

Ch. 2 Introduction to numerical methods

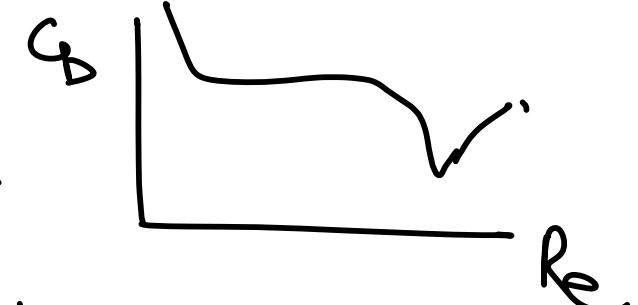
1. Approaches to fluid dynamical problems

- Dimensional analysis & experiments

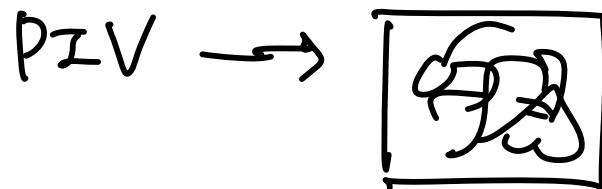
$$\xrightarrow{U} \textcircled{II} \uparrow d \quad \text{Drag} = D = f(\rho, \mu, U, d)$$

$$\rightarrow C_D = \frac{D}{\frac{1}{2} \rho U^2 A} = f\left(\frac{\rho U d}{\mu}\right) = f(Re)$$

drag coeff,



experiments: good for obtaining global parameters like drag, lift, pressure drop, --



cannot provide
3-D picture of flow.

⇒ Use CFD teraflop machine

large memory & disc space

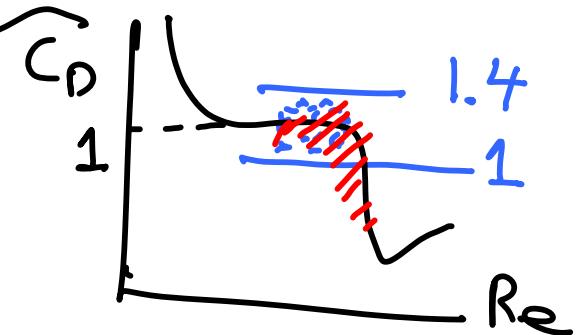
100 billion grid pts.

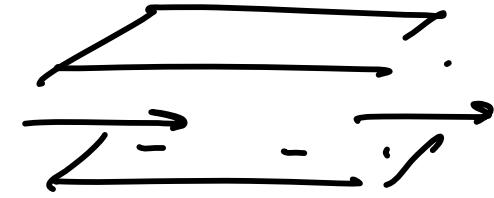
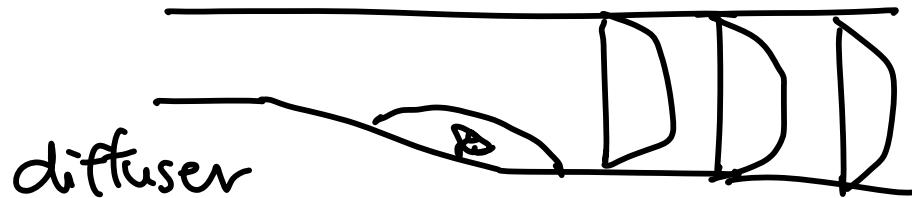
but errors in CFD

→ CFD requires good experiments.

flow over a circular cylinder

→ Drag





KMM
(1987)

2. Components of a numerical solution method

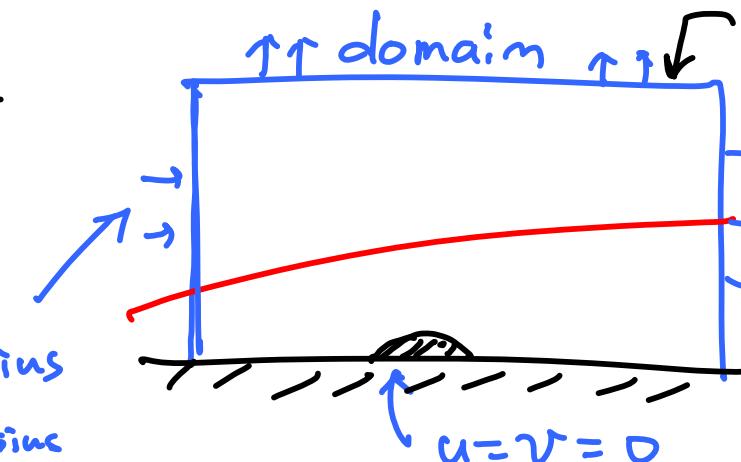
① mathematical model

Governing eqs. u, v

b.c's
i.c's

$$u = u_{\text{Blasius}}$$

$$v = v_{\text{Blasius}}$$



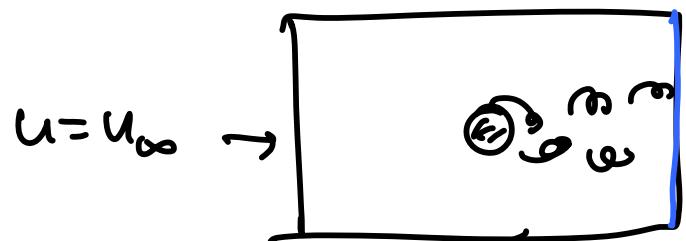
$$\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial v}{\partial x} = 0$$

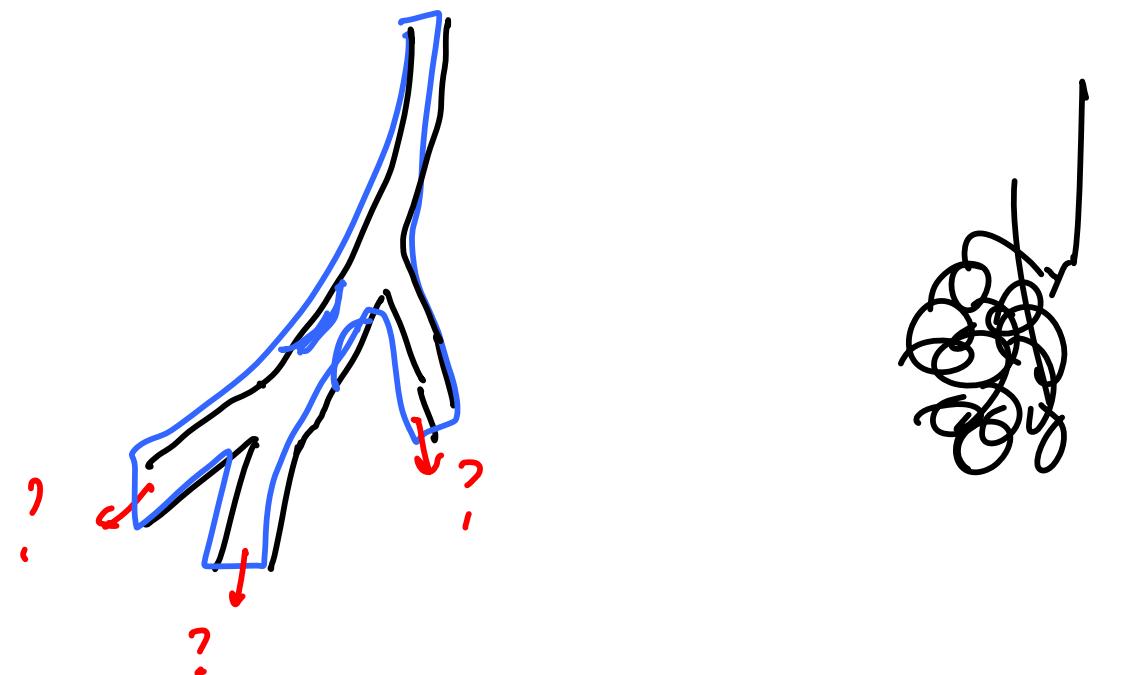
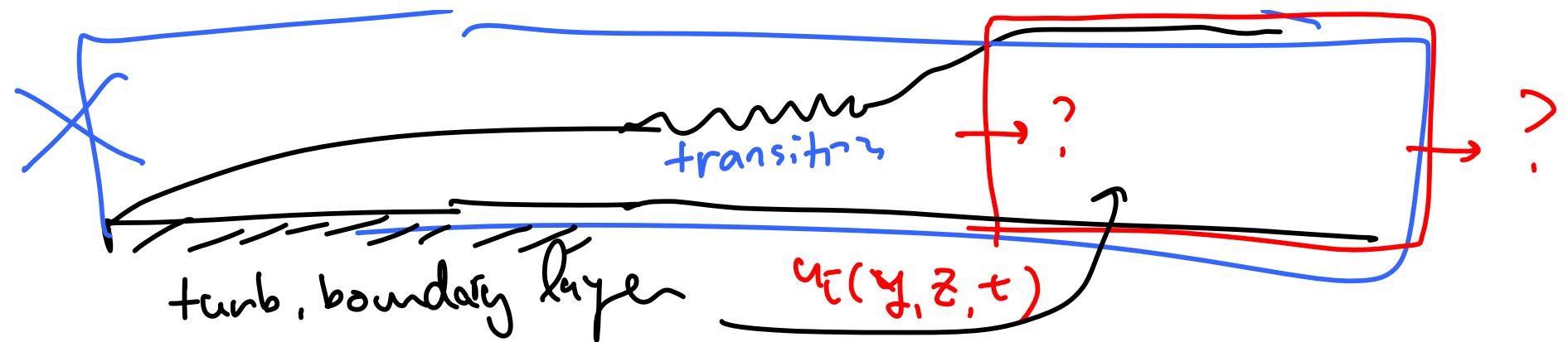
$$v = 0$$

$$\frac{\partial^2 u}{\partial x^2} = 0, \frac{\partial^2 v}{\partial x^2} = 0$$

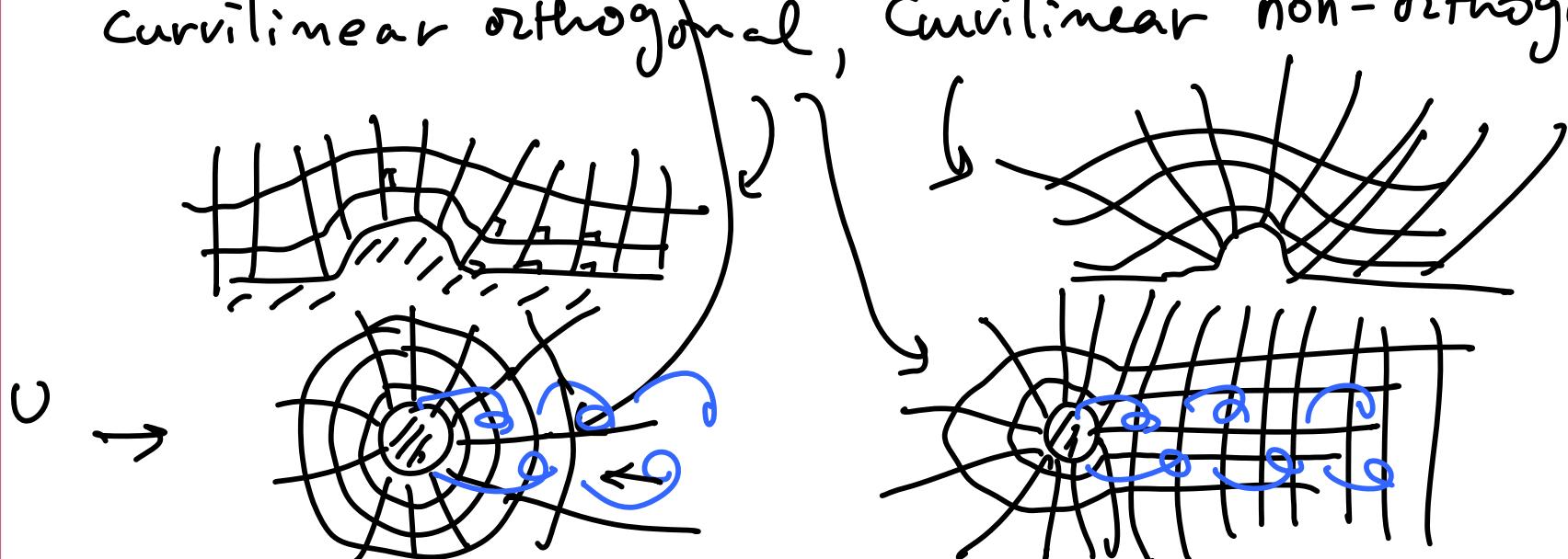


$$\frac{\partial u_i}{\partial t} + C \frac{\partial u_i}{\partial x} = 0 : \text{convective b.c.}$$

$$? C = f_{\text{ad}} y / L_y$$



- ② Discretization method - FEM, FDM, FVM, spectral method
- ③ coordinates, and basis vector systems
 - Cartesian, cylindrical, spherical
 - curvilinear orthogonal, curvilinear non-orthogonal



$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \rightarrow \alpha \frac{\partial^2 \phi}{\partial \xi^2} + \beta \frac{\partial^2 \phi}{\partial \xi \partial \eta} + \gamma \frac{\partial^2 \phi}{\partial \eta^2} + \dots = 0$$

if orthogonal
coord. sys.
is used.

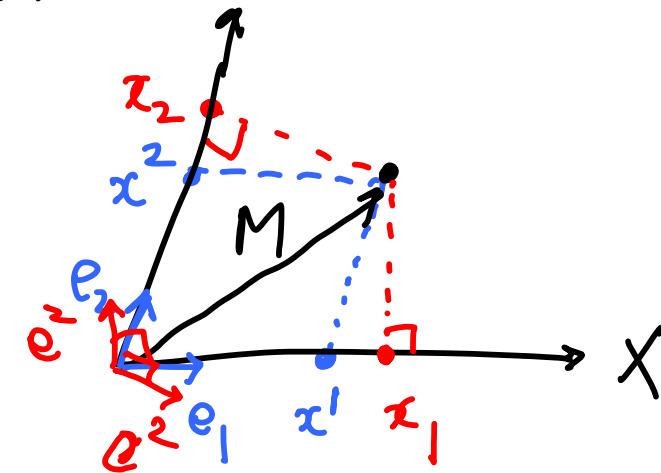
covariant, contra-variant

$$(x_1, x_2)$$

$$(x^1, x^2)$$

$$\underline{M} = \sum M^j \underline{e}_j = x^1 \underline{e}_1 + x^2 \underline{e}_2$$

$$= \sum M_j \underline{e}^j = x_1 \underline{e}^1 + x_2 \underline{e}^2$$



Numerical grid generation
Thompson et al. (1985)

④ Grids

- Structured grid

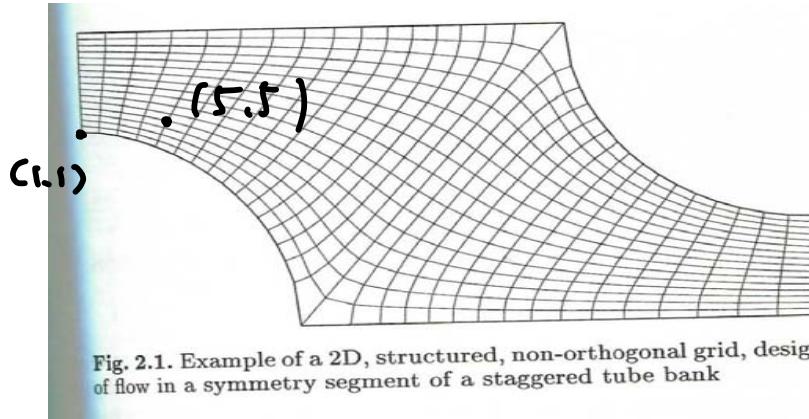


Fig. 2.1. Example of a 2D, structured, non-orthogonal grid, designed for calculation of flow in a symmetry segment of a staggered tube bank

- Block-structured grid

with matching interface

w/o " "

with overlapping blocks

(composite grid)
chimera "

can be treated in a fully
conservative manner

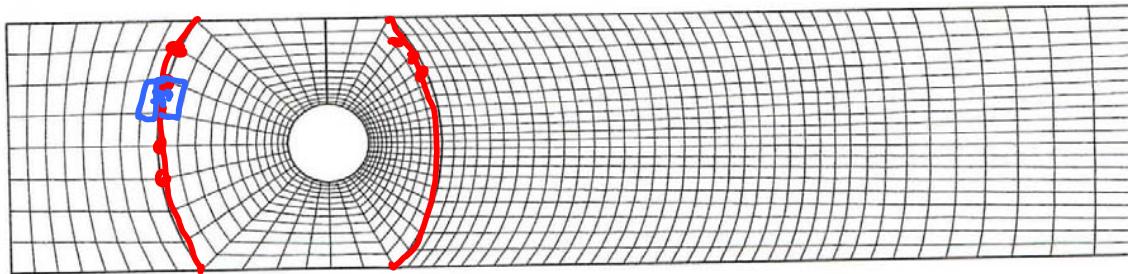


Fig. 2.2. Example of a 2D block-structured grid which matches at interfaces, used to calculate flow around a cylinder in a channel

C-grid

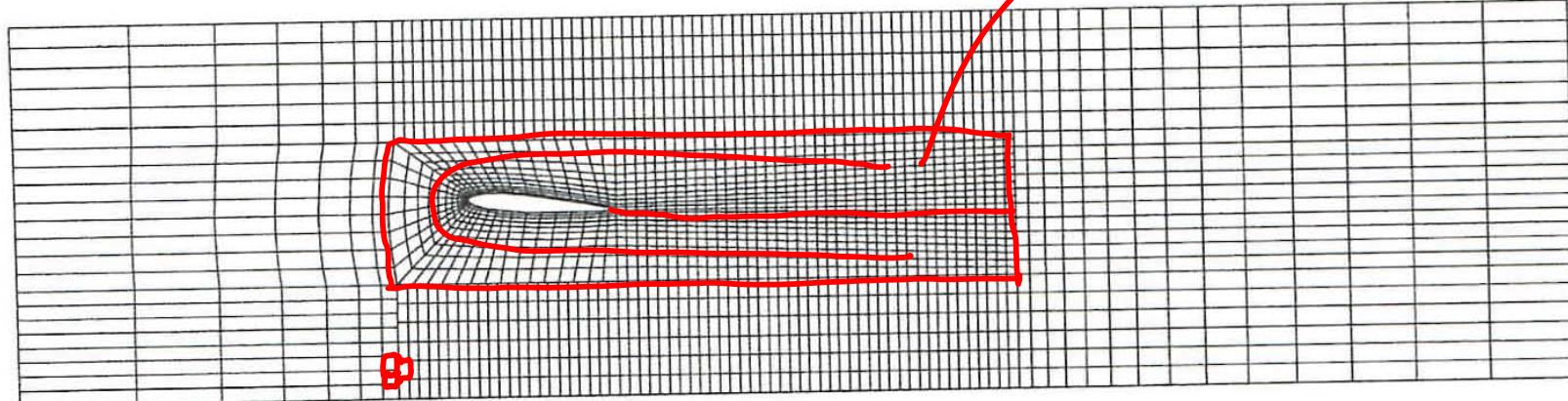


Fig. 2.3. Example of a 2D block-structured grid which does not match at interfaces, designed for calculation of flow around a hydrofoil under a water surface

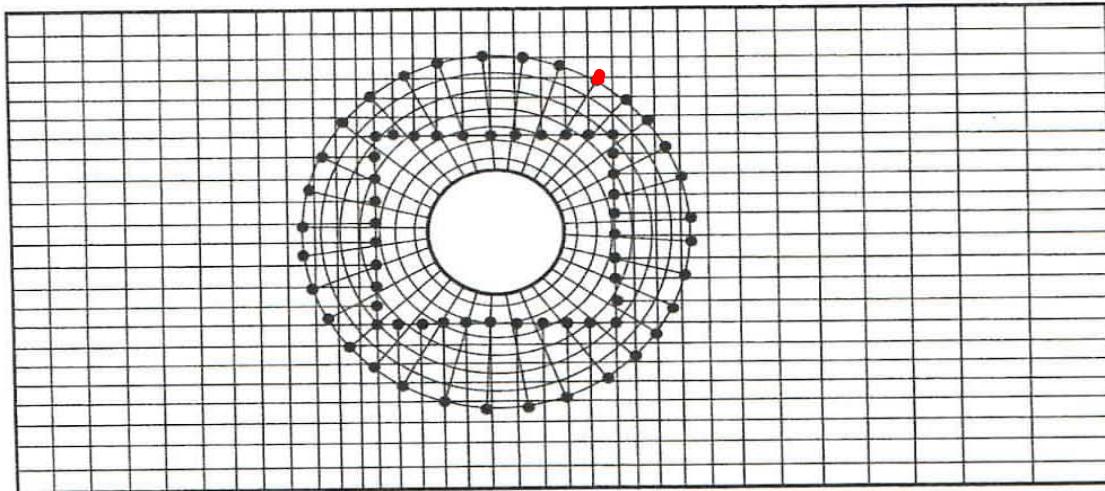


Fig. 2.4. A composite 2D grid, used to calculate flow around a cylinder in a channel

requires interpolation
 ↓
 difficulty in conserving
 properties
 but good for complex
 geometries!

- Unstructured grid
 - very complex geometries
 - good for FVM and FEM
 - irregularity of data structure

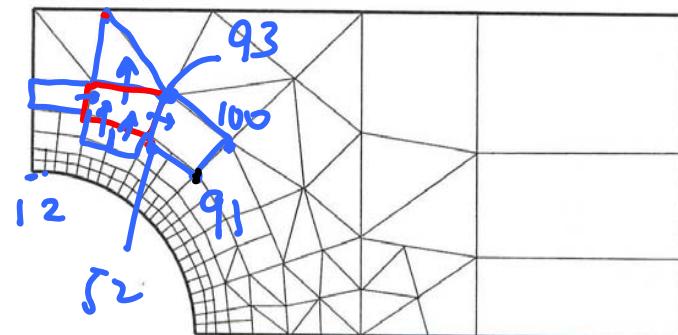


Fig. 2.5. Example of a 2D unstructured grid

PDE $\xrightarrow{\text{discretization}}$ ODE $\xrightarrow{\quad}$ algebraic eqs $Ax = b \neq x = A^{-1}b$

$$\begin{bmatrix} \text{---} & \text{---} & \text{---} \\ \text{---} & \ddots & \text{---} \\ \text{---} & \text{---} & \text{---} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{1000} \end{bmatrix} = \begin{bmatrix} \text{---} \end{bmatrix}$$

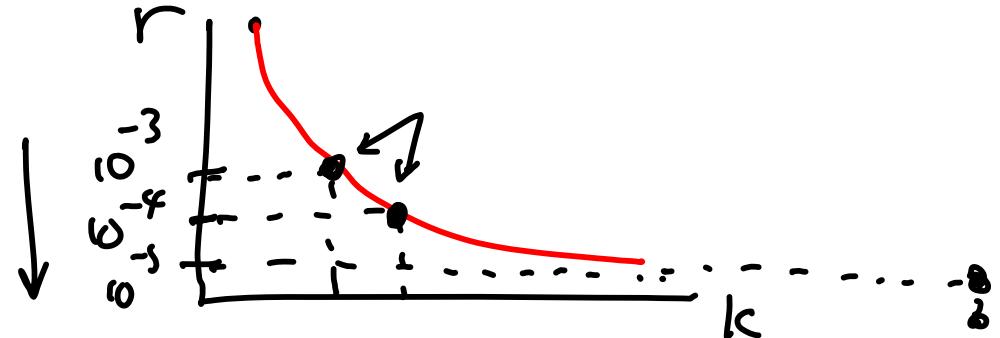
sparse matrix

requires good iterative solver

grid generation is difficult.

- ⑤ Finite approximations - accuracy, memory, ...
- ⑥ Solution method for nonlinear algebraic eqs.
- ⑦ convergence criteria $\cancel{\phi = A^{-1}b}$ residue
 $\nabla^2\phi = r \rightarrow A\phi = b \rightarrow$ iterative solver $r = b - A\phi^k$ $k:$ iteration number

$10^{-5} ?$
 10^{-1}
 $10^{-3} ?$
 10^{-10}



⑧ good physics.

3. Properties of numerical sol. methods

① Consistency: PDE $\xrightarrow[\text{method}]{\text{discretization}}$ Algebraic eqs.
 (N-seq.)

$$+ \frac{\partial x^2}{\partial t} \alpha - + \dots$$

should go to zero
 when $\partial x \rightarrow 0$
 $\partial t \rightarrow 0$

Algebraic eqs. $\xrightarrow{\text{modified PDE}}$

$$\rightarrow N\text{-Seq} + \underline{\partial t^2 \alpha + \partial x^2 \beta} + \dots$$

(Dufort-Frankel method is inconsistent) $\xrightarrow{\text{truncation error}}$

- ② Stability : (von Neumann stability analysis
modified wavenumber)
- ③ convergence : resolution test
- ④ conservation : strong conser. form of GE } conservative
+ FVM

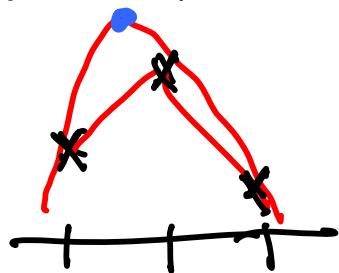
non-conservative scheme \rightarrow artificial source/sink



should go to zero as $\Delta x \rightarrow 0$



⑤ boundedness : difficult to guarantee



some first-order schemes guarantee this property
higher-order schemes can produce unbounded
sols.

⑥ realizability : models of phenomena should be correct.

ex) turbulence, combustion, multi-phase flow

⑦ accuracy

i) modeling error : difference bet. actual flow and exact sol. of
math. model

ii) discretization error : " " exact sol. of GE and
" " of algebraic eqs.

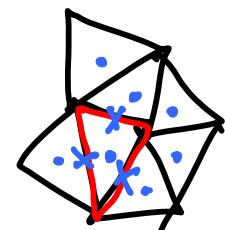
iii) iteration error : " " exact sol. of algebraic eqs
and iterative " " " "

4. Discretization approaches

노트 제목

2019-03-20

- ① FDM : finite difference method
- { oldest method
PDE → algebraic eqs.
simple and effective
easy to obtain high-order scheme on regular grids
conservation is not enforced
restricted to simple geometries
- ② FVM : finite volume method
- { integral eq. → algebraic eqs.
suitable for complex geometries
high-order schemes are difficult



③

FEM : finite element method

{ most popular in fluid mechanics because of conservation property

most popular in engineering
eqs. are multiplied by a weighting fn. in a way
that guarantees continuities of the sol. across
element boundaries

$$\int w \cdot G E \, dV = 0 \quad w=1 : FVM$$

$$w=\phi$$

lose conservation in general

arbitrary geometries \rightarrow sparse matrix

get high-order accuracy

④ Spectral method

$$u(x) \xrightarrow{\text{FT}} \hat{u}(k)$$

$$\underline{u(x)} = \sum_k \hat{u}(k) e^{ikx}$$

$$\underline{\frac{dy}{dx}(x)} = \sum_k ik \hat{u}(k) e^{ikx}$$

$$u \xrightarrow{\text{FT}} \hat{u}$$

$$\frac{dy}{dx} \xleftarrow{\text{IFT}}$$

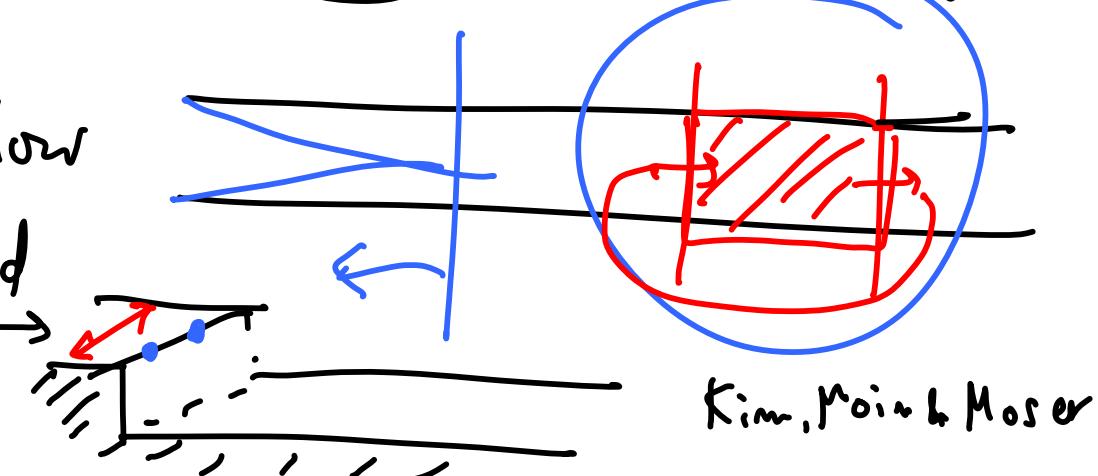
$$ik \hat{u}$$

very accurate
method.
requires
periodicity

periodicity of sol

fully developed pipe flow

flow over a backward
facing step



Kim, Poincaré, Moser