

HW3 : Do it again for  $\epsilon = \max |\phi_i^{\text{exact}} - \phi_i|$ .

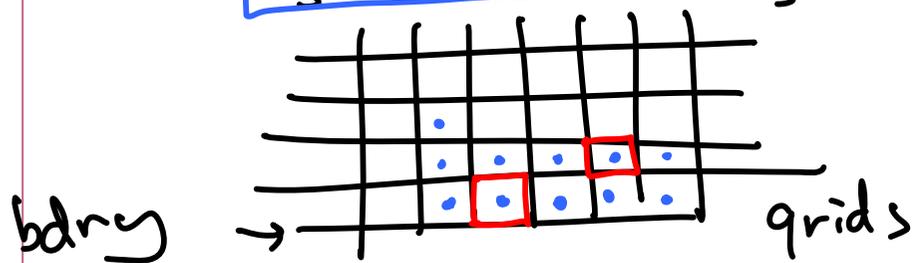
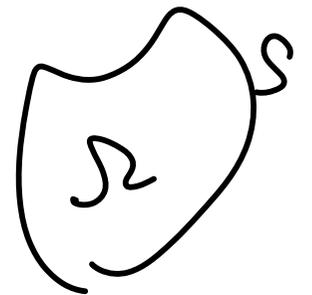
due : April 8 (Monday)

## Ch. 4. Finite volume methods (FVM)

FVM  $\rightarrow$  integral form of the G.E.

$$\frac{\partial}{\partial x_j} (\rho u_j \phi) = \frac{\partial}{\partial x_j} \left( \Gamma \frac{\partial \phi}{\partial x_j} \right) + g \phi$$

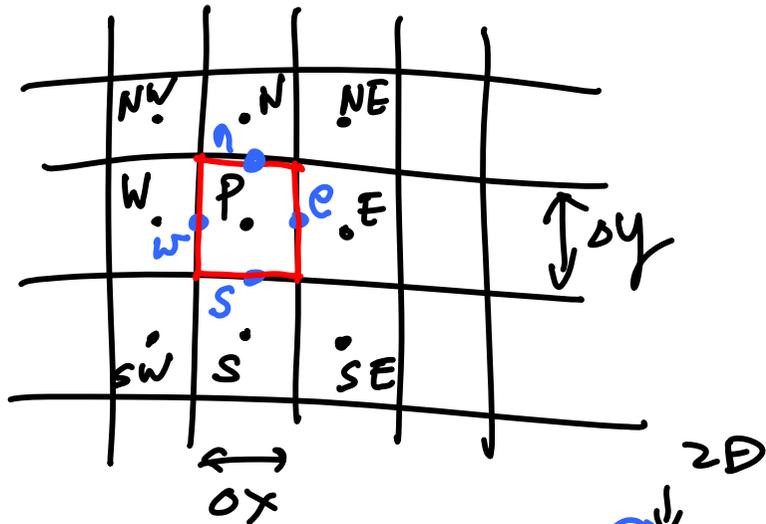
$$\rightarrow \int_S \rho \phi \underline{u} \cdot \underline{n} \, dS = \int_S \Gamma \nabla \phi \cdot \underline{n} \, dS + \int_{\Omega} g \phi \, d\Omega$$



• : location for variables

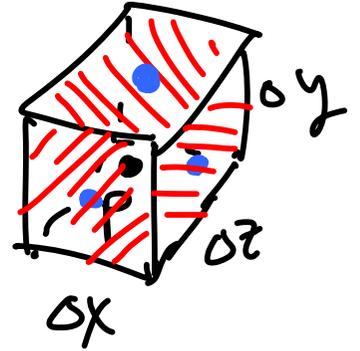
□ : control volume

2D



• : cell center  
 | or - : cell face

3D



$$\int_{S_e} f ds \approx f_e S_e = f_e \Delta y \quad , \quad \int_{\Omega} f d\Omega = f_p \Delta \Omega$$

(ΔxΔy)

how to determine ?

(variables are stored at cell center)

→ need interpolation

$$\int_S \rho \phi \underline{u} \cdot \underline{n} ds = \rho_e \phi_e (\underline{u} \cdot \underline{n})_e \Delta y$$

- upwind interpolation (UDS)

$$\phi_e = \begin{cases} \phi_p & \text{if } (u \cdot \eta)_e \geq 0 \\ \phi_E & \text{if } < 0 \end{cases}$$

→ This scheme never yields oscillatory sols.  
but is numerically diffusive. (dissipative)

Taylor series expansion about P

$$\boxed{\phi_e = \phi_p} + \underbrace{(x_e - x_p) \frac{\partial \phi}{\partial x} \Big|_p}_{\text{leading error term}} + \frac{1}{2} (x_e - x_p)^2 \frac{\partial^2 \phi}{\partial x^2} \Big|_p + \dots$$

UDS

leading error term

$$\sim \Gamma_n \frac{\partial \phi}{\partial x}$$

$$\Gamma_n = (\rho u) \frac{\Delta x}{2}$$

↳ numerical diffusion

$$\frac{\Delta x}{2} \sim x_e - x_p$$

$$\frac{\frac{\partial}{\partial x}(\rho u \phi)}{\frac{\partial}{\partial x}(\Gamma \frac{\partial \phi}{\partial x})}$$

- Linear interpolation (CAS)

$$\phi_e = \phi_E \lambda_e + \phi_p (1 - \lambda_e) \quad \text{2nd-order accurate}$$

$$\lambda_e = \frac{x_e - x_p}{x_E - x_p}$$

Taylor series expansion

$$\phi_e = \phi_E \lambda_e + \phi_p (1 - \lambda_e) - \frac{(x_e - x_p)(x_E - x_p)}{2} \frac{\partial^2 \phi}{\partial x^2} \Big|_p + \text{HOT}$$

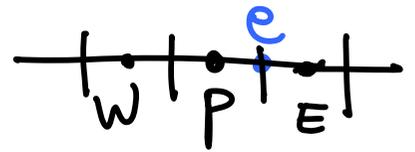
leading error term

$$\frac{\partial}{\partial x} (\text{err } \phi)$$

$$\frac{\partial^3 \phi}{\partial x^3} \Big|_p$$

sinusoidal oscillation ← dispersive error

For viscous term, we need  $\frac{\partial \phi}{\partial z}|_e$



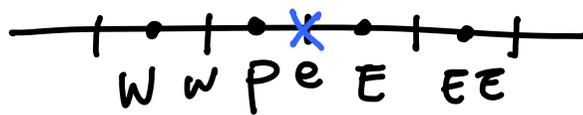
$$\frac{\partial}{\partial z} \left( \Gamma \frac{\partial \phi}{\partial z} \right)$$

$$\frac{\partial \phi}{\partial z}|_e = \frac{\phi_E - \phi_P}{x_E - x_P} + \frac{(x_e - x_P)^2 + (x_E - x_e)^2}{2(x_E - x_P)} \frac{\partial^2 \phi}{\partial x^2}|_P + \dots$$

CDS

2nd-order accurate on non-uniform grids with  $\Delta x_i = r_e \Delta x_{i-1}$ .

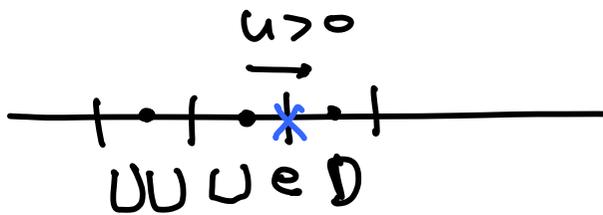
- QUICK (Quadratic Upwind Interpolation for Convective Kinematics) by Leonard (1979)
  - parabolic interpolation to evaluate variables at e.
    - ↳ 3 pts.



use data at one more point

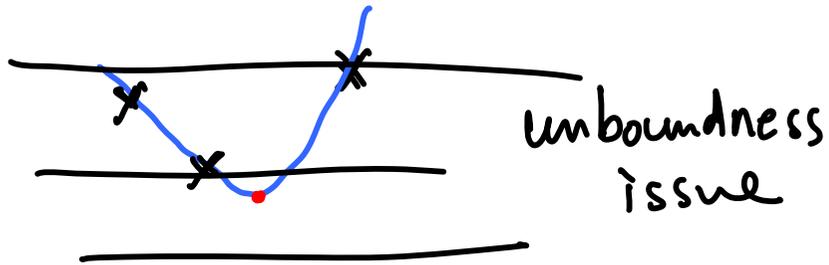
$W$  for  $(u \cdot \Delta)_e > 0$   
 $EE$  " "  $< 0$

QUICK



$$\phi_e = \phi_U + g_1(\phi_D - \phi_U) + g_2(\phi_U - \phi_{UU})$$

$$g_1 = \frac{(x_e - x_U)(x_e - x_{UU})}{(x_D - x_U)(x_P - x_{UU})}, \quad g_2 = \frac{(x_e - x_U)(x_D - x_e)}{(x_U - x_{UU})(x_P - x_{UU})}$$



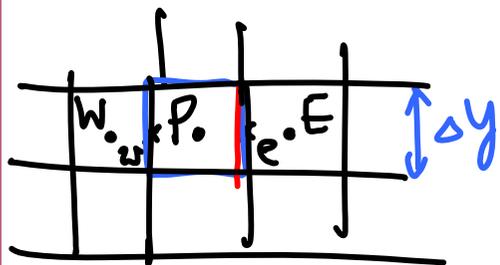
For uniform grids,

$$\phi_e = \begin{cases} \frac{3}{8}\phi_w + \frac{6}{8}\phi_p - \frac{1}{8}\phi_e & \text{if } u_e > 0 \\ \frac{3}{8}\phi_p - \frac{1}{8}\phi_{EE} + \frac{6}{8}\phi_E & \text{if } u_e < 0 \end{cases}$$

Taylor series exp.

$$\phi_e = \frac{3}{8}\phi_w + \frac{6}{8}\phi_p - \frac{1}{8}\phi_e - \frac{3}{64} \frac{\partial^3 \phi}{\partial x^3} \Big|_p + \text{HOT}$$

$O(\Delta^3)$

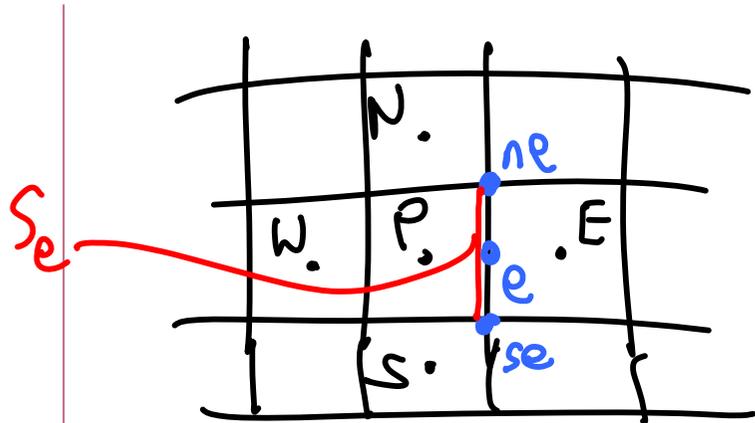


we need  $\int \phi_e dy \rightarrow \phi_e \Delta y \approx$  2nd-order accurate

$\therefore$  total 2nd-order accurate

$\Delta x$

- Higher-order schemes make sense only if integrals (i.e.  $\int_{\xi_e} f dy$ ) are approximated using higher-order formulae



$$\int_{S_e} f dy = \frac{S_e}{6} (f_{ne} + 4f_e + f_{se}) + O(\Delta y^k)$$

too much

For uniform mesh,  $\phi_e = \frac{1}{48} (27\phi_p + 27\phi_E - 3\phi_W - 3\phi_{EE}) + O(\Delta x^k)$

$$\frac{\partial \phi}{\partial x} \Big|_e = \frac{1}{24\Delta x} (27\phi_E - 27\phi_p + \phi_W - \phi_{EE}) + O(\Delta x^k)$$

CD4

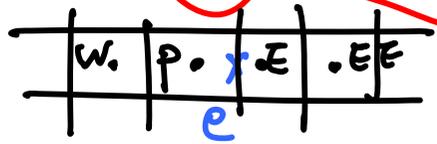
• Padé (compact) scheme

$\int \frac{\partial}{\partial x} (\rho u)$

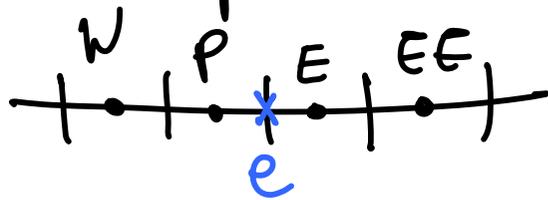
$$\phi_e = \frac{\phi_p + \phi_E}{2} + \frac{\Delta x}{8} \left[ \left( \frac{\partial \phi}{\partial x} \right)_p - \left( \frac{\partial \phi}{\partial x} \right)_E \right] + O(\Delta x^k)$$

$$\frac{\phi_E - \phi_W}{2\Delta x} - \frac{\phi_{EE} - \phi_p}{2\Delta x} + O(\Delta x^2) \quad O(\Delta x^3)$$

$\frac{df}{dx} \Big|_H \left( \frac{df}{dx} \Big|_j \right) \frac{df}{dx} \Big|_H$   
 $f_j \quad f_{j-1} \quad f_{j+1}$



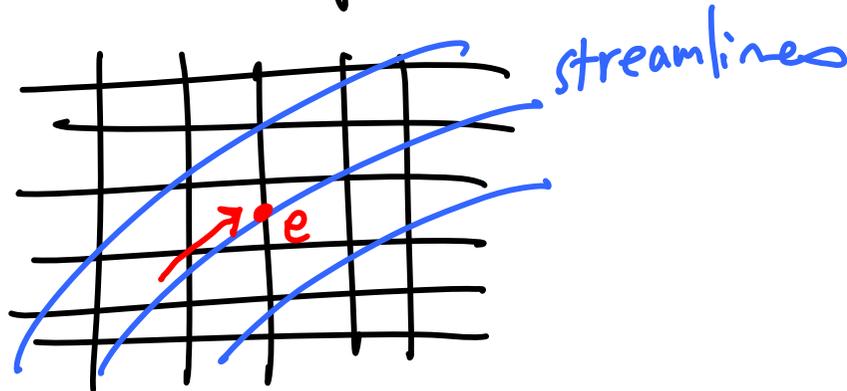
- Linear upwind scheme (LUDS) : 2nd-order upwind scheme



$\phi_e$  : linear extrapolation  
 from  $\begin{cases} \phi_P & \& \phi_W & \text{if } u_e > 0 \\ \phi_E & \& \phi_{EE} & \text{if } u_e < 0 \end{cases}$

→ may produce unboundedness  
 in sol.

- Skew upwind scheme (Raithby 1976)



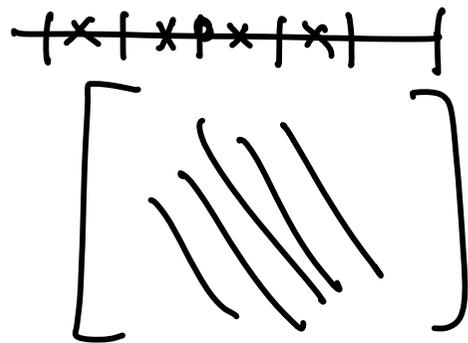
upstream nodes are from  
 streamlines rather than grid line.  
 → complex  
 may produce oscillatory sol.

- hybrid scheme (Spalding 1972)

UD  $\leftrightarrow$  CD depending on cell Peclet #.

- deferred correction

higher-order interpolation  $\rightarrow$  sparse matrix



low-order

high-order

hard to invert

$$\phi_e \approx \phi_e^L + (\phi_e^H - \phi_e^L)^{old}$$

usually upwind  
scheme  $\Rightarrow$  LHS

RHS

} iteration

• Example 1:  $\int_S \rho \phi (\underline{u} \cdot \underline{n}) dS = \int_S r (\nabla \phi \cdot \underline{n}) dS$

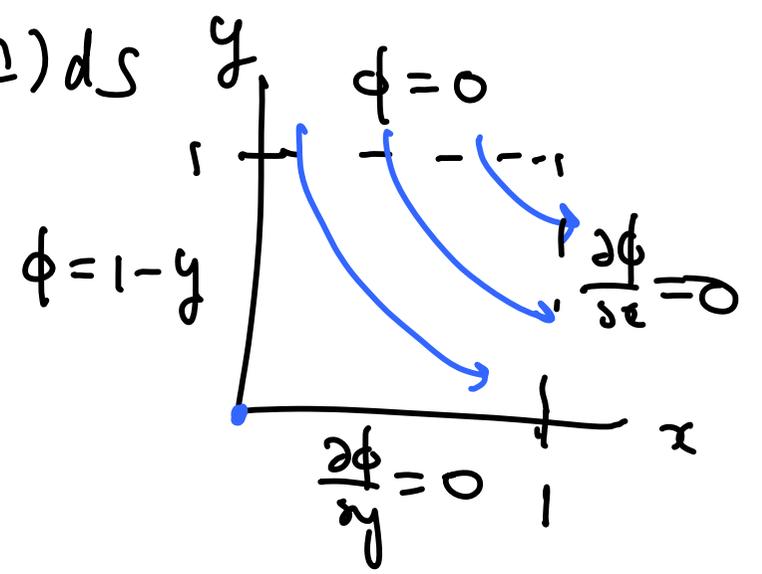
$\underline{u} = (u_x, u_y) = (x, -y)$   
 stagnation flow

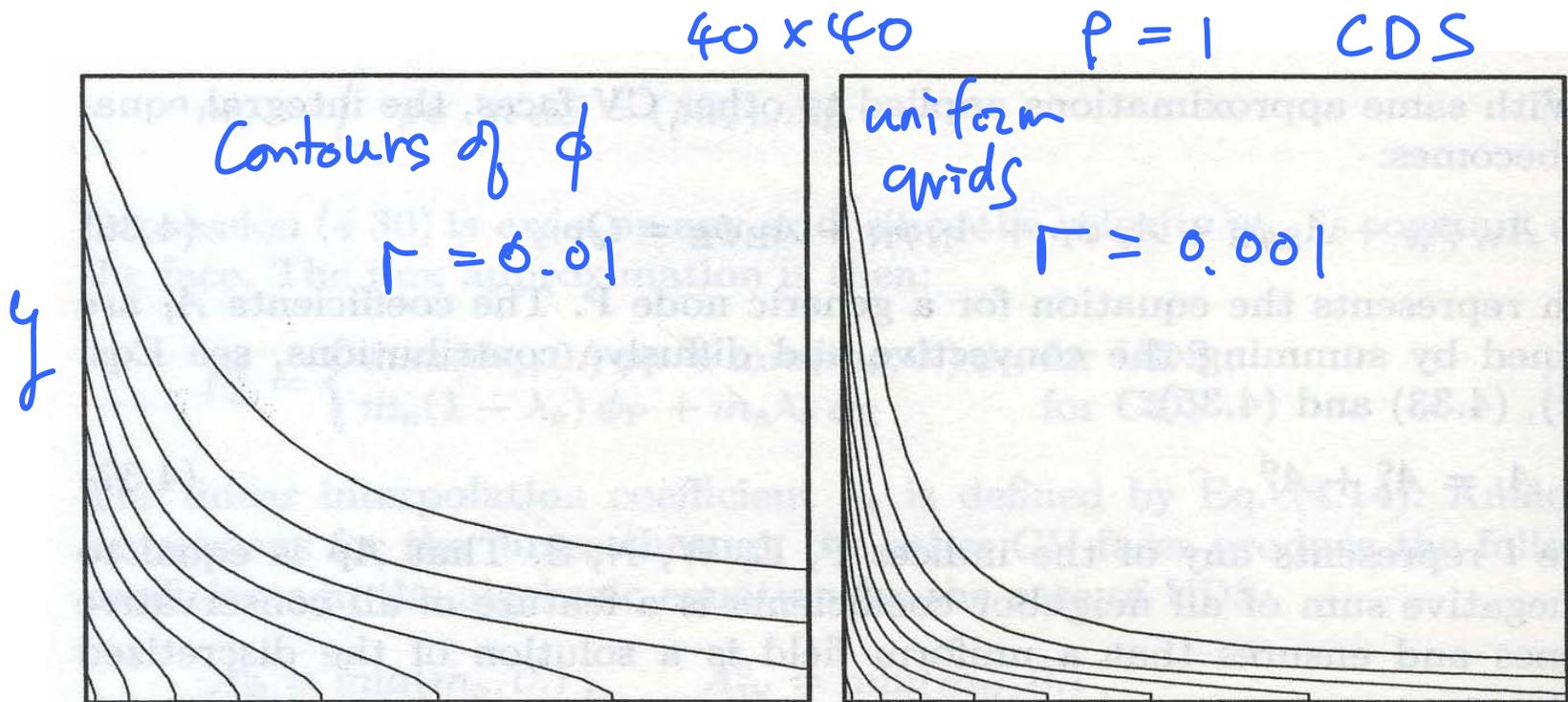
convection term: UDS or CDS

diffusion " : CDS

①  $x=1, \frac{\partial \phi}{\partial x} = 0 \leftarrow$  one-side difference

②  $y=0, \frac{\partial \phi}{\partial y} = 0 \leftarrow$  " "





**Fig. 4.5.** Isolines of  $\phi$ ,  $\alpha$  from 0.05 to 0.95 with step 0.1 (top to bottom), for  $\Gamma = 0.01$  (left) and  $\Gamma = 0.001$  (right)

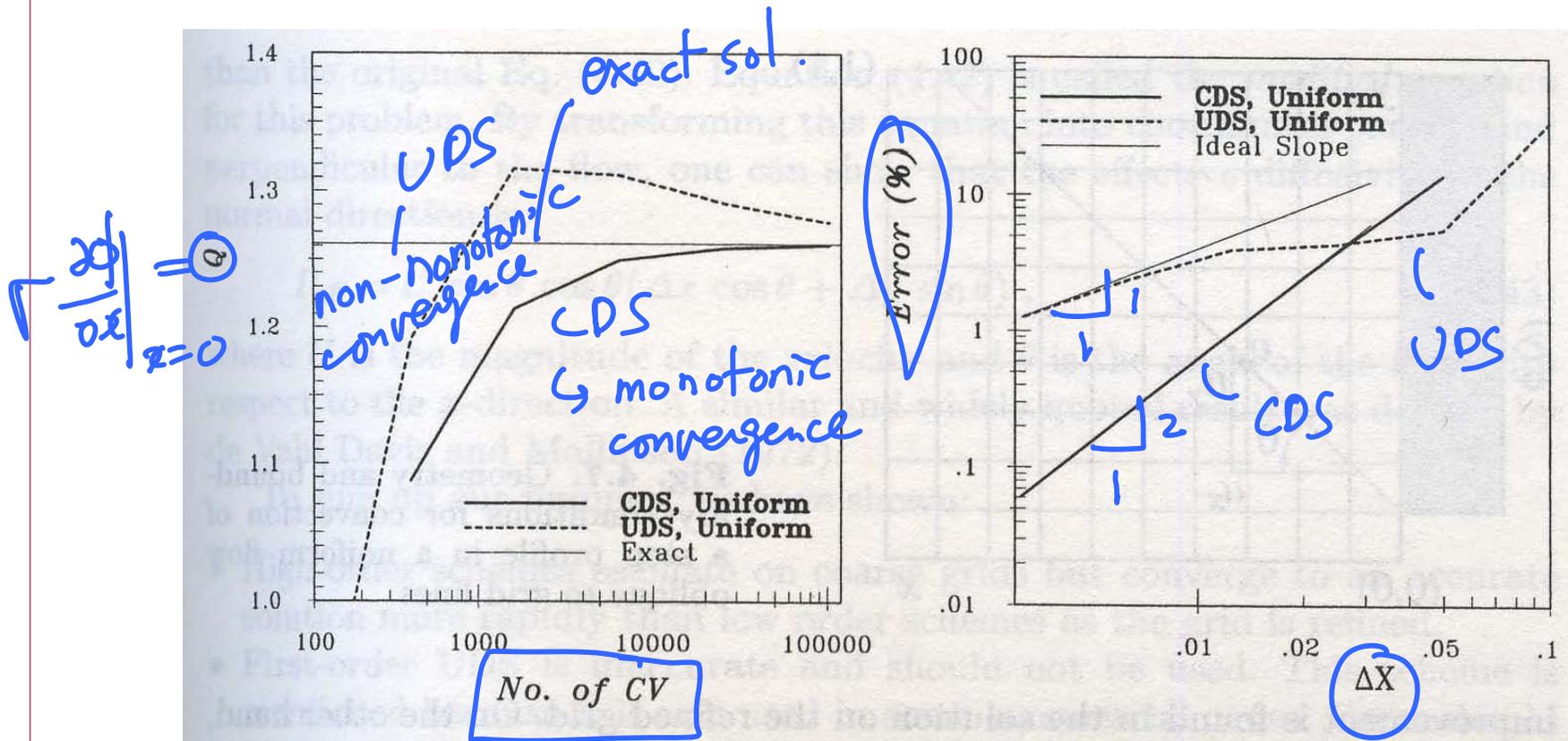
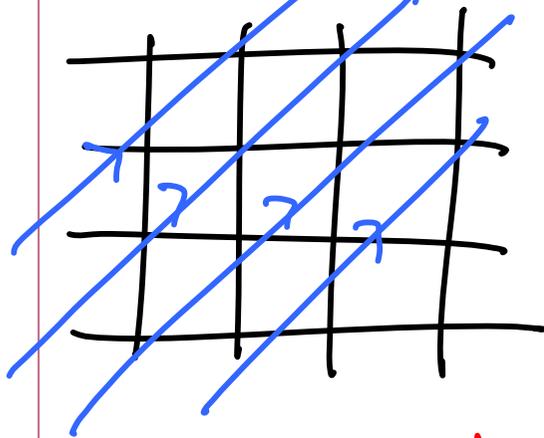


Fig. 4.6. Convergence of total flux of  $\phi$  through the west wall (left) and the error in computed flux as a function of grid spacing, for  $\Gamma = 0.001$

HW 4 : (due date : April 15)

Example 2:  $u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} = 0$  no diffusion



$u=v$  (const)

bad  
CDS

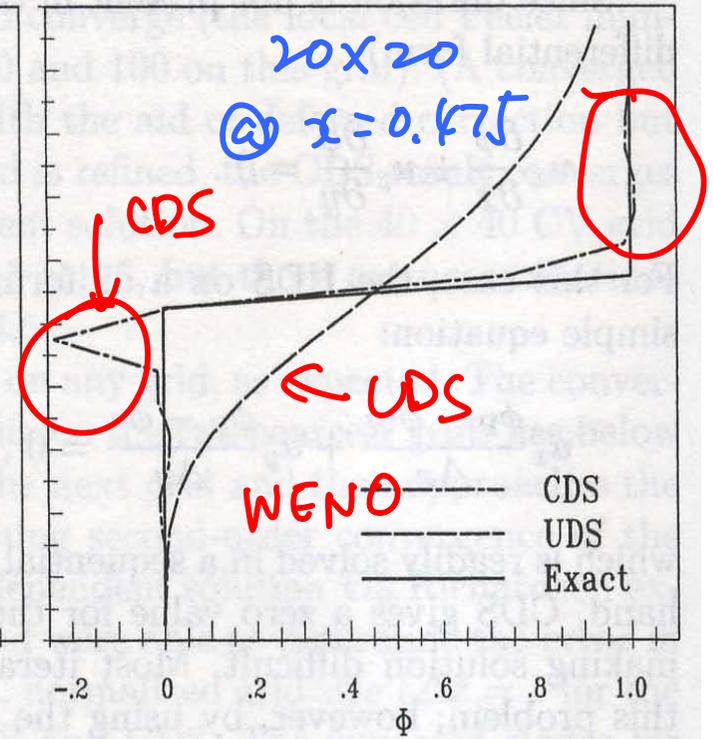
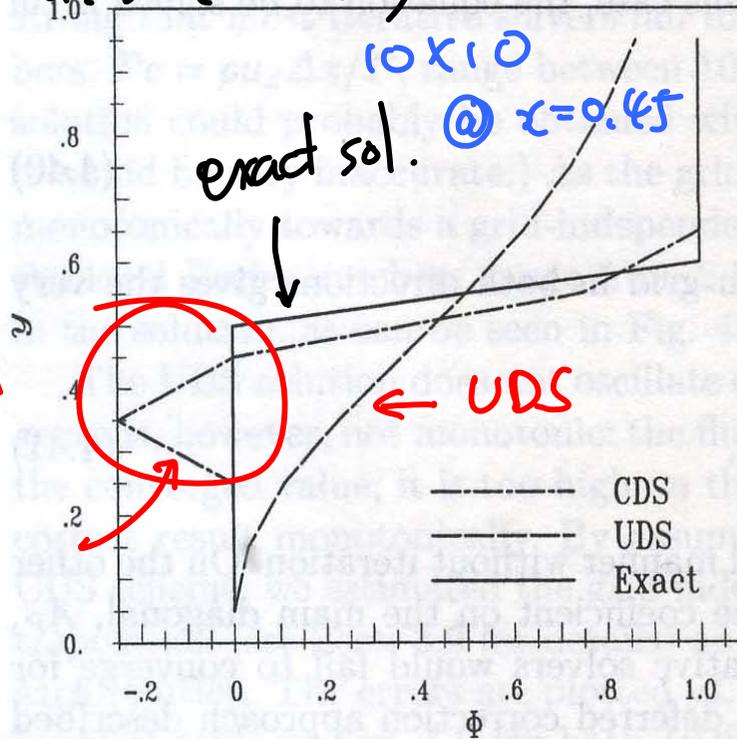
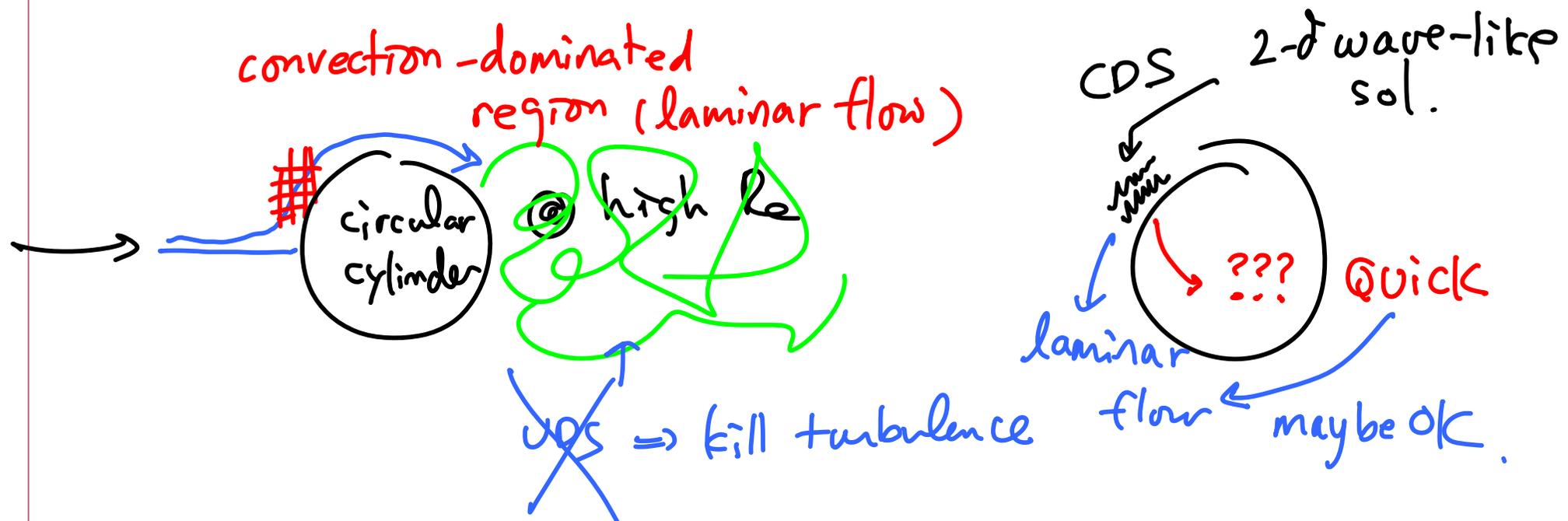


Fig. 4.8. Profile of  $\phi$  at  $x = 0.45$ , calculated on a  $10 \times 10$  CV grid (left), and at  $x = 0.475$ , calculated on a  $20 \times 20$  CV grid (right)



## Summary

1. High-order schemes oscillate on coarse grids but converge to an accurate sol. more rapidly than lower-order schemes as the grid is refined.

2. First-order CDS is inaccurate and should not be used.  
(but still used in commercial codes)

High accuracy cannot be obtained on affordable grids with this method, especially in 3D.

3. CDS is the simplest scheme of 2nd-order accuracy and offers a good compromise among accuracy, simplicity and efficiency.

Ch. 5 Solution of linear equation systems }  
Ch. 6 Method for unsteady problems } Numerical Method course.  
Ch. 7. Solution of the Navier-Stokes eqs. } (HW) Read them.