}{(2n+2)!}\frac{((n+1)!)^2}{(n!)^2}\right]=\lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)}=\frac{1}{4}$$

The series converges in the open disk $|z-3i|<\frac{1}{4}$ of radius $\frac{1}{4}$ and center $3i$.

15.2 Power Series

☑ Radius of Convergence R

$$R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad R = \frac{1}{\tilde{L}}, \ \tilde{L} = \lim_{n\to\infty}\sqrt[n]{|a_n|}$$

☑ Ex. 6 Extension of Theorem 2

Find the radius of convergence of the power series.

$$\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n = 3+\frac{1}{2}z+\left(2+\frac{1}{4}\right)z^2+\frac{1}{8}z^3+\left(2+\frac{1}{16}\right)z^5+\cdots$$

Sol) The sequence of the ratios $|a_{n+1}/a_n| = \frac{1}{6}, 2(2+\frac{1}{4}), 1/8(2+\frac{1}{4})$ does not converge.
Thus, we can't use Theorem 2 for this example.

$R = \frac{1}{\tilde{L}}, \ \tilde{L} = \lim_{n\to\infty}\sqrt[n]{|a_n|}$ For odd n, $\sqrt[n]{|a_n|} = \sqrt[n]{1/2^n} = \frac{1}{2}$

For even n, $\sqrt[n]{|a_n|} = \sqrt[n]{2+1/2^n} \to 1$ as $n \to \infty$

$R = \frac{1}{\tilde{l}}$, $\tilde{l}$: the greatest limit point of the sequence

$R = 1$, that is, the series converge for $|z| < 1$.

☑ Radius of Convergence R

$$R = \frac{1}{L*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \qquad R = \frac{1}{\tilde{L}},\ \tilde{L} = \lim_{n\to\infty}\sqrt[n]{|a_n|}$$

☑ Ex) Find the center and the radius of convergence

$$\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2}(z-2i)^n$$

Sol)

$$R = \lim_{n\to\infty}\left[\frac{\frac{(2n)!}{4^n(n!)^2}}{\frac{(2n+2)!}{4^{n+1}((n+1)!)^2}}\right] = \lim_{n\to\infty}\left[\frac{(2n)!}{(2n+2)!}\frac{4^{n+1}((n+1)!)^2}{4^n(n!)^2}\right] = \lim_{n\to\infty}\frac{4(n+1)^2}{(2n+2)(2n+1)} = 1$$

15.3 Functions Given by Power Series

☑ Terminology and Notation

Given power series $\sum_{n=0}^{\infty} a_n z^n$ has a nonzero radius of convergence R (thus $R > 0$)

→ Its sum is a function of z, say $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad \left(|z| < R\right)$$

→ $f(z)$ is represented by the power series.

☑ Uniqueness of a Power Series Representation

A function $f(z)$ cannot be represented by two different power series with the same center.

15.3 Functions Given by Power Series

Theorem 1 Continuity of the Sum of a Power Series

If a function $f(z)$ can be represented by a power series with radius of convergence $R > 0$, then $f(z)$ is continuous at $z = 0$.

Proof) $f(z)=\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$ converges for $|z| \le r < R \quad \Rightarrow \quad f(0) = a_0$

We must show that $\lim_{z\to 0} f(z) = a_0$

⇨ For a given $\varepsilon > 0$

Q: If not continuous?

There is a $\delta > 0$ such that $|z| < \delta$ implies $|f(z) - a_0| < \varepsilon$

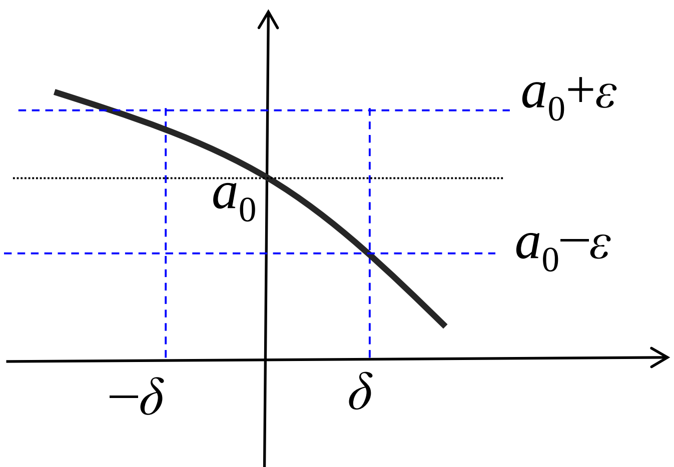
$$\sum_{n=0}^{\infty} |a_n| r^{n-1} = \frac{1}{r}\sum_{n=0}^{\infty} |a_n| r^n \equiv S$$

when $\delta < \varepsilon / S$

for $0<|z| \le r$ We can always find a $\delta > 0$ such that $|z| < \delta < \varepsilon / S$ which implies $|f(z) - a_0| < \varepsilon$.

$$|f(z) - a_0| = \left|\sum_{n=1}^{\infty} a_n z^n\right| \le |z| \sum_{n=1}^{\infty} |a_n| |z|^{n-1} \le |z| \sum_{n=1}^{\infty} |a_n| r^{n-1} = |z| S < \delta S < (\varepsilon / S) S = \varepsilon$$

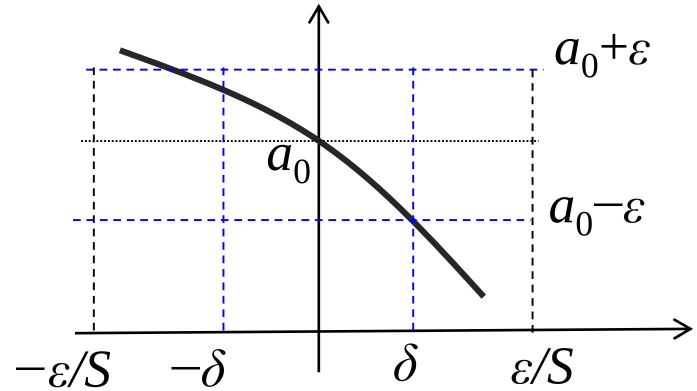
15.3 Functions Given by Power Series



Case I

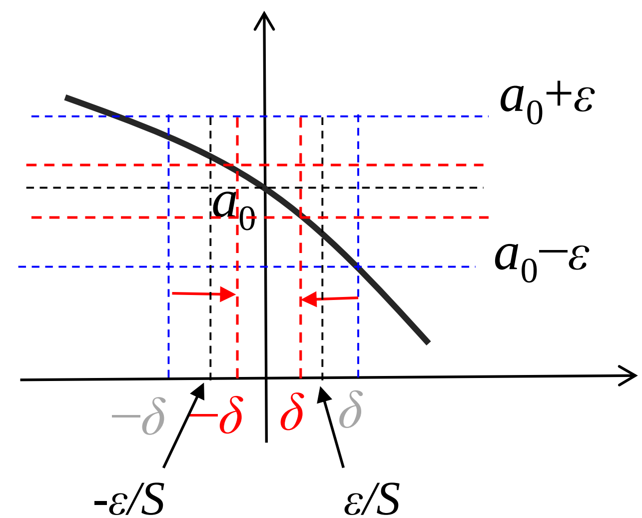
$\delta < \varepsilon / S$

We can find $|z| < \delta < \varepsilon / S$ which satisfies $|f(z) - a_0| < \varepsilon$



Case II

$\delta > \varepsilon / S$



For a given $\varepsilon > 0$

If $|f(z) - a_0| < \varepsilon$ for $|z| < \delta$

then for any δ' $|z| < \delta' < \delta$,
$|f(z) - a_0| < \varepsilon$ is always satisfied.

but for any δ' $|z| < \delta < \delta'$,
$|f(z) - a_0| < \varepsilon$ is not always satisfied.

We can find a new $\delta < \varepsilon / S$ which satisfies $|f(z) - a_0| \square \ \varepsilon$.

15.3 Functions Given by Power Series

☑ **Theorem 2 Identity Theorem for Power Series. Uniqueness**

Let the power series $a_0 + a_1 z + a_2 z^2 + \ldots$ and $b_0 + b_1 z + b_2 z^2 + \ldots$ both be convergence for $|z| < R$, where R is positive, and let them both have the same sum for all these z.

→ Then the series are identical, that is, $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, ...

Hence if a function $f(z)$ can be represented by a power series with any center z_0, this representation is unique.

Proof) We proceed by induction (귀납법). By assumption,

$$a_0 + a_1 z + a_2 z^2 + \ldots = b_0 + b_1 z + b_2 z^2 + \ldots$$

The sums of these two power series are continuous at $z = 0 \rightarrow a_0 = b_0$.

Now assume that $a_n = b_n$. For $n = 0, 1, \ldots, m$. Divide the result by z^{m+1}

$$a_{m+1} + a_{m+2} z + a_{m+3} z^2 + \ldots = b_{m+1} + b_{m+2} z + b_{m+3} z^2 + \ldots$$

Letting $z \rightarrow 0$, we concluded form this that $a_{m+1} = b_{m+1}$.

15.3 Functions Given by Power Series

☑ Operations on Power Series

- Termwise addition or subtraction

Termwise addition or subtraction of two power series with radii of convergence R_1 and R_2.
→ A power series with radius of convergence at least equal to the smaller of R_1 and R_2.

- Termwise multiplication

Termwise multiplication of two power series

$$f(z)=\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + \cdots \quad \text{and} \quad g(z)=\sum_{n=0}^{\infty} b_n z^n = b_0 + b_1 z + \cdots$$

Cauchy product of the two series:

$$\sum_{n=0}^{\infty}\left(a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0\right) z^n = a_0 b_0 + \left(a_0 b_1 + a_1 b_0\right) z + \left(a_0 b_2 + a_1 b_1 + a_2 b_0\right) z^2 + \cdots$$

- Termwise differentiation and integration

by termwise differentiation, that is,

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$

15.3 Functions Given by Power Series

☑ Theorem 3 Termwise Differentiation of a Power Series

The derived series (미분급수) of a power series has the same radius of convergence as the original series.

Proof)

$$\lim_{n\to\infty}\frac{n|a_n|}{(n+1)|a_{n+1}|}=\lim_{n\to\infty}\frac{n}{(n+1)}\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$R=\frac{1}{L^*}=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$\left(\sum_{n=0}^{\infty}a_nz^n\right)'=\sum_{n=1}^{\infty}na_nz^{n-1}$$

☑ Ex. 1 Application of Theorem 3

Radius of convergence of the power series

$$\sum_{n=2}^{\infty}\binom{n}{2}z^n=\sum_{n=2}^{\infty}\frac{n(n-1)}{2!}z^n=z^2+3z^3+6z^4+10z^5+\cdots.$$

$$\binom{n}{k}={}_nC_k=\frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$=\frac{n!}{(n-k)!k!}$$

Sol)

$$f(z)=\sum_{n=0}^{\infty}z^n=1+z^2+z^3+\cdots \quad\Rightarrow\quad f''(z)=n(n-1)\sum_{n=2}^{\infty}z^{n-2}$$

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=1 \qquad R=\lim_{n\to\infty}\left|\frac{(n-1)n}{n(n+1)}\right|=1$$

$$\frac{z^2}{2}f''(z)=\frac{n(n-1)}{2}\sum_{n=2}^{\infty}z^n=\sum_{n=2}^{\infty}\binom{n}{2}z^n \quad\Rightarrow\quad \therefore R=\lim_{n\to\infty}\left|\frac{(n-1)n}{n(n+1)}\right|=1$$

15.3 Functions Given by Power Series

☑ **Theorem 4 Termwise Integration of Power Series**

The power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots$

obtained by integrating the series $a_0 + a_1 z + a_2 z^2 + \ldots$ term by term has the

same radius of convergence as the original series.

☑ Power Series Represent Analytic Functions

"거듭제곱급수는 해석함수다"

☑ Theorem 5 Analytic Functions. Their Derivatives

- A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence.
- The derivatives of this function are obtained by differentiating the original series term by term.
- All the series thus obtained have the same radius of convergence as the original series.
- Hence, by the first statement, each of them (도함수) represents an analytic function.

15.3 Functions Given by Power Series

☑ **Ex)** Find the radius of convergence in two ways: (a) directly by the Cauchy-Hadamard formula in Sec. 15.2, and (b) from a series of simpler terms by using Theorem 3 or Theorem 4.

$$R = \frac{1}{L^*} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \quad (\textbf{Cauchy - Hadamard formula})$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

$$\sum_{n=0}^{\infty}\binom{n+k}{k}^{-1} z^{n+k}$$

Sol) (a)

$$\sum_{n=0}^{\infty}\binom{n+k}{k}^{-1} z^{n+k} = \sum_{n=0}^{\infty}\left\{\underbrace{\boxed{\binom{n+k}{k}^{-1} z^k}}_{a_n}\right\} z^n$$

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{z^k \Big/ \binom{n+k}{k}}{z^k \Big/ \binom{n+1+k}{k}}\right| = \left|\frac{\binom{n+1+k}{k}}{\binom{n+k}{k}}\right| \frac{(n+1+k)!/k!(n+1)!}{(n+k)!/k!n!} = \frac{n!(n+1+k)!}{(n+k)!(n+1)!} = \frac{n+k+1}{n+1} \to 1$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1$$

15.3 Functions Given by Power Series

☑ **Ex)** Find the radius of convergence in two ways: (a) directly by the Cauchy-Hadamard formula in Sec. 15.2, and (b) from a series of simpler terms by using Theorem 3 or Theorem 4.

$$\sum_{n=0}^{\infty} \binom{n+k}{k}^{-1} z^{n+k}$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

Sol) (b) $\displaystyle \sum_{n=0}^{\infty} \binom{n+k}{k}^{-1} z^{n+k} : \frac{(n)!k!}{(n+k)!} z^{k+n} = k!\frac{1}{(n+1)(n+2)\cdots(n+k)} z^{k+n}$

Integrate k times term by term

$k!z^k$

$\therefore$ Radius of convergence = 1

15.4 Taylor and Maclaurin Series

☑ Taylor series

Taylor series of a complex function $f(z)$:

$$f(z)=\sum_{n=1}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n=\frac{1}{n!}f^{(n)}(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$$

Integrate counterclockwise around a simple closed path C that contains z_0 in its interior.

$f(z)$ is analytic in a domain containing C and every point inside C.

☑ Maclaurin series: Taylor series with center z_0 = 0

☑ Taylor's formula

$$f(z)=f(z_0)+\frac{z-z_0}{1!}f'(z_0)+\frac{(z-z_0)^2}{2!}f''(z_0)+\cdots+\frac{(z-z_0)^n}{n!}f^{(n)}(z_0)+R_n(z)$$

Remainder:

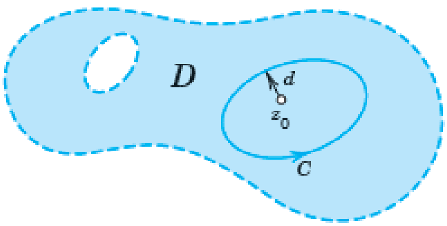
$$R_n(z)=\frac{(z-z_0)^{n+1}}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)}dz^*$$

☑ Theorem 1 Derivatives of an Analytic Function

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$



and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad (n=1,\ 2,\ \cdots)$$

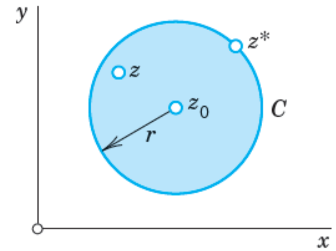
here C is any simply closed path in D that enclose z_0 and whose full interior belongs to D; and we integrate counterclockwise around C.

15.4 Taylor and Maclaurin Series

☑ Theorem 1 Taylor's Theorem

- Let $f(z)$ be analytic in a domain D, and let $z = z_0$ be any point in D.
- Then there exists precisely one Taylor series with center z_0 that represents $f(z)$.
- This representation is valid in the largest open disk with center z_0 in which $f(z)$ is analytic. The remainders $R_n(z)$ of the power series can be represented in the form

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)} dz^*$$



- The coefficients satisfy the inequality $|a_n| \le \frac{M}{r^n}$
- where M is the maximum of $|f(z)|$ on a circle $|z - z_0| = r$ in D whose interior is also in D.

Cauchy's Integral Formula

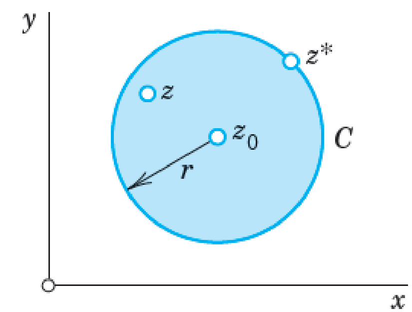
$$f(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$

☑ **Proof)** $f(z)=\dfrac{1}{2\pi i}\oint_C \dfrac{f(z^*)}{(z^*-z)}dz^*$

$$\frac{1}{(z^*-z)}=\frac{1}{z^*-z_0-(z-z_0)}=\frac{1}{(z^*-z_0)\left(1-\boxed{\dfrac{z-z_0}{z^*-z_0}}\right)}$$

$= q$

$$\left|\frac{z-z_0}{z^*-z_0}\right|<1$$



$$1+q+\ldots+q^n=\frac{1-q^{n+1}}{1-q}=\frac{1}{1-q}-\frac{q^{n+1}}{1-q}$$

$$\frac{1}{1-q}=1+q+\ldots+q^n+\frac{q^{n+1}-1}{1-q}$$

$$f(z)=\sum_{n=1}^{\infty}a_n(z-z_0)^n$$

$$a_n=\frac{1}{n!}f^{(n)}(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$$

$$q=\frac{z-z_0}{z^*-z}$$

$$\frac{1}{z^*-z}=\frac{1}{z^*-z_0}\left[1+\frac{z-z_0}{z^*-z_0}+\left(\frac{z-z_0}{z^*-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{z^*-z_0}\right)^n\right]+\frac{1}{z^*-z_0}\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}$$

$$\therefore f(z)=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z)}dz^*=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)}dz^*+\frac{z-z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^2}dz^*+\cdots+\frac{(z-z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^n}dz^*+R_n(z)$$

☑ Proof-continued

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)}dz^* + \frac{z-z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^2}dz^* + \cdots + \frac{(z-z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^n}dz^* + R_n(z)$$

$f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n$ will converge and represent $f(z)$ if and only if

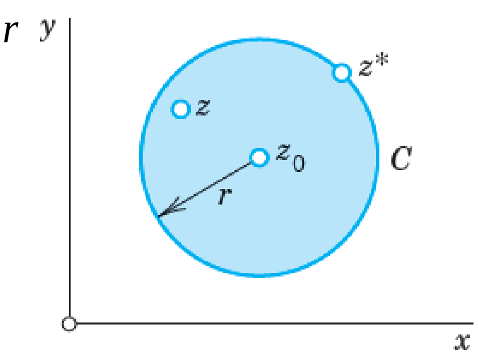
$$\lim_{n\to\infty} R_n(z) = 0$$

$f(z)$ is analytic inside and on C → $f(z^*)/(z^*- z)$ is analytic inside and on C.

$$\left|\frac{f(z^*)}{(z^*-z_0)}\right| \le \tilde{M}$$

$\left|\int_C f(z)\,dz\right| \le ML$ $\quad$ (**ML-inequality**)

$$|R_n(z)| = \frac{|z-z_0|^{n+1}}{2\pi}\left|\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*\right| \le \frac{|z-z_0|^{n+1}}{2\pi}\tilde{M}\frac{1}{r^{n+1}}2\pi r = \tilde{M}\left|\frac{z-z_0}{r}\right|^{n+1} r$$



$$|z-z_0| < r \;\Rightarrow |z-z_0|/r < 1$$

$$\therefore \lim_{n\to\infty} R_n(z) = 0$$

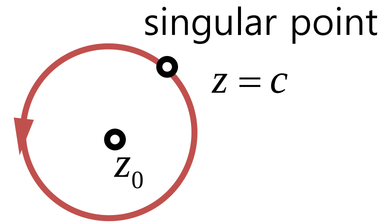
15.4 Taylor and Maclaurin Series

☑ Accuracy of Approximation.

We can achieve any preassigned accuracy in approximating $f(z)$ by a partial sum by choosing n large enough.

☑ Singularity, Radius of Convergence.

- Singular point: Point at which the function is not analytic
- On the circle of convergence there is at least one singular point $(z = c)$
- The radius of convergence R is usually equal to the distance from the center (z_0) to the nearest singular point.



☑ Theorem 2 Relation to the Last Section

A power series with a nonzero radius of convergence is the Taylor series of its sum.

Power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Taylor series

$$f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

15.4 Taylor and Maclaurin Series

☑ Maclaurin series

A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{n!} f^{(n)}(0) \ \text{ or } \ a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^{*\,n+1}}\,dz^*$$

☑ Important Special Taylor (Maclaurin) Series

☑ Ex. 1 Geometric Series

$$f(z) = (1-z)^{-1}$$

$$f'(z) = -1(1-z)^{-2}(-1) = (1-z)^{-2}$$

$$f''(z) = -2(1-z)^{-3}(-1) = 2!(1-z)^{-3}$$

$$f^{(n)}(z) = n!(1-z)^{-(n+1)} = \frac{n!}{(1-z)^{n+1}}$$

Let $f(z) = \dfrac{1}{1-z}$ then we have $f^{(n)}(z) = \dfrac{n!}{(1-z)^{n+1}}$, $f^{(n)}(0) = n!$.

Hence the Maclaurin expansion of $1/(1\text{-}z)$ is the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots. \qquad \left(\because a_n = \frac{1}{n!} f^{(n)}(0) = \frac{1}{n!} n! = 1\right)$$

$f(z)$ is singular at $z = 1$; this point lies on the circle of convergence.

15.4 Taylor and Maclaurin Series

☑ Maclaurin series

A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{n!} f^{(n)}(0) \text{ or } a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^{*n+1}} dz^*$$

☑ Important Special Taylor (Maclaurin) Series

☑ Ex. 2 Exponential Function

$$f(z) = e^z$$

We know that the exponential function e^z is analytic for all z, and $(e^z)' = e^z$.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \qquad \left(\because a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{n!}e^0 = \frac{1}{n!}\right)$$

15.4 Taylor and Maclaurin Series

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

☑ Maclaurin series

A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{n!} f^{(n)}(0) \ \text{ or } \ a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^{*\,n+1}}\,dz^*$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

Furthermore, by setting $z = iy$ and separating the series into the real and imaginary parts,

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = 1 + iy + -\frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} - \cdots.$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!}} + i\boxed{\sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}}$$

Maclaurin series of $\cos y$ Maclaurin series of $\sin y$

$\because e^{iy} = \cos y + i \sin y$

Euler's formula

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

15.4 Taylor and Maclaurin Series

☑ Maclaurin series

A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \quad a_n=\frac{1}{n!}f^{(n)}(0) \text{ or } a_n=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^{*n+1}}dz^*$$

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots.$$

☑ Important Special Taylor (Maclaurin) Series

☑ Ex. 3 Trigonometric and Hyperbolic Functions

Find the Maclaurin series of $\cos z$ and $\sin z$.

$$e^{iz}=\sum_{n=0}^{\infty}\frac{(iz)^n}{n!}=1+iz-\frac{z^2}{2!}-i\frac{z^3}{3!}+\frac{z^4}{4!}+i\frac{z^5}{5!}-+\cdots$$

$$e^{-iz}=\sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}=1-iz-\frac{z^2}{2!}+i\frac{z^3}{3!}-\frac{z^4}{4!}-i\frac{z^5}{5!}+-\cdots$$

$$\cos z=\frac{1}{2}(e^{iz}+e^{-iz})=1-\frac{z^2}{2!}+\frac{z^4}{4!}-+\cdots=\sum_{k=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$$

$$\sin z=\frac{1}{2i}(e^{iz}-e^{-iz})=z-\frac{z^3}{3!}+\frac{z^5}{5!}-+\cdots=\sum_{k=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

15.4 Taylor and Maclaurin Series

☑ Maclaurin series

A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,\quad a_n=\frac{1}{n!}f^{(n)}(0)\ \text{or}\ a_n=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^{*n+1}}dz^*$$

Find the Maclaurin series of cosh z and sinh z.

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots.$$

$$e^{-z}=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!}=1+(-z)+\frac{(-z)^2}{2!}+\frac{(-z)^3}{3!}+\cdots=1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\frac{z^4}{4!}-\frac{z^5}{5!}+-\cdots$$

$$\cosh z=\frac{1}{2}(e^z+e^{-z})=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots=\sum_{k=0}^{\infty}\frac{z^{2n}}{(2n)!}$$

$$\sinh z=\frac{1}{2}(e^z-e^{-z})=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots=\sum_{k=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

15.4 Taylor and Maclaurin Series

☑ Maclaurin series

A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{n!} f^{(n)}(0) \text{ or } a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^{*\,n+1}} dz^*$$

☑ Important Special Taylor (Maclaurin) Series

☑ Ex. 4 Logarithm

Find the Maclaurin series of $\mathrm{Ln}(1+z)$

$$\left.\begin{aligned} f'(0) &= 1 \\ f''(0) &= -1 \\ f'''(0) &= 2! \\ f^{(4)}(0) &= -3! \\ f^{(n)}(0) &= (-1)^{n+1}(n-1)! \end{aligned}\right\} \begin{aligned} f'(z) &= \frac{1}{1+z} = (1+z)^{-1} \\ f''(z) &= -(1+z)^{-2} \\ f'''(z) &= 2!(1+z)^{-3} \\ f^{(4)}(z) &= -3!(1+z)^{-4} \\ f^{(n)}(z) &= (-1)^{n+1}(n-1)!(1+z)^{-n} \end{aligned}$$

$$a_n = \frac{1}{n!} f^{(n)}(0) = \frac{1}{n!}(-1)^{n+1}(n-1)! = \frac{(-1)^{n+1}}{n}$$

$$\therefore \mathrm{Ln}\,(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

$$R = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{n}}{\frac{(-1)^{n+2}}{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = 1 \qquad \therefore |z| < 1$$

15.4 Taylor and Maclaurin Series

☑ Maclaurin series

A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{n!} f^{(n)}(0) \text{ or } a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^{*\,n+1}}\, dz^*$$

Find the Maclaurin series of $\mathrm{Ln}\,\frac{1+z}{1-z}$

$$\mathrm{Ln}\,(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + - \cdots$$

Replacing z by $-z$ and multiplying both sides by -1, we get

$$-\mathrm{Ln}\,(1-z) = \mathrm{Ln}\,\frac{1}{1-z} = -(-z) + \frac{(-z)^2}{2} - \frac{(-z)^3}{3} + \frac{(-z)^4}{4} - + \cdots = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots$$

By adding both series we obtain

$$\mathrm{Ln}\,\frac{1+z}{1-z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots \right) \qquad (|z| < 1)$$

15.4 Taylor and Maclaurin Series

☑ Maclaurin series

A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{n!} f^{(n)}(0) \ \text{ or } \ a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^{*\,n+1}}\,dz^*$$

Ex) $\sin\dfrac{z^2}{2}$

$$\sin z = \sum_{k=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$

$$\sin\frac{z^2}{2} = \frac{z^2}{2} - \frac{1}{3!}\left(\frac{z^2}{2}\right)^3 + \frac{1}{5!}\left(\frac{z^2}{2}\right)^5 - \frac{1}{7!}\left(\frac{z^2}{2}\right)^7 + \cdots$$

$$(R = \infty)$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots.$$

$\dfrac{z+2}{1-z^2}$

$$\frac{z+2}{1-z^2} = (z+2)\left(\frac{1}{1-z^2}\right) = (z+2)(1+z^2+z^4+z^6+\cdots)$$
$$= 2 + z + 2z^2 + z^3 + 2z^4 + z^5 + 2z^6 + z^7 + \cdots$$

$$(R = 1)$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots.$$

☑ Practical Methods

☑ Ex. 5 Substitution

Find the Maclaurin series of $f(z) = \dfrac{1}{1+z^2}$

$$f(z) = \frac{1}{1+z^2} = \frac{1}{1-\left(-z^2\right)} = \sum_{n=0}^{\infty} \left(-z^2\right)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots \qquad \left(|z| < 1\right)$$

☑ Ex. 6 Integration

Find the Maclaurin series of $f(z) = \arctan z$

$$f'(z) = \frac{1}{1+z^2} \quad \text{and} \quad f(0) = 0$$

- Integrate term by term

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

$$\Rightarrow \qquad \arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots \qquad \left(|z| < 1\right)$$

☑ Ex. 7 Development by Using the Geometric Series

Develop $\dfrac{1}{c-z}$ in powers of $z-z_0$, where $c-z_0 \neq 0$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots.$$

$$\frac{1}{c-z} = \frac{1}{c-z_0-(z-z_0)} = \frac{1}{(c-z_0)\left(1-\dfrac{z-z_0}{c-z_0}\right)}$$

$$= \frac{1}{(c-z_0)} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{c-z_0}\right)^n = \frac{1}{(c-z_0)}\left(1+\frac{z-z_0}{c-z_0}+\left(\frac{z-z_0}{c-z_0}\right)^2+\cdots\right)$$

This converges for $\left|\dfrac{z-z_0}{c-z_0}\right| < 1$, that is $|z-z_0| < |c-z_0|$

15.4 Taylor and Maclaurin Series

☑ Binomial Series (이항급수)

$$\frac{1}{(1+z)^m} = (1+z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n = 1 - mz + \frac{m(m+1)}{2!} z^2 - \frac{m(m+1)(m+2)}{3!} z^3 + \cdots$$

☑ Ex. 8 Binomial Series, Reduction by Partial Fractions

Find the Taylor series of the following with center $z_0 = 1$

$$f(z) = \frac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots.$$

Sol)

$$f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3} = \frac{1}{\left[3+(z-1)\right]^2} - \frac{2}{2-(z-1)}$$

$$= \frac{1}{9}\left(\frac{1}{\left[1+\frac{1}{3}(z-1)\right]^2}\right) - \frac{1}{1-\frac{1}{2}(z-1)} = \frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^n(n+1)}{3^{n+2}} - \frac{1}{2^n}\right](z-1)^n$$

$$= -\frac{8}{9} - \frac{31}{54}(z-1) - \frac{23}{108}(z-1)^2 - \frac{275}{1944}(z-1)^3 - \cdots$$

15.5 Uniform Convergence: SKIP

☑ Definition Uniform Convergence

A series with sum $s(z)$ is called uniformly convergent in a region G if for every $\varepsilon > 0$ we can find an $N = N(\varepsilon)$, not depending on z, such that

$$\left|s(z) - s_n(z)\right| < \varepsilon \qquad \text{for all } n > N(\varepsilon) \text{ and all } z \text{ in } G$$

Uniformity of convergence is thus a property that always refers to an infinite set in the z-plane, that is, a set consisting of infinitely many points

☑ Theorem 1 Uniform Convergence of Power Series

A power series

$$\sum_{m=0}^{\infty} a_m (z - z_0)^m$$

with a nonzero radius of convergence R is uniformly convergent in every circular disk $|z - z_0| \le r$ of radius $r < R$.

15.5 Uniform Convergence

☑ Properties of Uniformly Convergent Series

- Importance

1. If a series of continuous terms is uniformly convergent, its sum is also continuous.
2. Under the same assumption, termwise integration is permissible.

- Question

1. How can a converging series of continuous terms manage to have a discontinuous sum?
2. How can something go wrong in termwise integration?
3. What is the relation between absolute convergence and uniform convergence?

☑ Theorem 2 Continuity of the Sum

Let the series $\sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + \cdots$

be uniformly convergent in a region G. Let $F(z)$ be its sum. Then if each term $f_m(z)$ is continuous at a point z_1 in G, the function $F(z)$ is continuous at z_1

15.5 Uniform Convergence

☑ Ex. 2 Series of Continuous Terms with a Discontinuous Sum

Consider the series $x^2+\dfrac{x^2}{1+x^2}+\dfrac{x^2}{\left(1+x^2\right)^2}+\dfrac{x^2}{\left(1+x^2\right)^3}+\cdots \qquad (x \ \text{real})$

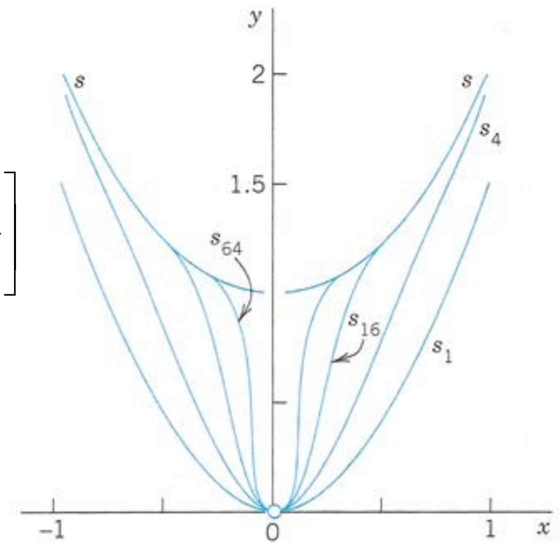
nth partial sum:

$$s_n=x^2\left[1+\frac{1}{1+x^2}+\frac{1}{\left(1+x^2\right)^2}+\cdots+\frac{1}{\left(1+x^2\right)^n}\right]$$

$$s_n-\frac{1}{1+x^2}s_n=x^2\left[1+\frac{1}{1+x^2}+\frac{1}{\left(1+x^2\right)^2}+\cdots+\frac{1}{\left(1+x^2\right)^n}\right]-x^2\left[\frac{1}{1+x^2}+\cdots+\frac{1}{\left(1+x^2\right)^n}+\frac{1}{\left(1+x^2\right)^{n+1}}\right]$$

$$\frac{x^2}{1+x^2}s_n=x^2\left[1-\frac{1}{\left(1+x^2\right)^{n+1}}\right] \qquad s_n=1+x^2-\frac{1}{\left(1+x^2\right)^n}$$

$$\therefore \quad s=\lim_{n\to\infty}s_n=\begin{cases}1+x^2 & (x\neq 0)\\ 0 & (x=0)\end{cases}$$



Partial sums

All the terms are continuous and the series converges even absolutely

Sum is discontinuous at x = 0.

The convergence cannot be uniform in an interval containing x = 0.

15.5 Uniform Convergence

☑ Termwise Integration

☑ Ex. 3 Series for which Termwise Integration is Not Permissible

Let $u_m(x) = mxe^{-mx^2}$ and consider the series

$$\sum_{m=0}^{\infty} f_m(x) \qquad \text{where} \quad f_m(x) = u_m(x) - u_{m-1}(x)$$

in the interval $0 \le x \le 1$.

Sol) (i) nth partial sum: $s_n = u_1 - u_0 + u_2 - u_1 + \cdots + u_n - u_{n-1} = u_n - u_0 = u_n$

The series has the sum $F(x) = \lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} u_n(x) = 0\,(0 \le x \le 1)$

$$\Rightarrow \int_0^1 F(x)\,dx = 0$$

(ii) By integrating term by term and using $s_n = f_1 + f_2 + \cdots + f_n = u_n$

$$\sum_{m=1}^{\infty} \int_0^1 f_m(x)\,dx = \lim_{n\to\infty} \sum_{m=1}^{n} \int_0^1 f_m(x)\,dx = \lim_{n\to\infty} \int_0^1 s_n(x)\,dx = \lim_{n\to\infty} \int_0^1 u_n(x)\,dx = \lim_{n\to\infty} \int_0^1 nxe^{-nx^2}\,dx$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1 - e^{-n}\right) = \frac{1}{2}$$

15.5 Uniform Convergence

☑ **Theorem 3 Termwise Integration**

Let $F(z) = \sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + \cdots$

be a uniformly convergent series of continuous functions in a region G. Let C be any path in G.

Then the series $\sum_{m=0}^{\infty} \int_C f_m(z)dz = \int_C f_0(z)dz + \int_C f_1(z)dz + \cdots$

is convergent and has the sum $\int_C F(z)dz$

☑ **Theorem 4 Termwise Differentiation**

Let the series $f_0(z) + f_1(z) + f_2(z) + \cdots$ be convergent in a region G and let $F(z)$ be its sum. Suppose that the series $f_0'(z) + f_1'(z) + f_2'(z) + \cdots$ converges uniformly in G and its terms are continuous in G. Then

$$F'(z) = f_0'(z) + f_1'(z) + f_2'(z) + \cdots \qquad \text{for all } z \text{ in } G$$

15.5 Uniform Convergence

☑ Test for Uniform Convergence

☑ Theorem 5 Weierstrass M-Test for Uniform Convergence

Consider a series of the form $\sum_{m=0}^{\infty} f_m(z) = f_0(z) + f_1(z) + \cdots$ in a region G of the z-plane. Suppose that one can find a convergent series of constant terms

$$M_0 + M_1 + M_2 + \ldots$$

such that $\left|f_m(z)\right| \leq M_m$ for all z in G and every $m = 0, 1, \ldots$. Then the series is uniformly convergent in G.

15.5 Uniform Convergence

☑ Ex. 4 Weierstrass M-Test

Does the following series converge uniformly in the disk

$$\sum_{m=1}^{\infty} \frac{z^m + 1}{m^2 + \cosh m|z|} \qquad |z| \le 1$$

Sol)

$$\left| \frac{z^m + 1}{m^2 + \cosh m|z|} \right| \le \frac{|z|^m + 1}{m^2} \le \frac{2}{m^2}$$

By the Weierstrass M-test and the convergence of $\sum_{m=1}^{\infty} \frac{1}{m^2}$ $\Rightarrow$ Uniform convergence

15.5 Uniform Convergence

☑ No Relation Between Absolute and Uniform Convergence

☑ Ex. 5 No Relation Between Absolute and Uniform Convergence

The series $\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{x^2+m} = \frac{1}{x^2+1} - \frac{1}{x^2+2} + \frac{1}{x^2+3} - + \cdots$ converges absolutely but not uniformly.

The series $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \cdots$ converge uniformly on the whole real line but not absolutely.

A series of alternating terms whose absolute values form a monotone decreasing sequence with limit zero.

By Leibniz test of calculus the remainder R_n does no exceed its first term in absolute value.

Given $e > 0$, for all x we have $|R_n(x)| \le \frac{1}{x^2+n+1} < \frac{1}{n} < \varepsilon$ if $n > N(\varepsilon) \ge \frac{1}{\varepsilon}$

N(e) does not depend on x ⇨ uniform convergence

For any fixed x we have $\left|\frac{(-1)^{m-1}}{x^2+m}\right| = \frac{1}{x^2+m} > \frac{k}{m}$

where k is a suitable constant, and $k\sum_{m=1}^{\infty}\frac{1}{m}$ diverges ⇨ The convergence is not absolute.