Ch. 16 Laurent Series.
Residue Integration

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※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

[Reference] Taylor and Maclaurin Series

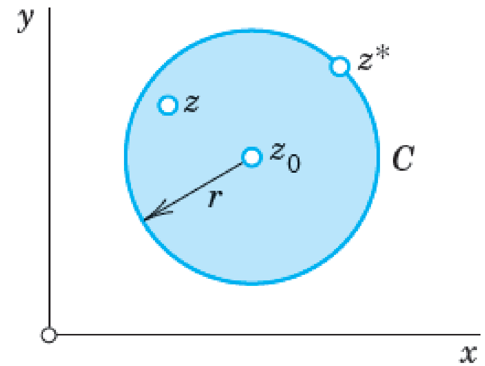
☑ **Theorem 1 Taylor's Theorem**

- Let $f(z)$ be analytic in a domain D, and let $z = z_0$ be any point in D.
- Then there exists precisely one Taylor series with center z_0 that represents $f(z)$.
- This representation is valid in the largest open disk with center z_0 in which $f(z)$ is analytic.

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$$

Q: If $f(z)$ is singular at z_0?

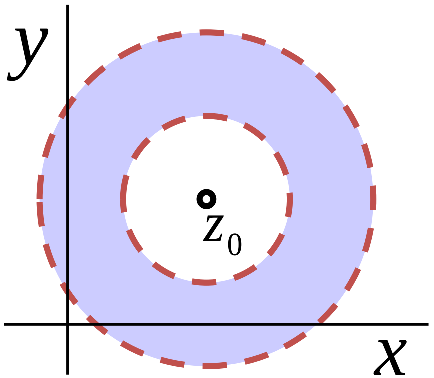
A: We cannot use a Taylor series.
Instead we may use Laurent series.



16.1 Laurent Series

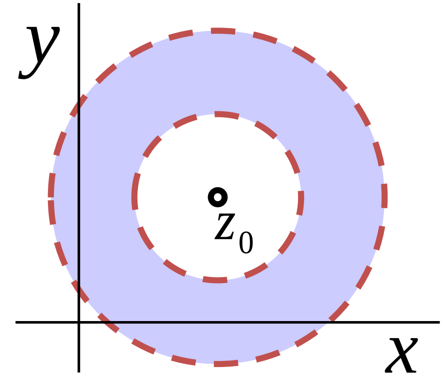
Taylor Series

$$f(z)=\sum_{n=1}^{\infty} a_n (z-z_0)^n \quad, \; a_n = \frac{1}{n!} f^{(n)}(z_0)$$



- Laurent series generalize Taylor series.
- Laurent series is a series of positive and negative integer powers of $z-z_0$ and converges in an annulus (a circular ring) with center z_0.
- By a Laurent series we can represent a given function $f(z)$ that is analytic in an annulus and may have singularities outside the ring as well as in the "hole" of the annulus.
- For a given function the Taylor series with a given center z_0 is unique.
- In contrast, a function $f(z)$ can have several Laurent series with the same center z_0 and valid in several concentric annuli.
- Laurent series converges for $0 < |z-z_0| < R$, that is, everywhere near the center z_0 except at z_0 itself, where z_0 is a singular point of $f(z)$.

16.1 Laurent Series



☑ Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \boxed{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}$$

$$= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \boxed{\frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2}} + \cdots$$

- The series (or finite sum) of the negative powers of this Laurent series is called the principal part (주부) of the singularity of $f(z)$ at z_0, and is used to classify this singularity (Sec. 16.2).
- The coefficient (b_1) of the power $1/(z - z_0)$ of this series is called the residue (유수) of $f(z)$ at z_0.
- If in an application we want to develop a function $f(z)$ in powers of $z - z_0$ when $f(z)$ is singular at z_0, we cannot use a Taylor series.
- Instead we may use Laurent series, consisting of positive integer powers of $z - z_0$ (and a constant) as well as negative integer powers of $z - z_0$.

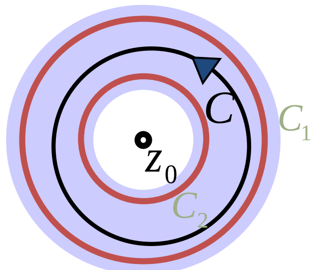
16.1 Laurent Series

☑ Theorem 1 Laurent's Theorem

Let $f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2, with center z_0 and the annulus between them (blue in the figure). Then $f(z)$ can be represented by the Laurent series

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \quad \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$



Laurent's theorem

consisting of nonnegative and negative powers.

The coefficients of this Laurent series are given by the integrals

$$(2) \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*, \quad b_n = \frac{1}{2\pi i} \oint_C (z^*-z_0)^{n-1} f(z^*) dz^*,$$

taken counterclockwise around any simple closed path C that lies in the annulus and encircles the inner circle. we may write (denoting b_n by a_{-n})

$$(1') \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad (2') \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*, \quad (n = 0, \pm 1, \pm 2, \cdots)$$

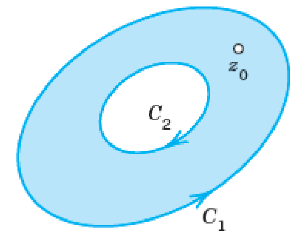
16.1 Laurent Series

Proof)

14.3 Cauchy's Integral Formula

(a) *The nonnegative powers* are those of a Taylor series

$$f(z) = g(z) + h(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z^*)}{z^*-z}dz^* \boxed{-} \frac{1}{2\pi i}\oint_{C_2}\frac{f(z^*)}{z^*-z}dz^*$$

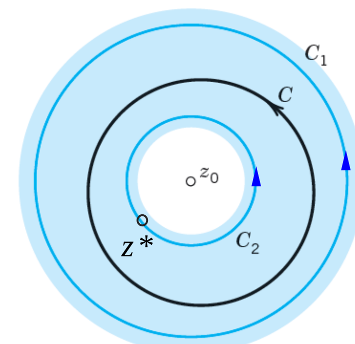


$$g(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z^*)}{z^*-z}dz^* = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$$

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}dz \boxed{+} \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}dz$$

We can replace C_1 by C, by the principle of deformation of path.

$$\therefore a_n = \frac{1}{2\pi i}\oint_{C}\frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$$



(b) *The negative powers*

Since z lies in the annulus, it lies in the exterior of the path C_2

$$\left|\frac{z-z_0}{z^*-z_0}\right| < 1 \text{ in the first integral} \Rightarrow \left|\frac{z^*-z_0}{z-z_0}\right| < 1 \text{ in the second integral.}$$

$$\frac{1}{(z^*-z)} = \frac{1}{z^*-z_0-(z-z_0)} = \frac{-1}{(z-z_0)\left(1-\frac{z^*-z_0}{z-z_0}\right)}$$

15.6 Taylor Series

$$f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{n!}f^{(n)}(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$$

16.1 Laurent Series

$$b_n = \frac{1}{2\pi i}\oint_C (z^*-z_0)^{n-1} f(z^*)dz^*$$

$$\frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \boxed{\frac{q^{n+1}}{1-q}}$$

$$\frac{1}{(z^*-z)} = \frac{1}{z^*-z_0-(z-z_0)} = \frac{-1}{(z-z_0)\left(1-\frac{z^*-z_0}{z-z_0}\right)}$$

$$\boxed{-\frac{1}{z-z_0}\frac{1}{1-\left(\frac{z^*-z_0}{z-z_0}\right)}\left(\frac{z^*-z_0}{z-z_0}\right)^{n+1} = \frac{-1}{z-z^*}\left(\frac{z^*-z_0}{z-z_0}\right)^{n+1}}$$

$$\frac{1}{z^*-z} = -\frac{1}{z-z_0}\left[1+\frac{z^*-z_0}{z-z_0}+\left(\frac{z^*-z_0}{z-z_0}\right)^2+\cdots+\left(\frac{z^*-z_0}{z-z_0}\right)^n\right] - \boxed{\frac{1}{z-z^*}\left(\frac{z^*-z_0}{z-z_0}\right)^{n+1}}$$

$$\therefore h(z) = -\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^*-z_0}dz^* = \frac{1}{2\pi i}\left\{\frac{1}{z-z_0}\oint_{C_2} f(z^*)dz^* + \frac{1}{(z-z_0)^2}\oint_{C_2}(z^*-z_0) f(z^*)dz^* + \cdots\right.$$

$$+\frac{1}{(z-z_0)^n}\boxed{\oint_{C_2}(z^*-z_0)^{n-1} f(z^*)dz^*} \;\; \mathbf{= b_n}$$

$$\left. + \frac{1}{(z-z_0)^{n+1}}\oint_{C_2}(z^*-z_0)^n f(z^*)dz^*\right\} + R_n^*(z)$$

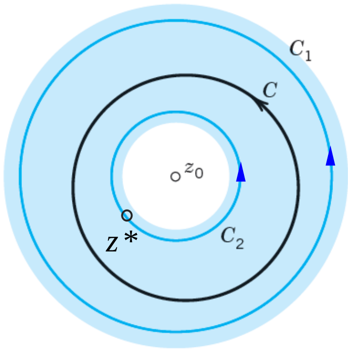
$$\boxed{R_n^*(z) = \frac{1}{2\pi i(z-z_0)^{n+1}}\oint_{C_2}\frac{(z^*-z_0)^{n+1}}{z-z^*} f(z^*)dz^*}$$

This establishes Laurent's theorem, provided $\lim_{n\to\infty} R_n^*(z) = 0$

16.1 Laurent Series

$$R_n^*(z) = \frac{1}{2\pi i (z-z_0)^{n+1}} \oint_{C_2} \frac{(z^*-z_0)^{n+1}}{z-z^*} f(z^*) dz^*$$

(c) *Convergence proof of* $\lim_{n\to\infty} R_n^*(z) = 0$



$$\left| \frac{f(z^*)}{z-z^*} \right| < \tilde{M} \quad \text{for all } z^* \text{ on } C_2$$

because $f(z^*)$ is analytic in the annulus and on C_2 and z^* lies on C_2 and z outside, so that $z - z^* \neq 0$

$$L = 2\pi r_2 = \text{length of } C_2,\ r_2 = \left| z^* - z_0 \right| = \text{radius of } C_2 = \text{const}$$

$$\left| R_n^*(z) \right| \le \frac{1}{2\pi \left| z - z_0 \right|^{n+1}} r_2^{n+1} \tilde{M} L = \frac{\tilde{M} L}{2\pi} \left(\frac{r_2}{\left| z - z_0 \right|} \right)^{n+1}$$

☑ Uniqueness.

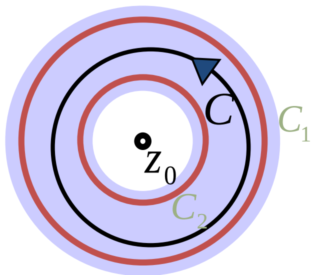
- The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique.
- $f(z)$ may have different Laurent series in two annuli with the same center.

  ⇒ The uniqueness is essential.

16.1 Laurent Series

Theorem 1) Laurent's Theorem

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$



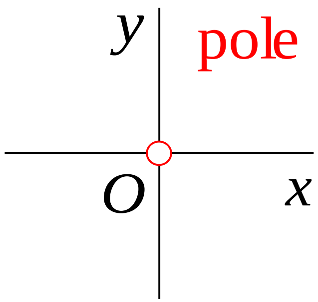
Laurent's theorem

☑ Ex. 1 Use of Maclaurin Series

Find the Laurent series of $z^{-5}\sin z$ with center 0.

Sol)

By (14) in Sec. 15.4, we obtain



Sec. 15.4. (14)

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

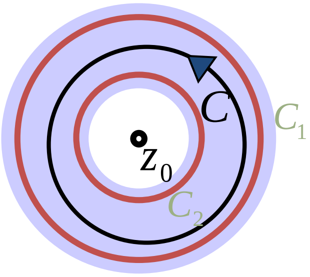
$$z^{-5} \sin z = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} = \boxed{\frac{1}{z^4} - \frac{1}{3!\,z^2}} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots$$

principal part of the series

16.1 Laurent Series

Theorem 1) Laurent's Theorem

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$



Laurent's theorem

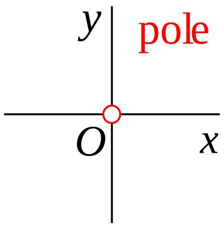
☑ Ex. 2 Substitution

Find the Laurent series of $z^2 e^{1/z}$ with center 0.

Sol) By (12) in Sec. 15.4, with z replaced by $1/z$ we obtain a Laurent series whose principal part is an infinite series,

Sec.15.4. (12)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

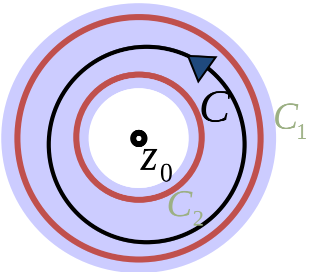


$$z^2 e^{1/z} = z^2 \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2-n}}{n!} = z^2 + z + \frac{1}{2} + \frac{1}{3!\,z} + \frac{1}{4!\,z^2} \cdots \quad (|z| > 0)$$

16.1 Laurent Series

Theorem 1) Laurent's Theorem

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$



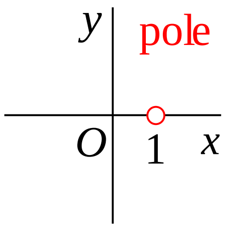
Laurent's theorem

☑ Ex. 3 Development of 1/(1-z)

Develop 1/(1- z) **(a) in nonnegative powers of z,** **(b) in negative powers of z.**

(valid if $|z| < 1$). (valid if $|z| > 1$).

Sol) **Harmonic series**



$$\text{(a)} \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \quad (\text{valid if } |z| < 1).$$

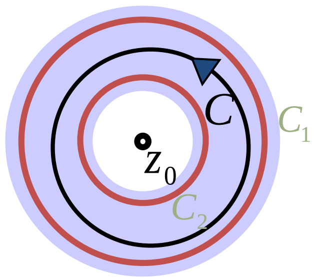
$$\text{(b)} \quad \frac{1}{1-z} = \frac{-1}{z} \cdot \frac{1}{1 - z^{-1}} = \frac{-1}{z} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{z} \right)^n = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \cdots$$

(valid if $|z| > 1$).

16.1 Laurent Series

Theorem 1) Laurent's Theorem

$$(1)\quad f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

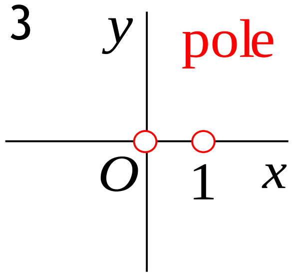


Laurent's theorem

☑ Ex. 4 Laurent Expansions in Different Concentric Annuli

Find all Laurent series of $1/(z^3 - z^4)$ with center 0.

Sol) Multiplying by $1/z^3$, we get from Example 3



Example 16.1-3

(a) $\dfrac{1}{1-z}=\sum_{n=0}^{\infty} z^n \quad (|z|<1)$

(b) $\dfrac{1}{1-z}=-\sum_{n=0}^{\infty} \dfrac{1}{z^{n+1}} \quad (|z|>1)$

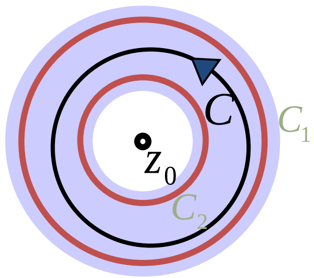
$$\text{(I)}\qquad \frac{1}{z^3}\frac{1}{(1-z)}=\sum_{n=0}^{\infty} z^{n-3}=\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+\cdots \qquad (0<|z|<1)$$

$$\text{(II)}\qquad \frac{1}{z^3}\frac{1}{(1-z)}=-\sum_{n=0}^{\infty} \frac{1}{z^{n+4}}=-\frac{1}{z^4}-\frac{1}{z^5}-\cdots \qquad (|z|>1)$$

16.1 Laurent Series

Theorem 1) Laurent's Theorem

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$



Laurent's theorem

☑ Ex. 5 Use of Partial Fractions

Find all Taylor and Laurent series of $f(z)$ with center 0.

$$f(z) = \frac{-2z + 3}{z^2 - 3z + 2}$$

Sol) In terms of partial fraction

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{1-z} - \frac{1}{z-2}$$

Example 16.1 - 3

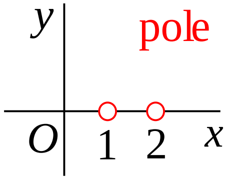
$$\text{(a)} \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

$$\text{(b)} \quad \frac{1}{1-z} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \quad (|z| > 1)$$

(a) and (b) in Example 3 take care of the first fraction.

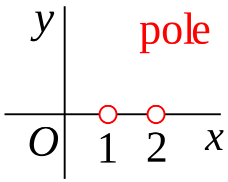
$$\text{(a)} \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

$$\text{(b)} \quad \frac{1}{1-z} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \quad (|z| > 1)$$



16.1 Laurent Series

For the second fraction,



$$\text{(c)}\quad -\frac{1}{z-2}=\frac{1}{2\left(1-\frac{z}{2}\right)}=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^{n}\quad(|z|<2)$$

$$\text{(d)}\quad -\frac{1}{z-2}=-\frac{1}{z\left(1-\frac{2}{z}\right)}=-\sum_{n=0}^{\infty}\frac{2^{n}}{z^{n+1}}\quad(|z|>2)$$

Example 16.1-3

$$\text{(a)}\quad \frac{1}{1-z}=\sum_{n=0}^{\infty}z^{n}\quad(|z|<1)$$

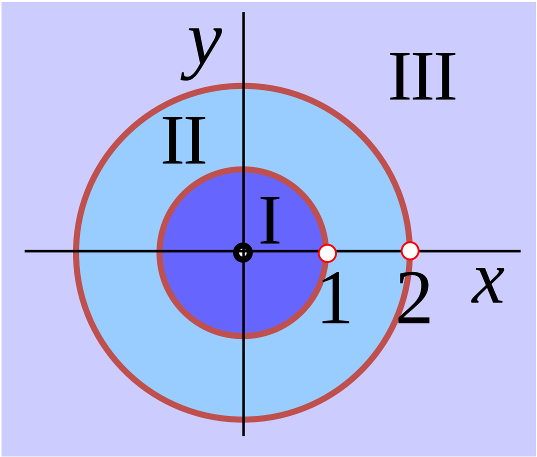
$$\text{(b)}\quad \frac{1}{1-z}=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\quad(|z|>1)$$

(I) From (a) and (c), valid for $|z|<1$,

$$f(z)=\sum_{n=0}^{\infty}\left(1+\frac{1}{2^{n+1}}\right)z^{n}=\frac{3}{2}+\frac{5}{4}z+\frac{9}{8}z^{2}+\cdots$$

(II) From (c) and (b), valid for $1<|z|<2$,

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^{n}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}=\frac{1}{2}+\frac{1}{4}z+\cdots-\frac{1}{z}-\frac{1}{z^{2}}-\cdots$$



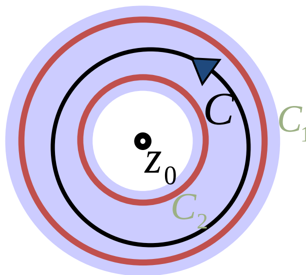
(III) From (d) and (b), valid for $|z|>2$,

$$f(z)=-\sum_{n=0}^{\infty}(2^{n}+1)\frac{1}{z^{n+1}}=-\frac{2}{z}-\frac{3}{z^{2}}-\frac{5}{z^{3}}-\frac{9}{z^{4}}\cdots$$

16.1 Laurent Series

Theorem 1) Laurent's Theorem

$$(1)\quad f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$



Laurent's theorem

☑ **Ex)** Expand the function in a Laurent series that converges for 0<|z|<R and determine the precise region of convergence. $\dfrac{e^z}{z^2-z^3}$

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$\frac{e^z}{z^2-z^3}=\frac{1}{z^2}\frac{e^z}{1-z}=\frac{1}{z^2}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(1+z+z^2+z^3+\cdots\right)$$

$$=\frac{1}{z^2}\left(1+2z+\frac{5}{2}z^2+\frac{8}{3}z^3+\cdots\right)$$

$$=\frac{1}{z^2}+\frac{1}{z}+\frac{5}{2}+\frac{8}{3}z+\cdots$$

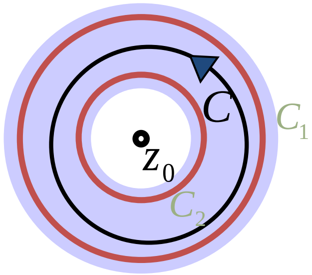
$R=1$

여기서는, 주어진 조건이 0 <|z|< R으로 주어졌기 때문에 1<|z|인 경우는 고려할 필요 없음

16.1 Laurent Series

Theorem 1) Laurent's Theorem

$$(1)\quad f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$



Laurent's theorem

☑ **Ex)** Find the Laurent series that converges for $0<|z-z_0|<R$ and determine the precise region of convergence. $\dfrac{\cos z}{(z-\pi)^2},\ z_0=\pi$

$$\frac{\cos z}{(z-\pi)^2}=\frac{-\cos(z-\pi)}{(z-\pi)^2}$$

$$=-\frac{1}{(z-\pi)^2}\left(1-\frac{(z-\pi)^2}{2!}+\frac{(z-\pi)^4}{4!}+\cdots\right)$$

$$=-\frac{1}{(z-\pi)^2}+\frac{1}{2}-\frac{(z-\pi)^2}{4!}+\cdots$$

Sec. 15.4. (14)

$$\cos z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!},$$

$$\sin z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

$$0<|z-\pi|<\infty$$

16.2 Singularities (특이점) and Zeros (영점). Infinity

☑ Singular Point

- $f(z)$ is singular or has a singularity at a point $z = z_0$ (a singular point of $f(z)$)

$\Longleftrightarrow$ $f(z)$ is not analytic at $z = z_0$

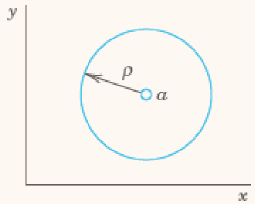
but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic.

- $z = z_0$ is an isolated singularity (고립특이점) of $f(z)$

$\Longleftrightarrow$ $z = z_0$ has a neighborhood without further singularities of $f(z)$.

- Neighborhood (근방) of a

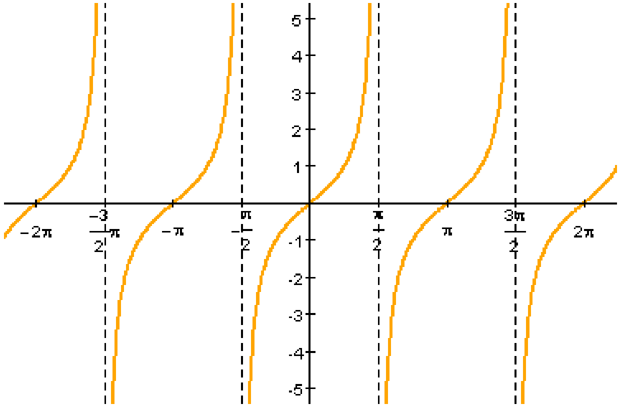
: An open circular disk, ρ-Neighborhood of a



☑ Ex. $\tan z$ has isolated singularities

at $\pm\pi/2,\ \pm 3\pi/2,\ \cdots$, etc.:

$\tan(1/z)$ has a nonisolated singularity at 0.



16.2 Singularities and Zeros. Infinity

☑ **Isolated singularities of $f(z)$ at $z - z_0$ can be classified by the Laurent series**

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

valid in the immediate neighborhood of the singular point $z - z_0$, except at z_0 itself, that is, in a region of the form $0 < |z - z_0| < R$.

- **Principal part**: The second series, containing the negative powers, of Laurent series.
- If the principal part has only finitely many terms, it is of the form

$$\frac{b_1}{z-z_0}+\cdots+\frac{b_m}{(z-z_0)^m} \qquad (b_m \neq 0,\ m:\text{order})$$

the singularity of $f(z)$ at $z = z_0$ is called a **pole (극)**, and *m* is called its **order (위수)**

- **Simple order (단순극)**: Poles of the first order (m = 1)
- **Isolated essential singular point (고립 진성 특이점):** If the principal part has infinitely many terms.

16.2 Singularities and Zeros. Infinity

☑ Ex. 1 Poles (극). Essential Singularities

- The function $f(z)=\dfrac{1}{z(z-2)^5}+\dfrac{3}{(z-2)^2}$ has a simple pole $z=0$ and a pole of fifth order at $z=2$.
- $e^{1/z}=\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!z^n}=1+\frac{1}{z}+\frac{1}{2!z^2}+\cdots$ $\sin\dfrac{1}{z}=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!z^{2n+1}}=\frac{1}{z}-\frac{1}{3!z^3}+\frac{1}{5!z^5}-+\cdots$

  ⇒ isolated essential singularity at $z=0$.
- $z^{-5}\sin z$: a fourth-order pole at 0
- $1/(z^3-z^4)$: a third-order pole at 0

Ex. 1

$$z^{-5}\sin z=\frac{1}{z^4}-\frac{1}{3!z^2}+\frac{1}{5!}-\frac{z^2}{7!}+\cdots$$

Ex. 4

$$\frac{1}{z^3}\frac{1}{(1-z)}=\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+\cdots\qquad(0<|z|<1)$$

☑ Ex. 2 Behavior Near a Pole

$f(z)=\dfrac{1}{z^2}$ has a pole at $z=0$, and $|f(z)|\to\infty$ as $z\to 0$ in any manner.

16.2 Singularities and Zeros. Infinity

☑ **Theorem 1 Poles (극)**

If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \longrightarrow z_0$ in any manner.

☑ Ex.3 Behavior Near an Essential Singularity (진성 특이점)

The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

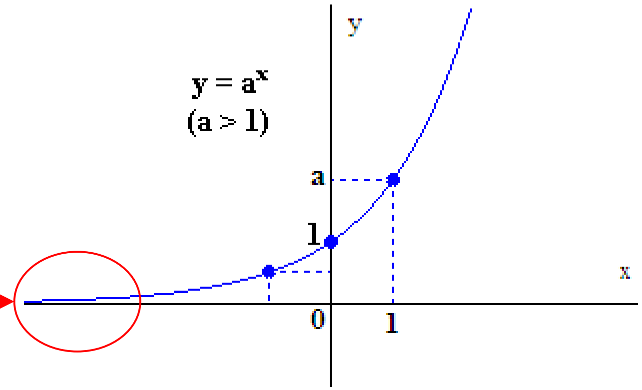
- It has no limit for approach along the imaginary axis.
- It approaches zero if $z \to 0$ ($1/z \to \infty$) through negative real values.
- It takes on any given value $c = c_0 e^{i\alpha} \neq 0$ in an arbitrarily small ε-neighborhood of $z = 0$.

$$z = re^{i\theta}, \quad \frac{1}{z} = \frac{e^{-i\theta}}{r} = e^{(\cos\theta - i\sin\theta)/r} = c_0 e^{i\alpha} \qquad e^{\cos\theta/r} = c_0$$

$$\cos\theta = r\ln c_0, \quad \text{and} \quad -\sin\theta = \alpha r$$

$$\cos^2\theta + \sin^2\theta = 1, \; \Rightarrow r^2(\ln c_0)^2 + \alpha^2 r^2 = 1$$

$$r^2 = \frac{1}{(\ln c_0)^2 + \alpha^2} \qquad \tan\theta = -\frac{\alpha}{\ln c_0}$$



r can be made arbitrarily small by adding multiples of 2π to α leaving *c* unaltered.

16.2 Singularities and Zeros. Infinity

☑ **Theorem 1 Poles**

If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \longrightarrow z_0$ in any manner.

☑ **Theorem 2 Picard's Theorem**

If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every value, with at most one exceptional value, in an arbitrarily small ε-neighborhood of z_0.

16.2 Singularities and Zeros. Infinity

☑ Zeros of Analytic Functions

Zero of an analytic function $f(z)$ in a domain D: a $z = z_0$ in D such that $f(z_0) = 0$

- A zero has **order** (위수) n
  : Not only f but also the derivatives $f', f'', \cdots, f^{(n-1)}$ are all 0 at $z = z_0$
  but $f^{(n)}(z_0) \neq 0$.
- Simple zero: A first-order zero (only $f(z_0) = 0$)

☑ Ex. 4 Zeros

- The function $1+z^2$ has simple zeros at $\pm i$.
- The function $(1-z^4)^2$ has second-order zeros at ± 1 and $\pm i$.
- The function e^z has no zeros.
- The function $\sin z$ has simple zeros at $0, \ \pm\pi, \ \pm 2\pi, \ \cdots$

  $\sin^2 z$ has second-order zeros.

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

16.2 Singularities and Zeros. Infinity

☑ Taylor Series at a Zero.

At an nth-order zero $z = z_0$ of $f(z)$ ⟺ The derivatives $f'(z_0),\ \cdots,\ f^{(n-1)}(z_0)$ are zero ⟹ The first few coefficients $a_0 = \ldots = a_{n-1} = 0$ of the Taylor series are zero, whereas $a_n \neq 0$

$$f(z) = a_n (z - z_0)^n + a_{n+1} (z - z_0)^{n+1} + \cdots = (z - z_0)^n \left[a_n + a_{n+1} (z - z_0) + a_{n+2} (z - z_0)^2 + \cdots \right]$$

☑ Theorem 3 Zeros

The zeros of an analytic function $f(z)$ $(\neq 0)$ are isolated; that is, each of them has a neighborhood that contains no further zeros of $f(z)$.

☑ Theorem 4 Poles and Zeros

Let $f(z)$ be analytic at $z = z_0$ and have a zero of nth order at $z = z_0$. Then $1/f(z)$ has a pole of *n*th order at $z = z_0$; and so does $h(z)/f(z)$, provided $h(z)$ is analytic at $z = z_0$ and $h(z) \neq 0$.

16.2 Singularities and Zeros. Infinity

☑ [Reference] Riemann Sphere. Point at Infinity

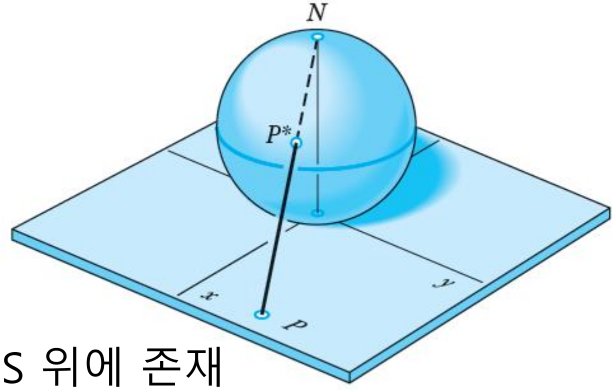
Riemann Sphere: A sphere S of diameter 1 touching the complex z-plane at $z = 0$

- Image (상) of a point P (a number *z* in the plane)

  : The intersection P* of the segment PN with S, where N is the "North Pole" diametrically opposite to the origin in the plane.

- Each point on S represents a complex number z, except for N, which does not correspond to any point in the complex plane.

- **Point at infinity** (denoted ∞): The image of N
- **Extended complex plane**: The complex plane with ∞.



z (복소평면, P)에 대응하는 점들이 S 위에 존재

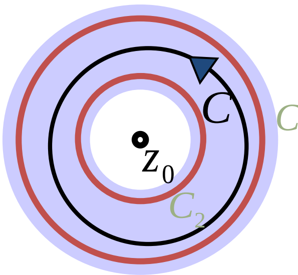
16.2 Singularities and Zeros. Infinity

☑ Analytic or Singular at Infinity

Set $z = 1/w$ and $f(1/w) = g(w)$

- $f(z)$ is analytic at infinity $\Longleftrightarrow$ $g(w)$ is analytic at $w = 0$.
- $f(z)$ is singular at infinity $\Longleftrightarrow$ $g(w)$ is singular at $w = 0$.
- $f(z)$ has an nth-order zero at infinity $\Longleftrightarrow$ $g(w)$ has such a zero at $w = 0$.
- Similarly for poles and essential singularities.

16.3 Residue Integration Method (유수적분)



Laurent's theorem

The purpose of Cauchy's residue integration: the evaluation of integrals

$$\oint_C f(z)dz$$

If $f(z)$ has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C and inside C, then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

that converges for all points near $z = z_0$ (except at $z = z_0$ itself), in some domain of the form $0 < |z - z_0| < R$.

Now comes the key idea. The coefficient b_1 of the first negative power $1/(z - z_0)$ of this Laurent series is given by the integral formula (2) with $n = 1$, namely,

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)dz \quad \Rightarrow \quad \oint_C f(z)dz = 2\pi i \cdot b_1$$

The coefficient b_1 is called the residue (유수) of $f(z)$ at $z = z_0$.

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

Sec.16.1 (2) $a_n = \dfrac{1}{2\pi i} \oint_C \dfrac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$

$$b_n = \frac{1}{2\pi i} \oint_C (z^* - z_0)^{n-1} f(z^*) dz^*$$

16.3 Residue Integration Method

$$\oint_C f(z)dz = 2\pi i \cdot b_1$$

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

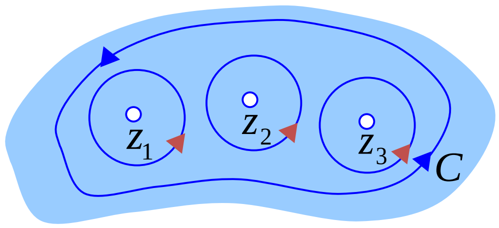
➢ **Several Singularities Inside the Contour**

☑ Theorem 1 Residue Theorem

Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C.

Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, z_2, \cdots, z_k$:

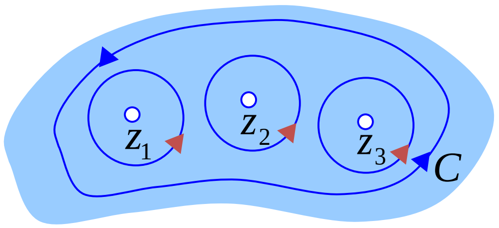
$$(6) \quad \oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$



16.3 Residue Integration Method

☑ Theorem 1 Residue Theorem

$$(6) \quad \oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$



☑ Ex. 1 Evaluation of an Integral by Means of a Residue

Integrate the function $f(z) = z^{-4} \sin z$ counterclockwise around the unit circle C.

Sol) **From (14) in Sec. 15.4 we obtain the Laurent series**

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} \boxed{-\frac{1}{3!\,z}} + \frac{z}{5!} - \frac{z^3}{7!} + - \cdots$$

$$(1) \quad \oint_C f(z)dz = 2\pi i b_1$$

Sec 15.4 (14)

$$\sin z = \sum_{k=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$

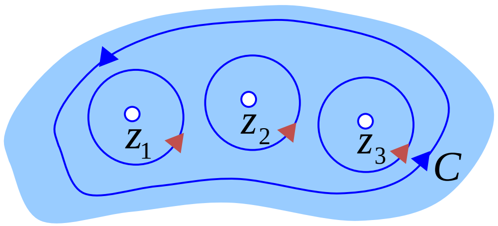
which converges for $|z| > 0$ (that is, for all $z \neq 0$). This series shows that $f(z)$ has a pole of third order at $z = 0$ and the residue $b_1 = -1/3!$. From (1) we thus obtain the answer.

$$\oint_C \frac{\sin z}{z^4} dz = 2\pi i b_1 = 2\pi i \left(-\frac{1}{6} \right) = -\frac{\pi i}{3}$$

16.3 Residue Integration Method

☑ Theorem 1 Residue Theorem

$$(6) \quad \oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$



☑ Ex. 2 Use the Right Laurent Series

Integrate the function $f(z) = 1/(z^3 - z^4)$ clockwise around the circle C: $|z| = 1/2$

Sol) $z^3 - z^4 = z^3(1 - z)$ shows that $f(z)$ is singular at $z = 0$ and $z = 1$.

Example 16.1 - 4

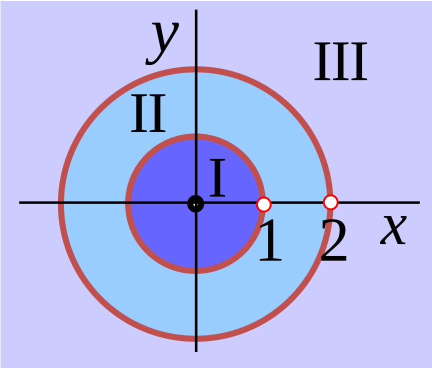
$$\text{(I)} \quad \frac{1}{z^3}\frac{1}{(1-z)} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \boxed{+\frac{1}{z}} + 1 + z + \cdots \quad (0 < |z| < 1)$$

$$\text{(II)} \quad \frac{1}{z^3}\frac{1}{(1-z)} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots \quad (|z| > 1)$$

Now $z = 1$ lies outside C. Hence it is of no interest here.
$0 < |z| < 1$. This is series (I) in Example 4, Sec. 16.1,

We see from it that this residue is 1. Clockwise integration thus yields

$$\oint_C \frac{dz}{z^3 - z^4} = \boxed{-}2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i$$



CAUTION! Had we used the wrong series (II) in Example 4, Sec. 16.1,
we would have obtained the wrong answer, 0, because this series has no power $1/z$.

16.3 Residue Integration Method

☑ Formulas for Residues

To calculate a residue at a pole, we need not produce a whole Laurent series, but, more economically, we can derive formulas for residues once and for all.

- Simple Poles

1. z_0 is a simple pole of $f(z)$: (3) $\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0}(z-z_0)f(z)$

Proof) $f(z) = \dfrac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots (0 <| z-z_0 |< R)$

$$\lim_{z\to z_0}(z-z_0)f(z) = b_1 + a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 + \cdots = b_1$$

2. Assume that $f(z) = \dfrac{p(z)}{q(z)}$, $p(z_0) \neq 0$, and a simple zero at z_0

$$(4) \quad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Proof) $q(z) = q(\cancel{z_0}) + (z-z_0)q'(z_0) + \dfrac{(z-z_0)^2}{2!}q''(z_0) + \dfrac{(z-z_0)^3}{3!}q''' + \cdots$

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)\frac{p(z)}{q(z)} = \lim_{z\to z_0}\frac{(z-z_0)p(z)}{(z-z_0)[q'(z_0)+(z-z_0)q''(z_0)/2+\cdots]} + \cdots = \frac{p(z)}{q'(z_0)}$$

16.3 Residue Integration Method

☑ Formulas for Residues

To calculate a residue at a pole, we need not produce a whole Laurent series, but, more economically, we can derive formulas for residues once and for all.

- **Poles of Any Order**

An m[th]-order pole: $$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right\}$$

A second-order pole: $$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}\left[(z-z_0)^2 f(z)\right]'$$

Proof) $$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \cdots + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \cdots + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + \cdots$$

$$b_1 = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

16.3 Residue Integration Method

☑ Simple Poles.

1. **z_0 is a simple pole of $f(z)$:** (3) $\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z)$
2. **Assume that** $f(z) = \dfrac{p(z)}{q(z)}$, $p(z_0) \neq 0$, **and a simple zero at z_0**

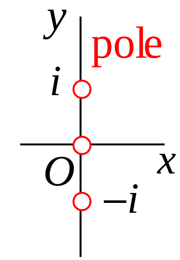
$$(4) \quad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

☑ Ex. 3 Residue at a Simple pole

$f(z)$ has some simple poles, and (3) gives the residues at the poles. Find the all residues of $f(z)$.

$$f(z) = \frac{9z+i}{z^3+z} = \frac{9z+i}{z(z+i)(z-i)}$$

Sol) $\text{Poles}: z = 0,\ i,\ -i$



By (4),

$$z=0, \quad \operatorname*{Res}_{z=0} \frac{9z+i}{z^3+z} = \left[\frac{9z+i}{3z^2+1}\right]_{z=0} = i$$

$$z=i, \quad \operatorname*{Res}_{z=i} \frac{9z+i}{z^3+z} = \left[\frac{9z+i}{3z^2+1}\right]_{z=i} = -5i$$

$$z=-i, \quad \operatorname*{Res}_{z=-i} \frac{9z+i}{z^3+z} = \left[\frac{9z+i}{3z^2+1}\right]_{z=-i} = 4i$$

By (3),

$$\operatorname*{Res}_{z=i} \frac{9z+i}{z^3+z}$$

$$= \lim_{z \to i} (z-i) \frac{9z+i}{z(z+i)(z-i)}$$

$$= \frac{9i+i}{i(i+i)} = -5i.$$

16.3 Residue Integration Method

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

☑ Simple Poles.

1. z_0 is a simple pole of $f(z)$: (3) $\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z)$

2. Assume that $f(z) = \dfrac{p(z)}{q(z)}$, $p(z_0) \neq 0$, **and a simple zero at z_0**

$$(4) \quad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

☑ Ex. 5 Residue at a Pole of Higher Order

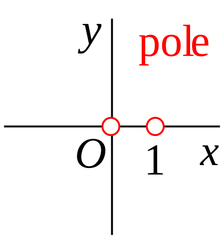
Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside (d) 0 and 1 are outside.

$$\oint_C \frac{4-3z}{z^2 - z} dz = \oint_C \frac{4-3z}{z(z-1)} dz$$

Sol) The integrand has simple poles at 0 and 1, with residues [by (3)]

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{(z-1)}\right]_{z=0} = -4,$$

$$\operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$



16.3 Residue Integration Method

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

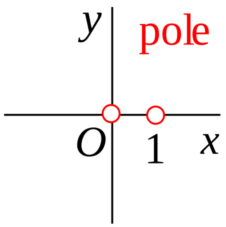
☑ Simple Poles.

1. z_0 is a simple pole of $f(z)$: (3) $\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0}(z-z_0)f(z)$
2. Assume that $f(z)=\dfrac{p(z)}{q(z)}$, $p(z_0)\neq 0$, and a simple zero at z_0

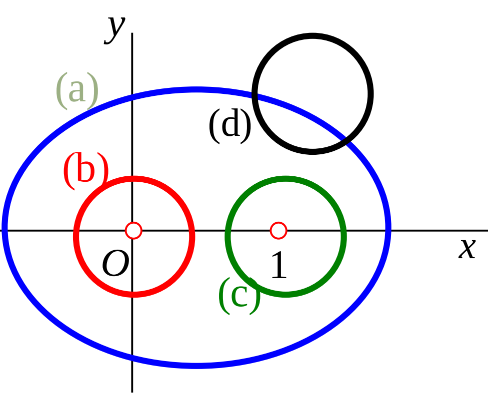
$$(4)\quad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

☑ Ex. 5 Residue at a Pole of Higher Order

Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside (d) 0 and 1 are outside.



$$\oint_C \frac{4-3z}{z^2-z}\,dz = \oint_C \frac{4-3z}{z(z-1)}\,dz$$



$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = -4, \qquad \operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = 1.$$

(a) $2\pi i(-4+1) = -6\pi i$ (b) $2\pi i(-4) = -8\pi i$

(c) $2\pi i(1) = 2\pi i$ (d) 0

16.3 Residue Integration Method

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

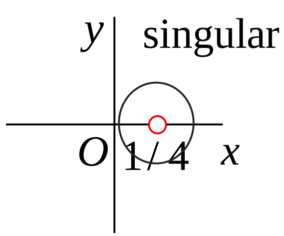
☑ Simple Poles.

1. **z_0 is a simple pole of $f(z)$:** (3) $\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z-z_0) f(z)$
2. **Assume that** $f(z) = \dfrac{p(z)}{q(z)}$, $p(z_0) \neq 0$, **and a simple zero at z_0**

$$(4) \quad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

☑ Ex. Evaluate (counterclockwise).

$$\oint_C \tan 2\pi z dz, \ C: |z-0.2| = 0.2$$



$$\operatorname*{Res}_{z=1/4} \text{f(z)} = \lim_{z\to 1/4} -\frac{\sin 2\pi z}{2\pi \sin 2\pi z} = -\frac{1}{2\pi},$$

$$\oint_C \tan 2\pi z dz = -2\pi i \frac{1}{2\pi} = -i$$

16.4 Residue Integration of Real Integrals

☑ Integrals of Rational Functions (유리함수) of cosθ and sinθ

Certain classes of complicated real integrals can be integrated by the residue theorem, as we shall see.

We first consider integrals of the type

$$(1)\quad J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta \qquad Ex)\ \frac{\sin^2\theta}{5-4\cos\theta}$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$.

Setting $e^{i\theta} = z,\ dz/d\theta = ie^{i\theta},\ d\theta = dz/iz$

Then,

$$(3)\quad J = \oint_C f(z)\frac{dz}{iz}$$

$$(2)\quad \cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right) = \frac{1}{2}\left(z+\frac{1}{z}\right)$$
$$\sin\theta = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right) = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

and, as θ ranges from 0 to 2π in (1), the variable $z = e^{i\theta}$ ranges counterclockwise once around the unit circle $|z| = 1$.

16.4 Residue Integration of Real Integrals

☑ Integrals of Rational Functions of cosθ and sinθ

$$(1)\quad J=\int_0^{2\pi}F(\cos\theta,\sin\theta)\,d\theta \quad\Rightarrow\quad (3)\quad J=\oint_C f(z)\frac{dz}{iz}$$

Real rational function

☑ Ex. 1. An Integral

Show by the present method that $\displaystyle\int_0^{2\pi}\frac{d\theta}{\sqrt{2}-\cos\theta}=2\pi.$

Sol) we use $\cos\theta=\frac{1}{2}(z+1/z)$ and $e^{i\theta}=z$ $(d\theta=dz/iz)$

$$(2)\quad \cos\theta=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)=\frac{1}{2}\left(z+\frac{1}{z}\right)$$
$$\sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)=\frac{1}{2i}\left(z-\frac{1}{z}\right)$$

Then the integral becomes

$$\int_0^{2\pi}\frac{d\theta}{\sqrt{2}-\cos\theta}=\oint_C\frac{dz/iz}{\sqrt{2}-\frac{1}{2}\left(z+\frac{1}{z}\right)}=-\frac{2}{i}\oint_C\frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}$$

***C*: counterclockwise once around the unit circle $|z|=1$**

16.4 Residue Integration of Real Integrals

☑ Integrals of Rational Functions of cosθ and sinθ

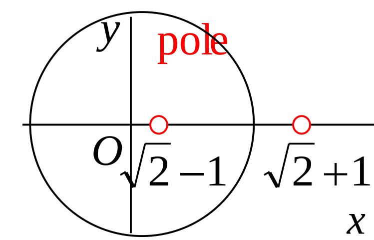
$$\text{(1)} \quad J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \quad \Rightarrow \quad \text{(3)} \quad J = \oint_C f(z) \frac{dz}{iz}$$

Real rational function

☑ Ex. 1 An Integral - continued

$$-\frac{2}{i} \oint_C \frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}$$

C: counterclockwise once around the unit circle |z| = 1



We see that the integrand has a simple pole at $z_1 = \sqrt{2}+1$ outside the unit circle C, so that it is of no interest here, and another simple pole at $z_2 = \sqrt{2}-1$.

(where $z-\sqrt{2}+1=0$) inside C with

$$\text{(3)} \quad \operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z-z_0) f(z)$$

$$\operatorname*{Res}_{z=z_2} \frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)} = \left[\frac{1}{z-\sqrt{2}-1}\right]_{z=\sqrt{2}-1} = -\frac{1}{2}$$

$$\therefore -\frac{2}{i} \oint_C \frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)} = -\frac{2}{i} \cdot 2\pi i \cdot \operatorname*{Res}_{z=z_2} \frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)} = -\frac{2}{i} \cdot 2\pi i \cdot \left(-\frac{1}{2}\right) = 2\pi$$

16.4 Residue Integration of Real Integrals

☑ Improper Integral

As another large class, let us consider real integrals of the form

$$(4) \qquad \int_{-\infty}^{\infty} f(x)dx$$

Such an integral, whose interval of integration is not finite is called an improper integral (**이상적분**), and it has the meaning

$$(5') \qquad \int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_{a}^{0} f(x)dx + \lim_{b\to\infty}\int_{0}^{b} f(x)dx$$

If both limits exist, we may couple the two independent passages to -∞ and ∞, and write

$$(5) \qquad \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

The limit in (5) is called the Cauchy principal value of the integral. It is written

$$\text{pr.v.} \int_{-\infty}^{\infty} f(x)dx$$

16.4 Residue Integration of Real Integrals

☑ Improper Integral

$$(5') \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx$$

$$(5) \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

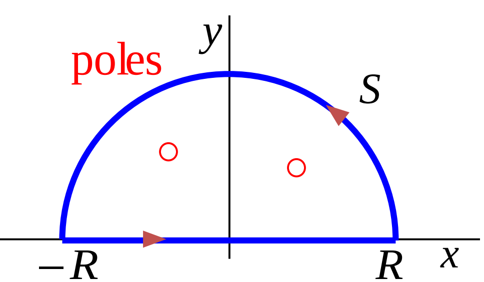
- We assume that the function $f(x)$ in (5') is a real rational function whose denominator(분모) is different from zero for all x and
- is of degree at least two units higher than the degree of the numerator(분자).
- Then the limits in (5') exist, and we may start from (5).

We consider the corresponding contour integral

$$(5^*) \quad \oint_C f(z)dz = \int_S f(z)dz + \int_{-R}^{R} f(x)dx$$

around a path C

$$\oint_C f(z)dz = 2\pi i \sum \operatorname{Res} f(z)$$



Path C of the contour integral in (5*)

Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper half-plane, and if we choose R large enough, then C encloses all these poles. By the residue theorem we then obtain

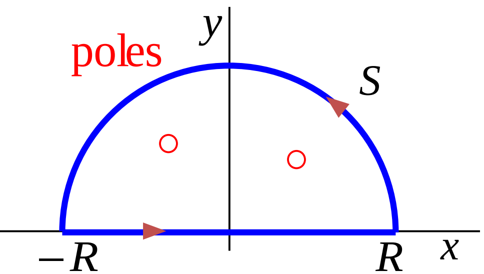
$$(6) \quad \int_{-R}^{R} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)dz$$

16.4 Residue Integration of Real Integrals

☑ Improper Integral

$$(5') \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{a \to -\infty} \int_{a}^{0} f(x)dx + \lim_{b \to \infty} \int_{0}^{b} f(x)dx$$

$$(5) \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx$$



Path *C* of the contour integral in (5*)

$$(5^*) \quad \oint_C f(z)dz = \int_S f(z)dz + \int_{-R}^{R} f(x)dx$$

around a path C

$$(6) \quad \int_{-R}^{R} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)dz$$

We prove that, if $R \to \infty$, the value of the integral over the semicircle S approaches zero.*

If we set $z=Re^{i\theta}$, S is represented by R=const.

$$|f(z)| < \frac{k}{|z|^2} \qquad \left(|z|^2 = R > R_0\right)$$

16.4 Residue Integration of Real Integrals

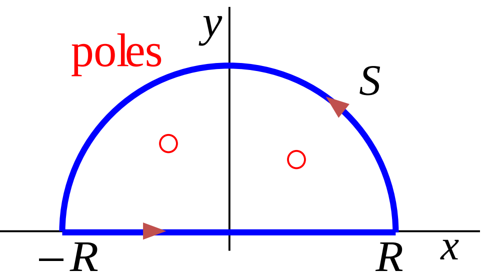
☑ Improper Integral

$$(5') \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx$$

$$(5) \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

By the ML-inequality

$$\left|\int_S f(z)dz\right| < \frac{k}{R^2}\pi R = \frac{k\pi}{R}$$



Path *C* of the contour integral in (5*)

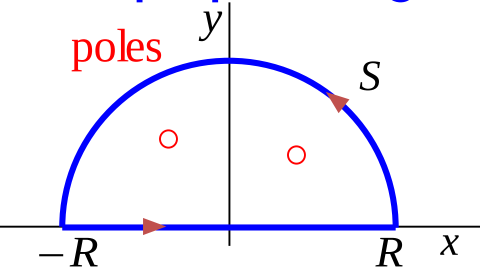
as *R* approaches infinity, the value of the integral over *S* approaches zero,

$$(7) \quad \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z)$$

where we sum over all the residues of $f(z)$ at the poles of $f(z)$ in the upper half-plane.

16.4 Residue Integration of Real Integrals

☑ Improper Integral



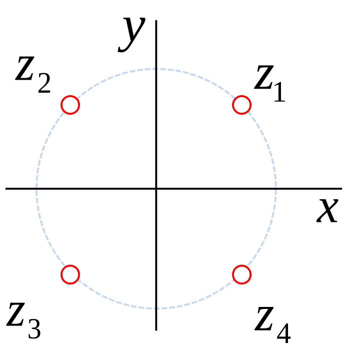
$$(7)\quad \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z)$$

☑ Ex. 2 An Improper Integral from 0 to

Sec.16.3 (4) $\underset{z=z_0}{\operatorname{Res}} f(z) = \underset{z=z_0}{\operatorname{Res}} \dfrac{p(z)}{q(z)} = \dfrac{p(z)}{q'(z)}$

Show that $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

Sol) $f(z) = \dfrac{1}{1+z^4}$ **has four simple poles at the points.**



$$z_1 = e^{\pi i/4},\quad z_2 = e^{3\pi i/4},\quad z_3 = e^{-3\pi i/4},\quad z_4 = e^{-\pi i/4}$$

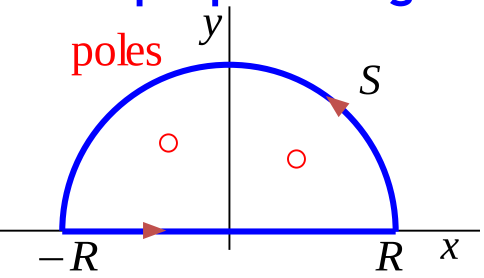
The first two of these poles lie in the upper half-plane. From (4) in the last section we find the residues.

$$\underset{z=z_1}{\operatorname{Res}} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4}e^{\pi i/4}$$

$$\underset{z=z_2}{\operatorname{Res}} f(z) = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4}e^{-\pi i/4}$$

16.4 Residue Integration of Real Integrals

☑ Improper Integral



$$(7) \quad \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z)$$

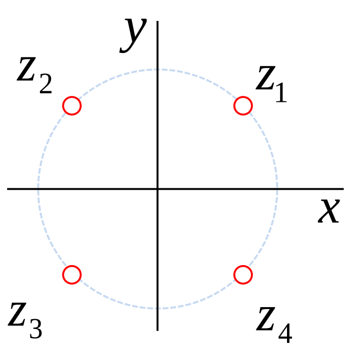
☑ Ex. 2 An Improper Integral from 0 to

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

Show that $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

Sol - continued)

$$\operatorname*{Res}_{z=z_1} f(z) = -\frac{1}{4}e^{\pi i/4} \qquad \operatorname*{Res}_{z=z_2} f(z) = \frac{1}{4}e^{-\pi i/4}$$



By (1) in Sec. 13.6 and (7) in this section,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i\left(-\frac{e^{\pi i/4} - e^{-\pi i/4}}{4}\right) = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}} \quad = -\frac{2\pi i}{4}\cdot 2i \cdot \sin\frac{\pi}{4}$$

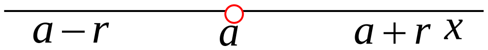
Since $1/(1+x^4)$ is an even function, we thus obtain, as asserted,

$$\therefore \int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

16.4 Residue Integration of Real Integrals

☑ Another Kind of Improper Integral

(11) $\displaystyle\int_A^B f(x)dx$



(12) $\displaystyle\int_A^B f(x)dx = \lim_{\varepsilon\to 0}\int_A^{a-\varepsilon} f(x)dx + \lim_{\eta\to 0}\int_{a+\eta}^{B} f(x)dx$

(13) $\displaystyle\lim_{\varepsilon\to 0}\left[\int_A^{a-\varepsilon} f(x)dx + \int_{a+\varepsilon}^{B} f(x)dx\right]$

This is called the Cauchy principal value (주값) of the integral. It is written

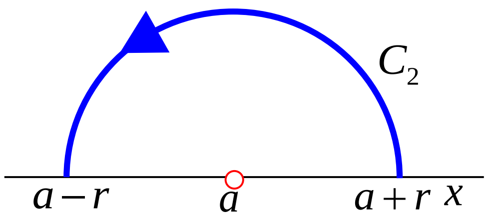
$$\text{pr. v.}\int_A^B f(x)dx.$$

16.4 Residue Integration of Real Integrals

☑ Another Kind of Improper Integral

☑ Theorem 1 Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then



$$\lim_{r\to 0}\int_{C_2} f(z)dz = \pi i \operatorname*{Res}_{z=a} f(z)$$

Proof) By the definition of a simple pole

$$f(z) = \frac{b_1}{z-a} + g(z), \quad b_1 = \operatorname*{Res}_{z=a} f(z) \qquad 0 < |z-a| < R$$

Here g(z) is analytic on the semicircle of integration

$$C_2 : z = a + re^{i\theta}, \quad 0 \le \theta \le \pi$$

And for all z between C_2 and the x-axis $\rightarrow$ $g(z) \le M$

$$\int_{C_2} f(z)dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}} ire^{i\theta} d\theta + \int_{C_2} g(z)dz = b_1 \pi i + \int_{C_2} g(z)dz$$

$$\int_{C_2} g(z)dz \le ML = M\pi r \to 0 \quad as \ r \to 0 \qquad \therefore \lim_{r\to 0}\int_{C_2} f(z)dz = \pi i \operatorname*{Res}_{z=a} f(z)$$

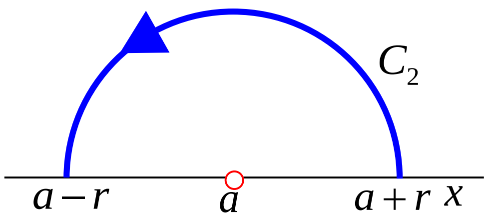
* Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p723

16.4 Residue Integration of Real Integrals

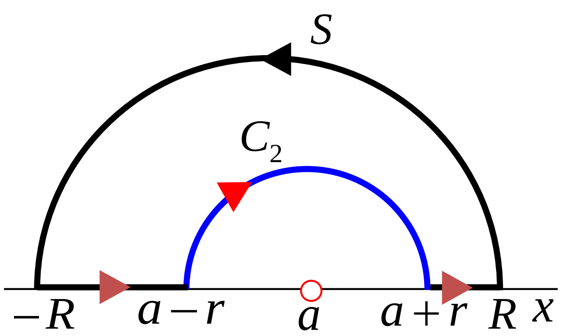
☑ Another Kind of Improper Integral

☑ Theorem 1 Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then



$$\lim_{r \to 0} \int_{C_2} f(z)dz = \pi i \operatorname*{Res}_{z=a} f(z)$$



For sufficiently large R the integral over the entire contour has the value J given by $2\pi i$ times the sum of the residues of $f(z)$ at the singularities in the upper half-plane.

$$J = 2\pi i \sum \text{Res } f(z)$$

We assume that $f(z) \rightarrow 0$, as x goes infinite then the value of the integral over the large semicircle S approaches 0 as $R \rightarrow \infty$.
For $r \rightarrow 0$ the integral over C_2 (clockwise!) approaches the value.

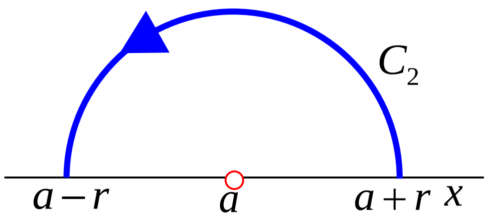
$$K = -\pi i \operatorname*{Res}_{z=a} f(z)$$

16.4 Residue Integration of Real Integrals

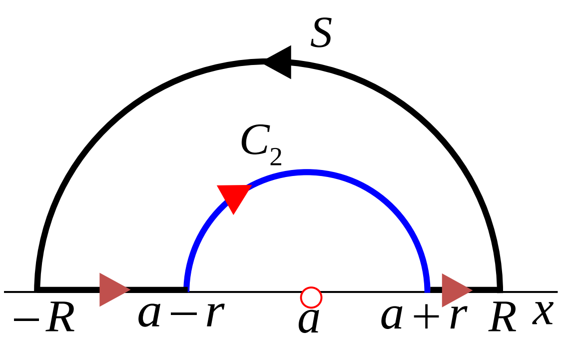
☑ Another Kind of Improper Integral

☑ Theorem 1 Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then



$$\lim_{r\to 0}\int_{C_2} f(z)dz = \pi i \operatorname*{Res}_{z=a} f(z)$$



$$J = 2\pi i \sum \operatorname{Res} f(z) \qquad K = -\pi i \operatorname*{Res}_{z=a} f(z)$$

Together this show that the principal value P of the integral from -∞ to ∞ Plus K equals J.

Hence $\quad P = J - K = 2\pi i \sum Res\, f(z) + \pi i \operatorname*{Res}_{z=a} f(z)$

If $f(z)$ has several simple poles on the real axis, then

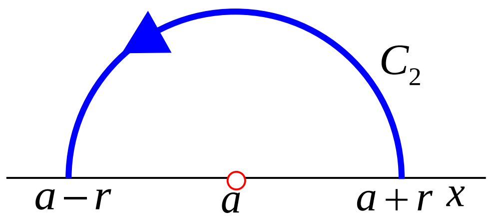
$$K = -\pi i \sum \operatorname{Res} f(z).$$

16.4 Residue Integration of Real Integrals

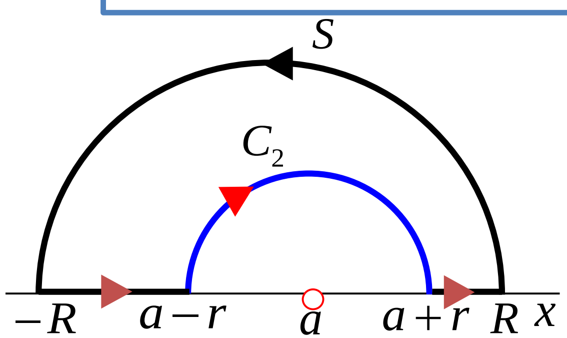
☑ Another Kind of Improper Integral

☑ Theorem 1 Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then



$$\lim_{r\to 0}\int_{C_2} f(z)dz = \pi i \operatorname*{Res}_{z=a} f(z)$$



$$P = J - K = 2\pi i \sum \operatorname{Res} f(z) + \pi i \operatorname*{Res}_{z=a} f(z).$$

$$J = 2\pi i \sum \operatorname{Res} f(z) \qquad K = -\pi i \sum \operatorname{Res} f(z).$$

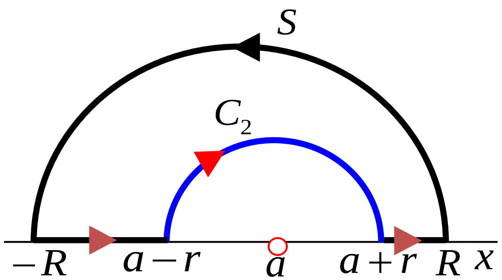
Hence the desired formula is

$$\text{(14)} \quad \text{pr. v.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

16.4 Residue Integration of Real Integrals

☑ Another Kind of Improper Integral



$$(14) \quad \text{pr. v.}\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

☑ Ex. 4 Poles on the Real Axis

$$(3) \quad \operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0}(z-z_0)f(z)$$

Find the principal value $\text{pr. v.}\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$

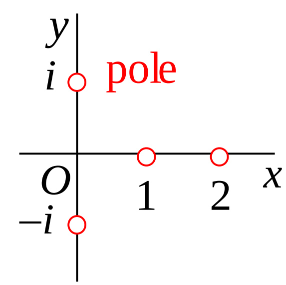
Sol) Since $f(x) = \dfrac{1}{(x^2-3x+2)(x^2+1)} = \dfrac{1}{(x-1)(x-2)(x+i)(x-i)}$,

the integrand $f(x)$, considered for complex z, has simple poles at

$$z=1, \quad \operatorname*{Res}_{z=1} f(z) = \left[\frac{1}{(z-2)(z^2+1)}\right]_{z=1} = -\frac{1}{2}$$

$$z=2, \quad \operatorname*{Res}_{z=2} f(z) = \left[\frac{1}{(z-1)(z^2+1)}\right]_{z=2} = \frac{1}{5}$$

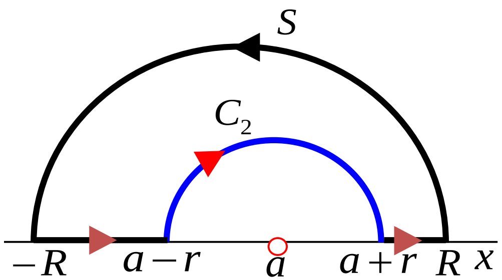
$$z=i, \quad \operatorname*{Res}_{z=i} f(z) = \left[\frac{1}{(z^2-3z+2)(z+i)}\right]_{z=i} = \frac{1}{6+2i} = \frac{3-i}{20}$$



$z = -i$ in the lower half-plane, which is of no interest.

16.4 Residue Integration of Real Integrals

☑ Another Kind of Improper Integral



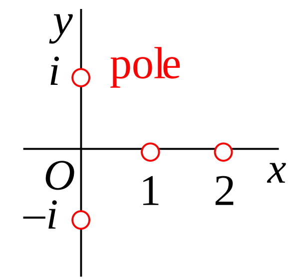
$$(14)\quad \text{pr. v.}\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

☑ Ex. 4 Poles on the Real Axis

Find the principal value $\quad \text{pr. v.}\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$

Sol-continued) Since $f(x) = \dfrac{1}{(x^2-3x+2)(x^2+1)} = \dfrac{1}{(x-1)(x-2)(x+i)(x-i)}$,



$$\underset{z=1}{\operatorname{Res}} f(z) = -\frac{1}{2}, \quad \underset{z=2}{\operatorname{Res}} f(z) = \frac{1}{5}, \quad \underset{z=i}{\operatorname{Res}} f(z) = \frac{3-i}{20}$$

real axis **upper half-plane**

z = - i in the lower half-plane, which is of no interest.

$$\therefore \text{pr. v.}\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)} = 2\pi i\left(\frac{3-i}{20}\right) + \pi i\left(-\frac{1}{2}+\frac{1}{5}\right) = \frac{\pi}{10}$$