

Ch. 7. Solution of the Navier-Stokes eqs.

노트 제목

2019-04-08

1. Conservation properties

- mass conservation : continuity eq.

mtm " : Navier-Stokes eq.

energy " : thermal eq.

→ conservative approx. \Rightarrow FVM

- How about kinetic energy? \leftarrow biggest problem.

$$\frac{1}{2} \rho u^2$$

incomp. flow \rightarrow kinetic energy

comp. " \rightarrow kinetic & thermal energy.

$$\cdot \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) - \frac{2}{3} \frac{\partial}{\partial x_j} \left(\mu \frac{\partial u_k}{\partial x_k}\right) \delta_{ij} + \rho b_i \quad (1)$$

multiply $\cancel{\rho}$ by u_i :

$$u_i \frac{\partial}{\partial t}(\rho u_i) + u_i \frac{\partial}{\partial x_j}(\rho u_i u_j) = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) - \frac{2}{3} u_i \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_k}{\partial x_k}\right) \delta_{ij} + \rho u_i b_i$$

$$u^2 = u_i u_i \rightarrow \frac{1}{2} \rho u^2 : \text{kinetic energy}$$

$$\begin{aligned} \rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2} \rho u^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2} \rho u^2 u_j\right) &= -\frac{\partial}{\partial x_i}(\rho u_i) + \rho \frac{\partial u_i}{\partial x_i} + \frac{\partial}{\partial x_j}(u_i \mu \frac{\partial u_i}{\partial x_j}) \\ &\quad - \mu \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} - \frac{2}{3} \frac{\partial}{\partial x_j} \left(\mu u_i \frac{\partial u_k}{\partial x_k}\right) + \frac{2}{3} \mu \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_k} \delta_{ij} + \rho u_i b_i \end{aligned}$$

$\int_{S_2} :$

$$\begin{aligned} \frac{\partial}{\partial t} \int_{S_2} \frac{1}{2} \rho u^2 dA &= - \int_S \frac{1}{2} \rho u^2 u_j n_j dA - \int_S \rho u_i n_i dA + \int_S u_i \left(\mu \frac{\partial u_i}{\partial x_j} - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) n_j \\ &\quad + \int_{S_2} \left[\rho \frac{\partial u_i}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \left(\mu \frac{\partial u_i}{\partial x_j} - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) + \rho u_i b_i \right] dA \end{aligned}$$

Discussion

- ① first 3 terms on RHS : kinetic energy in Ω is not changed by the action of convection and pressure within the CV.
if there are no viscosity
no internal energy ($\frac{\partial u_k}{\partial x_k} = 0$) → then globally kinetic energy is
no mtn forcing. conserved.
- property that we want to
preserve in a numerical method.
- ② We solve mtn eq. (not kinetic energy eq.)
conservative scheme for mtn eq. does not
guarantee conservation of energy.
→ difficult to conserve energy numerically.

(3) If numerical method is energy conservative,
total kinetic energy does not grow in time.

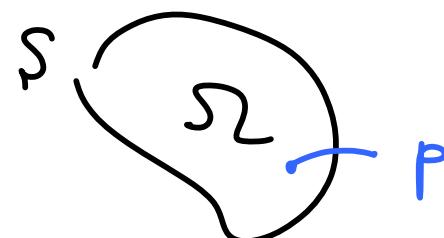
- velocity at every pt. in the domain must remain bounded.
- guarantees numerical stability (but not accuracy)
- kinetic energy conservation is important in computing unsteady flows.

(4) pressure gradient term

$$u_i \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i} (\rho u_i) - p \frac{\partial u_i}{\partial x_i}$$

if incomp. flow, $\frac{\partial u_i}{\partial x_i} = 0$

$$\hookrightarrow \int_{S_2} u_i \frac{\partial p}{\partial x_i} dA = \int_{S_2} \frac{\partial}{\partial x_i} (\rho u_i) dA = \int_S \rho u_i n_i dA$$



→ pressure influences the overall kinetic energy budget only by its action at the surface.

→ We have to retain this property.

$G_i P$ ($= \frac{\partial p}{\partial x_i}$) : numerical approx. of the press. grad.

$$\sum u_i \cdot (u_c - m_{in}) \rightarrow \sum_{i=1}^N u_i G_i P \Delta \Omega . \quad \left(u_i \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i} (p u_i) - p \frac{\partial u_i}{\partial x_i} \right)$$

$$\rightarrow \textcircled{=} \sum_{S_b} p u_n \Delta S - \sum_{i=1}^N \underbrace{P D_i u_i}_{\text{num. approx. of } \frac{\partial u_i}{\partial x_i}} \Delta \Omega$$

equality is ensured only if

G_i & D_i are compatible.

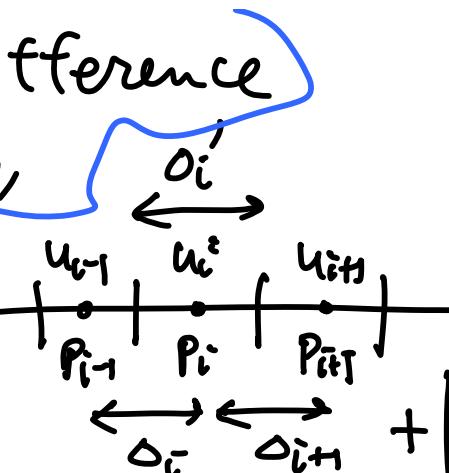
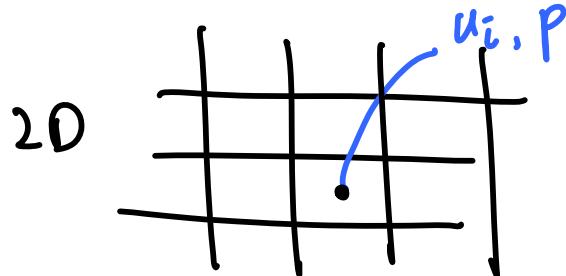
num. approx. of $\frac{\partial u_i}{\partial x_i}$

$$\Rightarrow \text{i.e. } \sum_{i=1}^N (u_i G_i P + p D_i u_i) \Delta \Omega = \text{surface terms.}$$

① G_i : backward difference

D_i : forward " "

i) collocated mesh



$$u_i G_i P \Delta \Omega |_i = u_i - \frac{P_i - P_{i-1}}{\delta_i} \cdot \Delta'_i$$

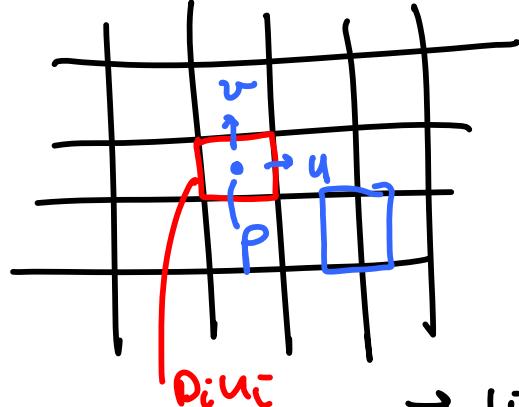
$$P D_i u_i \Delta \Omega |_i = P_i \frac{u_{i+1} - u_i}{\delta_{i+1}} \cdot D'_i$$

$$(u_i G_i P \Delta \Omega + P D_i u_i \Delta \Omega)_i = (-u_{i-1} P_{i-1} + P_i u_{i+1})$$

(if $\Delta'_i = \Delta_i = \delta_{i+1}$)

$$\sum_i (-) = -u_0 P_0 + u_{N+1} P_N \therefore \text{OK}$$

∴) Staggered mesh



$$\begin{aligned}
 & u_i G_i p \Delta R \Big|_* = u_i \frac{p_i - p_{i-1}}{\delta_i''} \cdot \delta_i \\
 & + p D_i u_i \Delta R \Big|_0 = p_i \frac{u_{i+1} - u_i}{\delta_{i+1}''} \cdot \delta_i' \\
 \rightarrow u_i G_i p \Delta R + p D_i u_i \Delta R &= -u_i p_{i-1} + p_i u_{i+1} \quad \text{if } \delta_i'' = \delta_i' = \delta_i
 \end{aligned}$$

$$\sum_i () = -u_i p_0 + u_N p_N \quad \underline{\text{OK}}$$

② G_i : CD2 collocated mesh staggered mesh

D_i : CD2

↓
not good

↓
good

③ G_i : forward
 D_i : "

→ not good.

Hw5
(Apr. 15)

* The requirement that only bdry terms remain when the sum over all control volumes is taken is not easily satisfied for other two terms (conv. & diff.). Especially difficult for arbitrary and unstructured meshes.

$$(5) \text{ pressure : } \nabla \cdot (\text{N-S. eqs.}) \rightarrow \nabla^2 p = r : \text{Poisson eq.}$$

(incomp. flow)

$$-\nabla p \quad \begin{matrix} \downarrow \\ \nabla \cdot (\nabla p) \end{matrix}$$

divergence gradient

D_i G_i

numerical operators

for D_i & G_i should be consistent if mass conservation is to obtain.

⑥ Incomp. flow w/o body force b_i ,
the only remaining vol. integral is the viscous term

$$-\int_{\Omega} \mu \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} d\Omega < 0$$

: dissipation \rightarrow to thermal energy.

⑦ $\frac{d}{dt}(\rho u_i) \rightarrow \oint \frac{u_c^{n+1} - u_c^n}{\Delta t}$ n : time step

$$\cancel{\frac{u_c^{n+1} - u_c^n}{2\Delta t}}$$

If the Crank-Nicolson method is used,
(CN)

$$(y' = f(y, t)) \rightarrow CN : \frac{y^{n+1} - y^n}{\Delta t} = \frac{1}{2} [f(y^n, t^n) + f(y^*, t^*)]$$

$$\oint_{\Omega} u_i \frac{\partial}{\partial t} (\rho u_i) : u_i^{n+\frac{1}{2}} \cdot \oint \frac{u_c^{n+1} - u_c^n}{\Delta t} \Delta \Omega = \frac{\rho \Delta \Omega}{\Delta t} \left[\frac{1}{2} (u_i^2)^{n+1} - \frac{1}{2} (u_i^2)^n \right]$$

$$\stackrel{?}{=} \frac{1}{2} (u_c^{n+1} + u_i^n)$$

$$\sum_{\text{time}} (\quad) = \frac{1}{2} u_f^2 - \frac{1}{2} u_i^2$$

$$y^l = f(y, t)$$

$$\text{EE: } \frac{y^{m+1} - y^l}{\Delta t} = f(y^l, t)$$

\therefore CN is energy conservative.

$$\text{If EE: } \int_{\Omega} u_i \frac{\partial}{\partial t} (f u_i) : \underbrace{u_i^{n+1} \int \frac{u_i^{n+1} - u_i^n}{\Delta t}}_{\Sigma(\) X} \circ \text{IE: } .. = f(y^m, t^{n+1})$$

$$\text{IE: } \int_{\Omega} u_i \frac{\partial}{\partial t} (f u_i) : \underbrace{u_c^{n+1} \int \frac{u_c^{n+1} - u_c^n}{\Delta t} . \Delta \Omega}_{\Sigma(\) X}$$

\rightarrow EE and IE are not energy conservative.

\rightarrow time differen (or integration) method can destroy the energy conservation property.

$$\frac{\partial}{\partial t} (\rho u_i) + \underbrace{H_i}_{\text{nonlinear term}} = - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \frac{\partial u_i}{\partial x_j} \right) : \text{incomp. flow}$$

노트 제목

2019-04-10

nonlinear term

convection "

$$\nabla \cdot (\rho u u)$$

$$H_i = \frac{\partial}{\partial x_j} (\rho u_i u_j) : \text{divergence form - conservative form}$$

$$H_i = u_j \frac{\partial}{\partial x_j} (\rho u_i) : \text{convective form - non-conservative form}$$

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = \dots$$

$$H_i = \frac{1}{2} \left(\frac{\partial}{\partial x_j} (\rho u_i u_j) + u_j \frac{\partial}{\partial x_j} (\rho u_i) \right) : \text{skew-symmetric form}$$

(best in stability)

$$H_i = u_j \left(\frac{\partial}{\partial x_j} (\rho u_i) - \frac{\partial}{\partial x_i} (\rho u_j) \right) + \frac{\partial}{\partial x_i} \left(\frac{1}{2} \rho u_j u_j \right) : \text{rotational form}$$

$$= \epsilon_{ijk} \rho u_j \omega_k + \frac{\partial}{\partial x_i} \left(\frac{1}{2} \rho u_j u_j \right)$$

$$\epsilon_{ijk} = \begin{cases} 1 \\ -1 \\ 0 \end{cases} \quad \epsilon_{123} = 1 \quad \epsilon_{132} = -1 \quad \epsilon_{112} = 0$$

These four H_i 's are mathematically identical
but numerically different.

rotational form

$$\frac{\partial}{\partial t}(fu_i) + \cancel{P\epsilon_{ijk}u_j\omega_k} = -\frac{\partial}{\partial x_i}(P + \frac{1}{2}\rho u_j u_j) + \frac{\partial}{\partial x_j}(u_i \frac{\partial u_i}{\partial x_j}) \quad \text{⊗}$$

" "

u_i

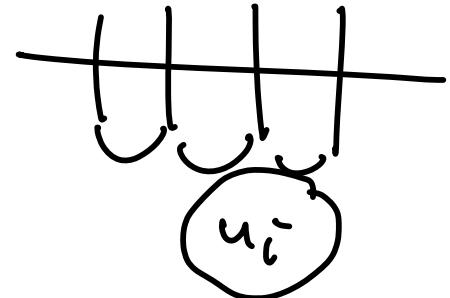
$$\cancel{P\epsilon_{ijk}u_i u_j \omega_k} = 0 \rightarrow \text{nonlinear term has no effect.}$$

anti-sym Sym

However, ⊗ is non-conservative form for mtm.

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} (u_i u_j) - u_i \frac{\partial u_j}{\partial x_j}$$

aliasing error & truncation error



Kravchenko & Moin (1997, JCP) #542 times
turbulent channel flow

$u_i u_j$

Nonlinear terms	aliased			dealiased		
	Spectral	FD 2	Pade 6	Spectral	FD 2	Pade 6
rotational	↓	•	•	•	•	•
divergence	↑	↑ +	↑	•	↑ +	↑
skew-sym.	•	•	•	•	•	•

•, stable ; ↑, unstable ; ↑+, unstable on collocated grid, but stable on staggered grid

\downarrow , flow laminarizes.

* kinetic energy conservation \rightarrow turbulence, weather prediction

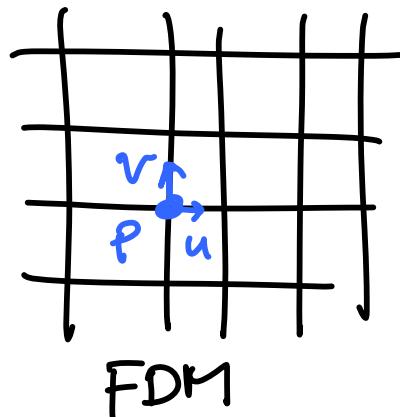
CD2 on staggered mesh \rightarrow mtm } conserving
energy } for divergence form
of N-S. eq.

Also, angular mtm conservation

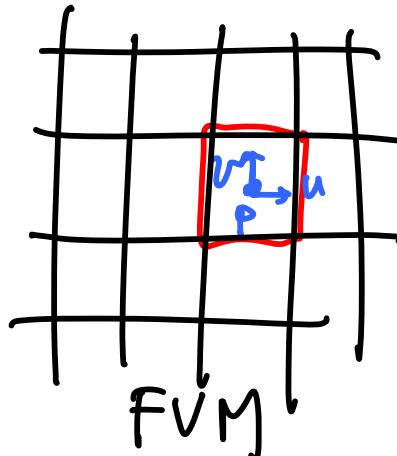
↳ for turbomachinery
↳ CD2 is much better than UDS.

2. choice of grid system

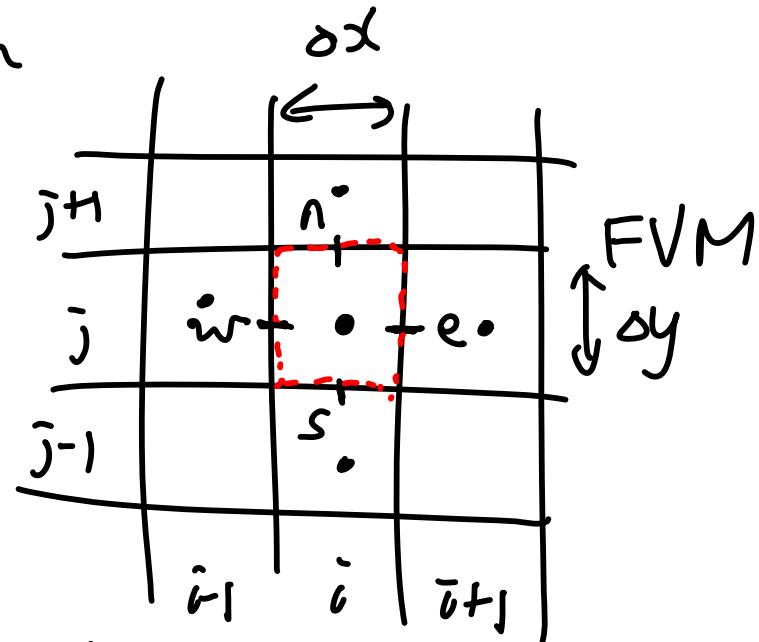
① Collocated (non-staggered) mesh



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



$$\text{CDZ : } \frac{u_e - u_w}{\Delta x} + \frac{v_n - v_s}{\Delta y} = 0$$

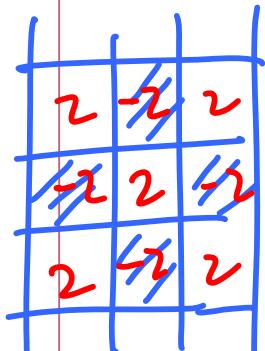
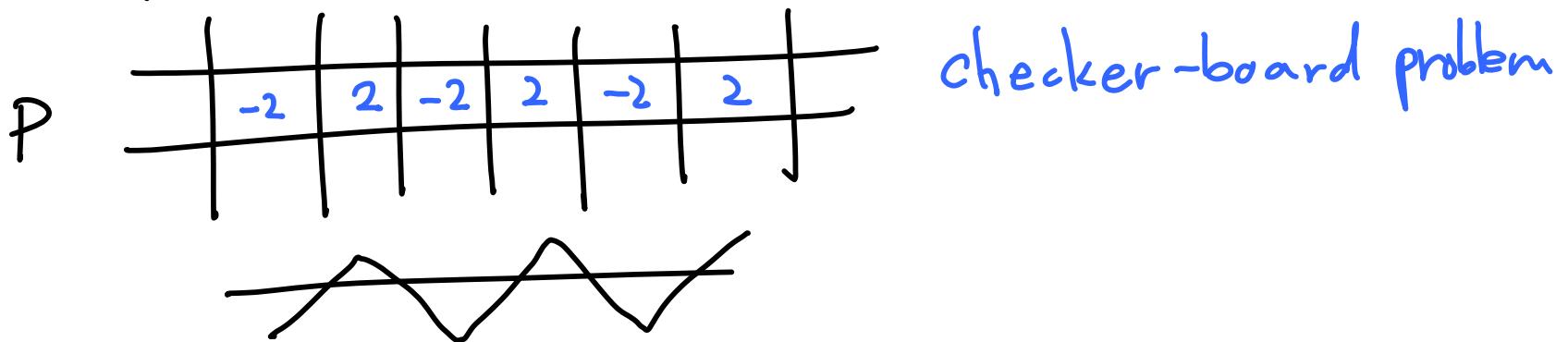


$$\left. \begin{aligned} u_e &= \frac{1}{2} (u_{i+1,j} + u_{i,j}) \\ u_w &= \frac{1}{2} (u_{i-1,j} + u_{i,j}) \end{aligned} \right\} \quad \left. \begin{aligned} u_e - u_w &= \frac{1}{2} (u_{i+1,j} - u_{i-1,j}) \end{aligned} \right\} \quad \begin{matrix} \text{no } u_{i,j} \\ v_{i,j} \end{matrix}$$

likewise, $v_n - v_s = \frac{1}{2} (v_{i,j+1} - v_{i,j-1})$ ↗

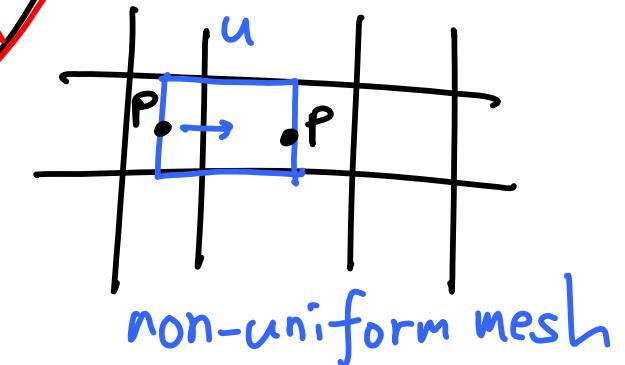
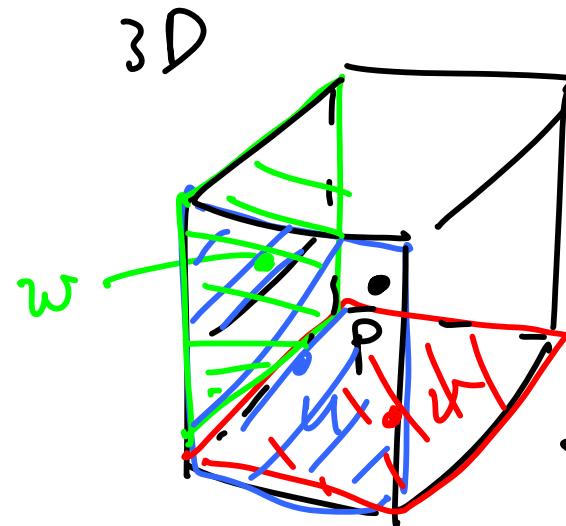
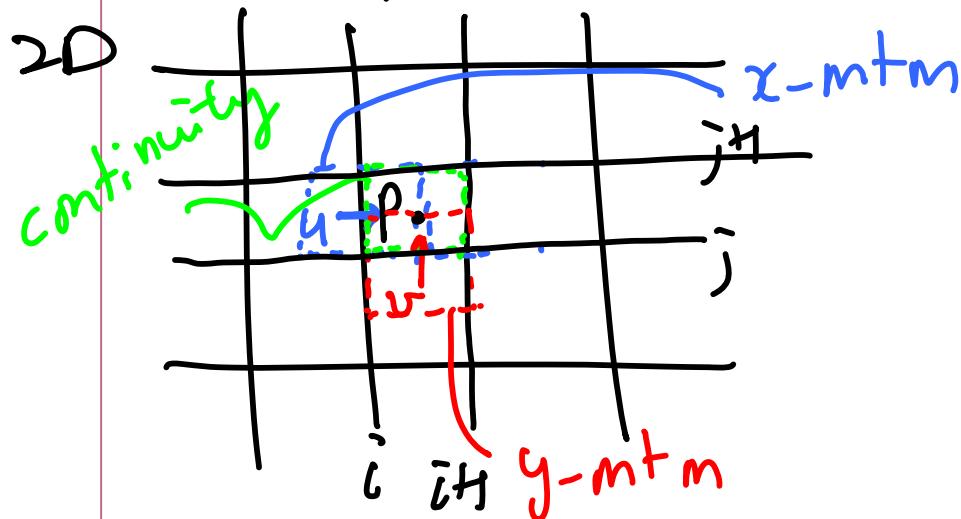
x-mtm eq. : $\frac{\partial p}{\partial x} = \frac{p_e - p_w}{\Delta x} = \frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x}$ no $p_{0,j}$

- p 's at adjacent grids do not talk each other
- decoupling bet. p and u_i .



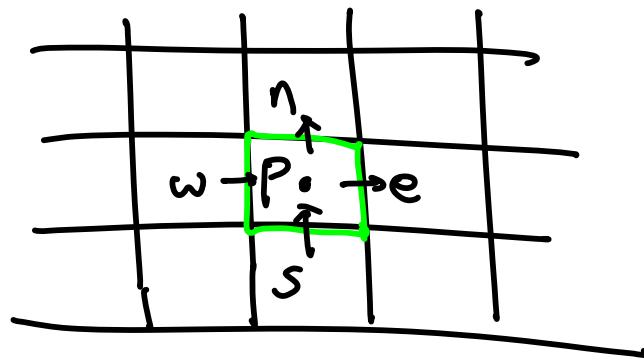
out of favor for incomp. flow
due to the difficulties w/ press-rel. decoupling
and occurrence of "spurious" pressure oscillations.

② staggered mesh



Harlow & Welch (1965) # 6859 times
 Numerical calculation of time-dependent
 viscous incomp. flow of fluid w/ free surface. Phys. Fluids.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 : \square$$



$$\frac{U_e - U_w}{\partial x} + \frac{V_n - V_s}{\partial y} = 0$$

$$\rightarrow \frac{U_{\bar{i}+1,\bar{j}} - U_{\bar{i},\bar{j}}}{\partial x} + \frac{V_{\bar{i},\bar{j}+1} - V_{\bar{i},\bar{j}}}{\partial y} = 0$$

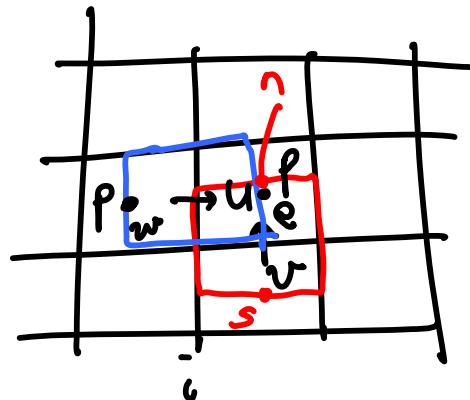
(no interpolation is required)
(compact)

x -mtm eq.

$$\frac{\partial P}{\partial x} = \frac{P_e - P_w}{\partial x} = \frac{P_{\bar{i},\bar{j}} - P_{\bar{i}-1,\bar{j}}}{\partial x}$$

again,

no interpolations



y -mtm eq

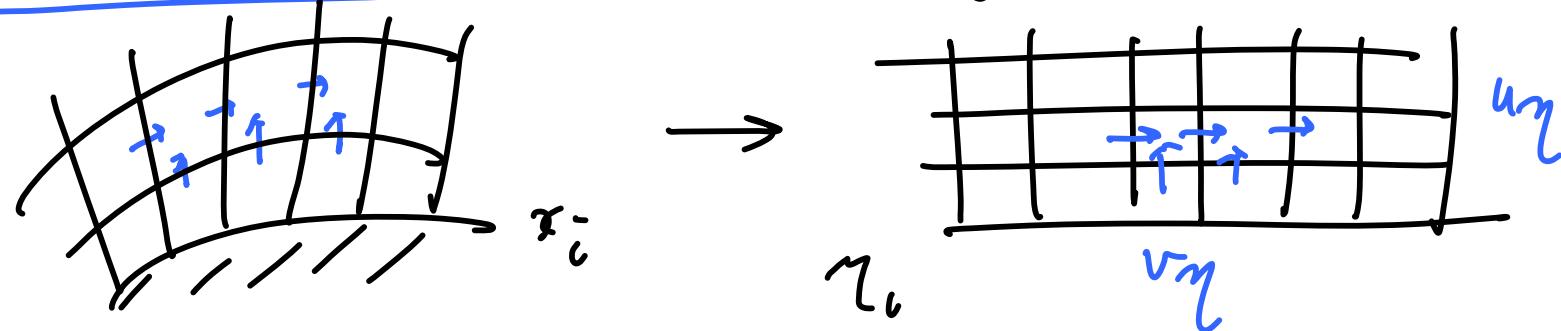
$$\frac{\partial P}{\partial y} = \frac{P_n - P_s}{\partial y} = \frac{P_{\bar{i},\bar{j}} - P_{\bar{i},\bar{j}-1}}{\partial y}$$

,,

\Rightarrow strong coupling between u_i & p .

\rightarrow no oscillation in p & u_i .

Owing to this property, staggered mesh is dominantly used.
however, for complex geometry, one has to transform N-S eq.
in generalized coords. w/ dependent variables of
contravariant velocity to satisfy $\nabla \cdot \underline{u} = 0$.



\rightarrow very complicated especially in 3D.

Choi, Moim & Kim (1993, JFM) - 2D # 649



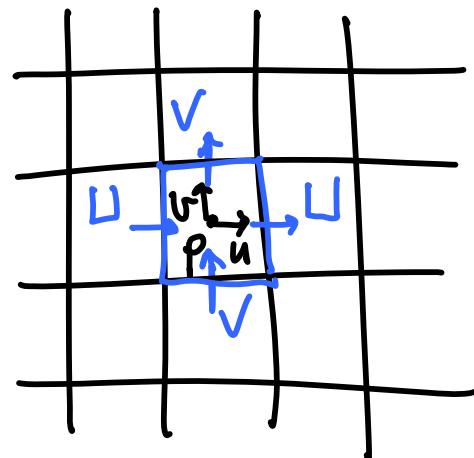
→ back to collocated mesh

with improved P-U_i coupling algorithm since 1980's.

Rhie & Chow, AIAA J. 21, 1525 (1983) → # 5103

collocated mesh,

momentum interpolation method

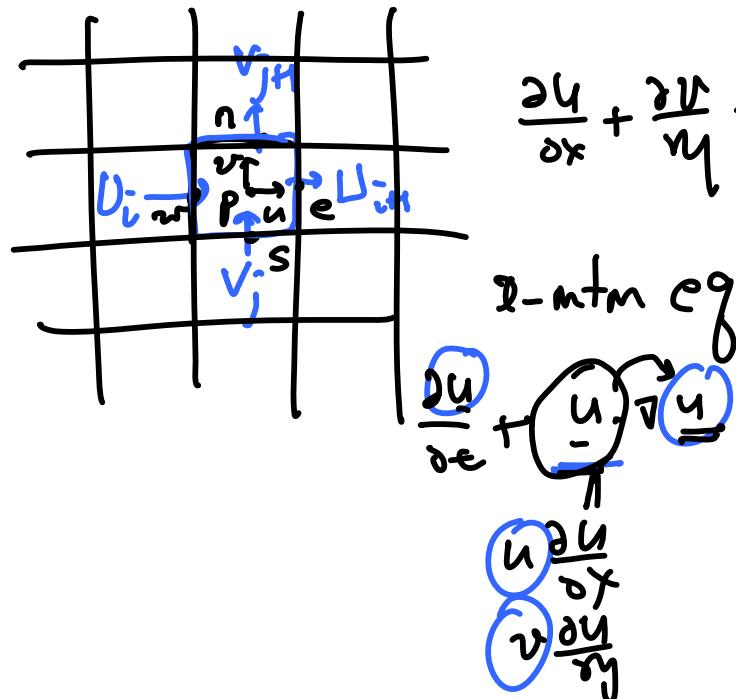


$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

$$\rightarrow \frac{U_{i+1,j} - U_{i,j}}{\Delta x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y} = 0$$

Rhie & Chow . momentum interpolation method collocated mesh

2019-04-15



$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \rightarrow \frac{U_{i+1,j} - U_{i,j}}{\Delta x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y} = 0$$

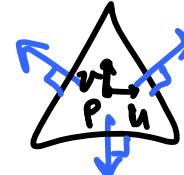
2-mtm eq : $\frac{\partial}{\partial x}(UU) = \frac{U_{i+1,j}U_e - U_{i,j}U_w}{\Delta x}$

$$\frac{\partial}{\partial y}(UV) = \frac{V_{i,j+1}U_n - V_{i,j}U_s}{\Delta y}$$

continuity eq. is not satisfied for U & V , but U & V :

Complex → unstructured mesh + FVM
+ collocated mesh

Kim & Choi (2000, JCP)

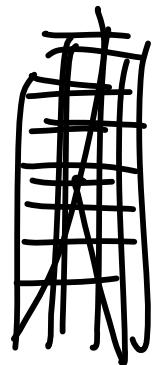


or immersed boundary method in Cartesian coord.

Peskin (JCP, 1972) #2034



Kim, Kim & Choi (JCP, 2001) #1156



3. Calculation of the pressure

Incomp. flow

p : pressure - a mathematical quantity to ensure the continuity.

① pressure eq.

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \frac{\partial u_i}{\partial x_j} \right)$$

$$\frac{\partial}{\partial x_i} ("") \Rightarrow$$

$$\left(\frac{\partial u_i}{\partial x_i} = 0 \right)$$

$$\frac{\partial}{\partial x_i} \frac{\partial p}{\partial x_i} = - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (\rho u_i u_j)$$

div. grad.

Poisson eq.

use consistent discretization method
otherwise, continuity is not satisfied.

② Simple explicit time advance scheme

$$\frac{\partial}{\partial t}(\rho u_i) = -\frac{\partial}{\partial x_j}(\rho u_i u_j) - \frac{\partial p}{\partial x_i} + \frac{\partial T_{tj}}{\partial x_j}$$

$$EE: \frac{(\rho u_i)^{n+1} - (\rho u_i)^n}{\Delta t} = -\frac{\partial}{\partial x_j}(\rho u_i u_j)^n - \frac{\partial p^n}{\partial x_i} + \frac{\partial T_{tj}^n}{\partial x_j}$$

$$(assume \frac{\partial u_i}{\partial x_i} = 0)$$

$$obtain u_i^{n+1} \rightarrow \frac{\partial}{\partial x_i}(\rho u_i)^{n+1} \neq 0$$

$$\frac{\partial}{\partial x_i}(*) : \frac{\partial}{\partial x_i}(\rho u_i)^{n+1} - \frac{\partial}{\partial x_i}(\rho u_i)^n = \Delta t \frac{\partial}{\partial x_i} \left[-\frac{\partial}{\partial x_j}(\rho u_i u_j)^n - \frac{\partial p^n}{\partial x_i} + \frac{\partial T_{tj}^n}{\partial x_j} \right]$$

$$We require \frac{\partial}{\partial x_i}(\rho u_i)^{n+1} = 0.$$

$$\rightarrow \frac{\partial}{\partial x_i} \left(\frac{\partial p^n}{\partial x_i} \right) = - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (\rho u_i u_j)^n \Rightarrow obtain p^n$$

$$\frac{du}{dt} = f$$

$$EE: \frac{u^{n+1} - u^n}{\Delta t} = f^n$$

✗

→ obtain u_i^{n+1} w/ p^n obtained.
from \star

$$\rightarrow \frac{\delta}{\delta x_i} (\rho u_i)^{n+1} = 0 \Rightarrow \Delta t \text{ is very small.}$$

EE. $\rightarrow \Theta(\Delta t)$ not good for time-accurate sol.

③ Simple implicit time advance method

$$IE: \frac{(p u)^{n+1} - (p u)^n}{\Delta t} = - \frac{\delta}{\delta x_j} (p u_i u_j)^{n+1} - \frac{\delta p^{n+1}}{\delta x_i} + \frac{\delta q_j}{\delta x_j}$$

$$\text{Assume } \frac{\delta (p u_i)}{\delta x_i} = 0$$

$$\text{we require } \frac{\delta}{\delta x_i} (p u_i)^{n+1} = 0$$

$$\frac{\delta}{\delta x_i} (\star): \frac{\delta}{\delta x_i} \frac{\delta p^{n+1}}{\delta x_i} = - \frac{\delta}{\delta x_i} \frac{\delta}{\delta x_j} (p u_i u_j)^{n+1} \xrightarrow{\text{unknown}}$$

$$\frac{du}{dt} = f$$

$$IE: \frac{u^{n+1} - u^n}{\Delta t} = f^{n+1}$$

$$+ \Theta(\Delta t) - \star$$

Thus, one has to solve $\textcircled{1}$ & $\textcircled{2}$ simultaneously.
 → iterative procedure.

Even if we know P^{n+1} , $\textcircled{2}$ contains nonlinear term.
 → iterative procedure.

or, linearization $u_i^{n+1} = u_i^n + \alpha t \frac{\partial u_i}{\partial t} + \dots$

$$= u_i^n + \Delta u_i + \dots \quad \mathcal{O}(\alpha t)$$

$$\begin{aligned} u_i^{n+1} u_j^{n+1} &= (u_v^n + \Delta u_v^n)(u_j^n + \Delta u_j^n) + \dots \\ &= u_v^n u_j^n + u_v^n \Delta u_j + u_j^n \Delta u_v + \underline{\Delta u_i \Delta u_j + \dots} \quad \mathcal{O}(\alpha t^2) \end{aligned}$$

$\therefore \text{neglect!}$

$$\begin{aligned} \textcircled{2} \longrightarrow \int \Delta u_i &= \alpha t \left[-\frac{\partial}{\partial x_j} (\rho u_i u_j)^n - \frac{\partial}{\partial x_j} (\rho u_i^n \Delta u_j) - \frac{\partial}{\partial x_j} (\rho u_j^n \Delta u_i) \right. \\ &\quad \left. - \frac{\partial P^n}{\partial x_i} - \frac{\partial \Delta P}{\partial x_i} + \frac{\partial \tau_{ij}^n}{\partial x_j} + \frac{\partial \Delta \tau_{ij}}{\partial x_j} \right] \end{aligned}$$

~~$\textcircled{1}$~~

direct sol. of ~~***~~ is still expensive.

use ADI to split eq. into a series of 1D problems.

So, the procedure is (without $\frac{\partial p}{\partial x_i}$) :

$$\text{Solve } \rho u_c^* = \alpha t \left[-\frac{\partial}{\partial x_j} (\rho u_i c_j)^n - \frac{\partial}{\partial x_j} (\rho u_i^n \Delta u_j^*) - \frac{\partial}{\partial x_j} (\rho u_j^n \Delta u_i^*) - \frac{\partial p^n}{\partial x_i} + \frac{\partial c_i^n}{\partial x_j} + \frac{\partial \Delta c_j^*}{\partial x_j} \right] \quad \rightarrow \text{***}$$

together with ADI.

$$\text{to obtain } \Delta u_i^* = u_i^* - u_i^n. \Rightarrow \frac{\partial}{\partial x_i} (\rho u_c^*) \neq 0.$$

$$\text{We require } \frac{\partial}{\partial x_i} (\rho u_c)^{n+1} = 0 \quad (\Delta u_i = u_i^{n+1} - u_i^n)$$

$$\text{From } \text{***} \text{ & } \text{****}, \rho u_i^{n+1} - \rho u_c^* = -\alpha t \frac{\partial \Delta p}{\partial x_i} + \text{error terms}.$$

$$\Rightarrow \frac{\partial}{\partial t}(\cdot) \Rightarrow \frac{\partial}{\partial x_i} \frac{\delta \phi}{\delta x_i} = \frac{1}{\delta t} \frac{\delta (\rho u_i^{*})}{\delta x_i}. \text{ Solve this eq. to get } \phi.$$

$$\Rightarrow (\rho u_i)^{*+} = (\rho u_i)^{*} - \delta t \frac{\delta \phi}{\delta x_i} \rightarrow \frac{\delta}{\delta x_i} (\rho u_i)^{*+} = 0$$

with implicit method, large δt can be used. stable!
but still $O(\delta t)$.

$$\frac{\partial u_i}{\partial t} = 0$$

$$\frac{\partial (\rho u_i)}{\partial t} + \dots .$$

$$\left[\begin{array}{l} \text{nonlinear} \\ \left[\begin{array}{l} u_{11} \\ u_{12} \\ \vdots \\ u_{NN} \\ v_{11} \\ v_{12} \\ \vdots \\ v_{NN} \\ p_{11} \\ \vdots \\ p_{NN} \end{array} \right] \end{array} \right] = \left[\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right]$$

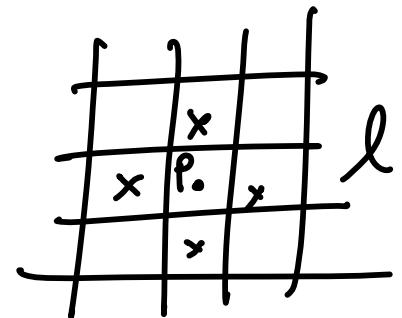
direct method
 \Downarrow
nearly impossible!

④ Implicit pressure-correction method

To get steady-state flow, use implicit method \rightarrow large Δt .

mtm eq. \leftarrow time & space discretizations

$$\rightarrow A_p u_i^{n+1} + \sum_l A_l u_{i+l}^{n+1} = Q_{u_i} - \frac{\partial p^{n+1}}{\partial x_i} \Big|_P \text{ & } \frac{\partial u_i^{n+1}}{\partial x_i} = 0$$



$A_p(u_i)$

all terms treated explicitly
as well as body force term
or other linearized term

iterative

Solver

(neglect Q_{u_i} from now on)

$$A_p u_i^{(m)} + \sum A_\ell u_{i,\ell}^{(m)} = - \frac{\delta p^{(m)}}{\delta x_i} |_p$$

m : iteration index

$$\rightarrow A_p u_i^{(m+1)} + \sum A_\ell u_{i,\ell}^{(m+1)} = - \frac{\delta p^{(m+1)}}{\delta x_i} |_p$$

$$\rightarrow u_i^{(m+1)} = - \frac{1}{A_p} \sum A_\ell u_{i,\ell}^{(m+1)} - \frac{1}{A_p} \frac{\delta p^{(m+1)}}{\delta x_i} |_p$$

$$\downarrow \\ = \tilde{u}_i^{(m+1)}$$

$$\frac{\delta}{\delta x_i} u_i^{(m+1)} \neq 0$$

$$\text{we require } \frac{\delta}{\delta x_i} u_i^{(m)} = 0$$

$$\rightarrow u_i^{(m)} = \tilde{u}_i^{(m+1)} - \frac{1}{A_p} \frac{\delta \phi}{\delta x_i} |_p$$

$$\begin{aligned} u_i^{(m)} &= u_i^{(m+1)} \\ &+ \frac{1}{A_p} \frac{\delta p^{(m+1)}}{\delta x_i} |_p - \frac{1}{A_p} \frac{\delta \phi}{\delta x_i} |_p \end{aligned}$$

$$\begin{aligned} u_i^{(m+1)} &= u_i^{(m)} + \frac{1}{A_p} \frac{\delta \phi}{\delta x_i} |_p - \frac{1}{A_p} \frac{\delta p^{(m+1)}}{\delta x_i} |_p \end{aligned}$$

$$\hookrightarrow A_p \tilde{u}_{i,p}^{m+1} = - \sum A_l u_{i,l}^{m+1} - \frac{\delta p^M}{\delta x_i} \Big|_p$$

$$\rightarrow A_p \tilde{u}_{i,p}^m + \sum A_l u_{i,l}^m = - \frac{\delta \phi}{\delta x_i} \Big|_p - \sum_l \left[\frac{\partial \phi}{\partial x_i} \Big|_l - \frac{\delta p^M}{\delta x_i} \Big|_l \right]$$

$\rightarrow \phi = p^M$ once converged

$$\rightarrow u_{i,p}^m = \tilde{u}_{i,p}^{m+1} - \frac{1}{A_p} \frac{\delta p^M}{\delta x_i} \Big|_p$$

$$\frac{\partial}{\partial x_i} (\cdot) \rightarrow \frac{\delta}{\delta x_i} \left(\frac{\delta p^M}{A_p \delta x_i} \right)_p = \frac{\delta \tilde{u}_c}{\delta x_i} \Big|_p \text{ by requiring } \frac{\delta u_i^m}{\delta x_i} = 0$$

\rightarrow obtain p^M

\rightarrow update u_i^m keep iteration

called "pressure-correction" method.

or "projection" method.

⑥ SIMPLE (Semi-implicit method for pressure-limited eq.)

$$u_i^M = u_i^{n+} + u_i', \quad p^M = p^{n+} + p'$$

$$\text{mtm: } A_p u_{i,p}^M + \sum_{\ell} A_{\ell} u_{i,\ell}^M = - \frac{\delta p^M}{\delta x_i}|_p \quad \frac{\delta u_i^M}{\delta x_i} = 0$$

$$-\boxed{A_p u_{i,p}^{n+} + \sum_{\ell} A_{\ell} u_{i,\ell}^{n+}} = - \frac{\delta p^{n+}}{\delta x_i}|_p \rightarrow \text{obtain } u_i^{n+}$$

$$A_p u_{i,p}' + \sum_{\ell} A_{\ell} u_{i,\ell}' = - \frac{\delta p'}{\delta x_i}|_p \quad \frac{\delta u_i^{n+}}{\delta x_i} \neq 0$$

$$\rightarrow u_{i,p}' = \underbrace{\frac{1}{A_p} \left(- \sum_{\ell} A_{\ell} u_{i,\ell}' \right)}_{\equiv \tilde{u}_{i,p}'} - \frac{1}{A_p} \frac{\delta p'}{\delta x_i}|_p - \textcircled{*}$$

(From continuity, $\frac{\partial u_i^m}{\partial x_i} = \frac{\partial u_r^{m*}}{\partial x_i} + \frac{\partial u_i^l}{\partial x_i^-} = 0$)

$$\frac{\partial p}{\partial x_i} \rightarrow \frac{\partial}{\partial x_i} \left(\frac{1}{A_p} \frac{\partial p'}{\partial x_i} \right)_p = \underbrace{\frac{\partial u_r^{m*}}{\partial x_i}}_{\text{known}} \Big|_p + \underbrace{\frac{\partial u_r}{\partial x_i}}_{\text{unknown}} \Big|_p : \text{pressure correction e.g.}$$

\rightarrow obtain p'

~~$\rightarrow u_i' = -\frac{1}{A_p} \frac{\partial p'}{\partial x_i} \Big|_p + u_i^l$~~

w/o justification
 \rightarrow causes slow convergence

update u_i' $\rightarrow u_i^m = u_i^{m*} + u_i'$

$$p^M = p^{M-1} + p'$$

& iterate.

under-relaxation

$$u_i^m = \alpha u_i^{m*} + (1-\alpha) u_i^{m-1}$$

$$(0 < \alpha \leq 1)$$

⑥

SIMPLEX

Do not neglect $\frac{\partial}{\partial x_i} (\rho \tilde{u}_i')$.

Do approximate as $\sum_l A_l u_{il}' \simeq u_{ip}' \sum_l A_l$

$$\text{Then, } \tilde{u}_{ip}' = -\frac{1}{A_p} \sum_l A_l u_{il}' \simeq -u_{ip}' \frac{\sum_l A_l}{A_p}$$

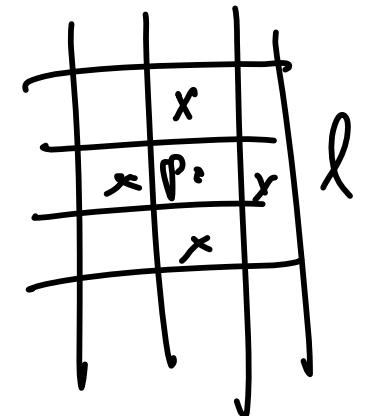
$$\rightarrow u_{ip}' = \tilde{u}_{ip}' - \frac{1}{A_p} \frac{\partial \rho'}{\partial x_i} \Big|_p = -\frac{1}{A_p + \sum_l A_l} \frac{\partial \rho'}{\partial x_i} \Big|_p$$

$$\rightarrow \frac{\partial}{\partial x_i} \left(\frac{1}{A_p + \sum_l A_l} \frac{\partial \rho'}{\partial x_i} \right)_p = \frac{\partial u_i^{mt}}{\partial x_i} \Big|_p \quad \rightarrow \text{obtain } \rho' \quad \begin{matrix} u_i' \rightarrow u_i^m \\ \rho' \rightarrow \rho^m \end{matrix}$$

iterate

$$(u_i^m = u_i^{mt} + u_i')$$

$$\text{b. } \frac{\partial u_i^m}{\partial x_i} = 0$$



c) PISO (Pressure implicit with split operator)

Neglect $\frac{\partial}{\partial x_i}(\hat{u}_i')$ as in SIMPLE

and obtain $p' \approx u_c'$ as in SIMPLE.

but introduce second correction such as

$$u_i^{m*} = u_e^{m*} + u_i' + u_c''$$

$$u_{ip}'' = \hat{u}_{ip}' - \left(\frac{1}{A_p} \frac{\delta p''}{\delta x_i} \right)_p$$

$$\rightarrow \frac{\partial}{\partial x_i} \left[\frac{1}{A_p} \frac{\delta p''}{\delta x_i} \right]_p = \frac{\partial}{\partial x_i} (\hat{u}_e')_p \rightarrow \text{obtain } p'' \rightarrow p^m \quad \left. \begin{array}{l} \text{update } u_c'' \rightarrow u_c^m \\ \text{iterate} \end{array} \right)$$

d) SIMPLE R

SIMPLEX

These methods are fairly efficient
for solving steady state problems!!!

SIMPLZ in different expression.

$$\frac{\partial}{\partial x_j} u_i u_j = -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \nabla^2 u_i$$

Given u_i^k and p^k k : iteration index

$$\frac{\partial}{\partial x_j} u_i^{k+1} u_j^{k+1} = -\frac{\partial p^{k+1}}{\partial x_i} + \frac{1}{Re} \nabla^2 u_i^{k+1}$$

$$\rightarrow \frac{\partial}{\partial x_j} (u_j^k \hat{u}_i^{k+1}) = -\frac{\partial p^k}{\partial x_i} + \frac{1}{Re} \nabla^2 \hat{u}_i^{k+1}$$

$$\rightarrow (u_j^k \frac{\partial}{\partial x_j} - \frac{1}{Re} \nabla^2) \hat{u}_i^{k+1} = -\frac{\partial p^k}{\partial x_i} \Rightarrow \text{obtain } \hat{u}_i^{k+1}$$

$$p^{k+1} = p^k + p' , \quad u_i^{k+1} = \hat{u}_i^{k+1} + u_i'$$

$$\rightarrow (u_j^k \frac{\partial}{\partial x_j} - \frac{1}{Re} \nabla^2) (u_i^{k+1} - u_i') = -\frac{\partial}{\partial x_i} (p^{k+1} - p')$$

~~original eq:~~ $(u_j^{k+1} \frac{\partial}{\partial x_j} - \frac{1}{Re} \nabla^2) u_i^{k+1} = - \frac{\partial p^{k+1}}{\partial x_i}$

$= u_j^k - u_j$ **neglect in SIMPLE**

$$(u_j^k \frac{\partial}{\partial x_j} - \frac{1}{Re} \nabla^2) u_i^r = - \frac{\partial p'}{\partial x_i}$$

iterate

obtain $u_i^r \rightarrow u_i^{k+1}$

$$\rightarrow u_i^r = - F(u^k)^{-1} \frac{\partial p'}{\partial x_i}$$

$$\frac{\partial u_i^{k+1}}{\partial x_i} = \frac{\partial u_i^{k+1}}{\partial x_e} + \frac{\partial u_i^r}{\partial x_e} = 0$$

Patankar

only for
steady flow.

4

other methods

(1) Fractional step methods for incompressible flow

Roger Temam
Chorin } Projectm method $O(\delta t)$

Kim and Moin (1985, JCP) $O(\delta t^2)$ if explicit method is taken,
 $\delta t \sim \delta x^2$
#3200

$$\left\{ \frac{\partial u_i}{\partial x_i} = 0 \right.$$

$$\left. \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} (u_i u_j) = - \frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right.$$

2nd-order Adams-Basforth method (AB2) - $O(\delta t^2)$, explicit

$$\left(\frac{du}{dt} = f \rightarrow \frac{u^{n+1} - u^n}{\delta t} = \frac{1}{2} (3f^n - f^{n-1}) : AB2 \right. \\ \left. \frac{1}{2} (f^{n+1} + f^n) : CN \right)$$

Crank-Nicolson method (CN) - $O(\delta t^2)$

implicit
 $(\delta t \rightarrow \infty)$

$\delta t \sim \delta x$ semi-implicit scheme

$$\rightarrow \frac{u_i^{n+1} - u_i^n}{\delta t} + \frac{1}{2} \left[3 \frac{\partial}{\partial x_j} (u_i^n \cdot u_j^n) - \frac{\partial}{\partial x_j} (u_i^{n+1} u_j^{n+1}) \right] = -\frac{1}{2} \left(\frac{\partial p^{n+1}}{\partial x_i} + \frac{\partial p^n}{\partial x_i} \right) + \frac{1}{2} \frac{1}{Re} \nabla^2 (u_i^{n+1} + u_i^n)$$

◎ Fractional step method (Kim & Moin, 1985, JCP)

노트 제목

2019-04-22

Apply AB2 + CN to N-S eqs.
 (convection) (diffusion)

$$\left\{ \begin{array}{l} \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{1}{2} \left[3 \frac{\partial}{\partial x_j} (u_i^n u_j^n) - \frac{\partial}{\partial x_j} (u_i^{n+1} u_j^{n+1}) \right] \\ = -\frac{1}{2} \left(\frac{\partial p^{n+1}}{\partial x_i} + \frac{\partial p^n}{\partial x_i} \right) + \frac{1}{2} \frac{1}{Re} \nabla^2 (u_i^{n+1} + u_i^n) + O(\Delta t^2) \\ \frac{\partial u_i^{n+1}}{\partial x_i} = 0 \end{array} \right. \quad (1)$$

decouple the velocity from the pressure.

$$\rightarrow \frac{\hat{u}_c - u_c^n}{\Delta t} + \frac{1}{2} \left[3 \frac{\partial}{\partial x_j} (u_c^n u_j^n) - \frac{\partial}{\partial x_j} (u_c^{n+1} u_j^{n+1}) \right] = \frac{1}{2Re} \nabla^2 (\hat{u}_c + u_c^n) \quad (2)$$

\hat{u}_i : intermediate velocity

② Second-order approx. of $\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} u_i u_j = \frac{1}{Re} \nabla^2 u_i$

However, $\frac{\partial \hat{u}_i}{\partial x_i} \neq 0$.

$\frac{\hat{u}_i^{n+1} - \hat{u}_i^n}{\Delta t} = - \frac{\partial \phi^{n+1}}{\partial x_i}$ — (3) ϕ : pseudo-pressure

Force u_i^{n+1} to satisfy the continuity, $\frac{\partial \hat{u}_i}{\partial x_i} = 0$ — (4)

→ $\nabla^2 \phi^{n+1} = \frac{1}{\Delta t} \frac{\partial \hat{u}_i}{\partial x_i}$ — (5)

Solution procedure

i) obtain \hat{u}_i from ② using u_i^n & u_i^{n-1} .

∴ obtain ϕ^{n+1} from ⑤ using \hat{u}_i

∴ obtain u_i^n from ③ using \hat{u}_i and p^n .

How about p^{n+1} ?

$$③ : \hat{u}_i = u_i^{NH} + \text{ot } \frac{\partial \phi}{\partial x_i}^{NH} \quad - ⑥$$

$$\textcircled{6} \rightarrow \textcircled{2} : \frac{\underline{u_i^{NM} + \sigma t \frac{\partial \phi^{NM}}{\partial x_i}} - u_i^N}{\sigma t} + \frac{1}{2} \left[3 \frac{\partial}{\partial k_j} (u_i^N u_j^N) - \frac{\partial}{\partial x_j} (u_i^{N-1} u_j^{N-1}) \right] = \frac{1}{2R_0} \sigma^2 \left(u_i^{NM} + \sigma t \frac{\partial \phi^{NM}}{\partial x_i} + u_i^N \right). \quad \Gamma_{+u_i^N}$$

$$\rightarrow \frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{1}{2} \left[3 \frac{\partial}{\partial x_j} \right] = - \underbrace{\frac{\partial \phi^{n+1}}{\partial x_i}}_{- \frac{2}{\partial x_i} (\phi^{n+1} - \frac{\Delta t}{2Re} \nabla^2 \phi^{n+1})} + \frac{1}{2Re} \nabla^2 (U_i^{n+1})$$

$$\textcircled{1} \Rightarrow \phi^{n+1} - \frac{\alpha t}{2Re} \nabla^2 \phi^{n+1} = \frac{1}{2} (p^{nn} + p^n)$$

$$\rightarrow \underbrace{p^{n+1}}_{\text{update } p^{n+1} \text{ from } \phi^{n+1} \text{ in } p^n} = -p^n + 2\phi^{n+1} - \frac{\alpha t}{Re} \nabla^2 \phi^{n+1}$$

- \textcircled{2}

Fractional-step
method!

How to solve \textcircled{2} ?
eq.

$$\frac{\hat{u}_i - u_i^n}{\alpha t} + \frac{1}{2} \left[3 \frac{\partial}{\partial x_j} (u_i^n u_j^n) - \frac{\partial}{\partial x_j} (u_i^{n+1} u_j^{n+1}) \right] = \frac{1}{2Re} \nabla^2 (\hat{u}_i + u_i^n) + O(\alpha t^2)$$

Apply some discretization method such as CD2.

$$\frac{\hat{u}_i - u_i^n}{\alpha t} + \frac{1}{2} \left[3 \frac{\delta}{\delta x_j} (u_i^n u_j^n) - \frac{\delta}{\delta x_j} (u_i^{n+1} u_j^{n+1}) \right] = \frac{1}{2Re} \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} \right) (\hat{u}_i + u_i^n) + O(\alpha t^2)$$

$$\rightarrow \left[1 - \frac{\sigma t}{2R_e} \left(\frac{\delta^2}{\partial x^2} + \frac{\delta^2}{\partial y^2} + \frac{\delta^2}{\partial z^2} \right) \right] (\hat{u}_c - \hat{u}_e^\top)$$

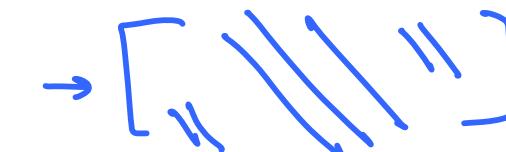
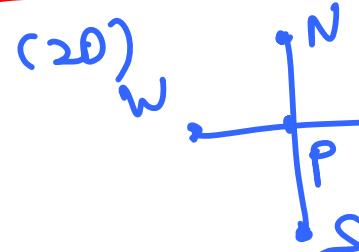
$$= -\frac{\sigma t}{2} \left[3 \frac{\delta}{\partial x_j} (u_i^\top u_j^\top) - \frac{\delta}{\partial x_j} (u_c^\top u_j^\top) \right] + \frac{\sigma t}{R_e} \left(\frac{\delta^2}{\partial x^2} + \frac{\delta^2}{\partial y^2} + \frac{\delta^2}{\partial z^2} \right) u_i^\top$$

$+ O(\sigma t^3)$

$$(A_x = \frac{1}{2R_e} \frac{\delta^2}{\partial x^2}, \quad A_y = \frac{1}{2R_e} \frac{\delta^2}{\partial y^2}, \quad A_z = \frac{1}{2R_e} \frac{\delta^2}{\partial z^2})$$

$$\rightarrow (1 - \sigma t A_x - \sigma t A_y - \sigma t A_z) (\hat{u}_c - \hat{u}_e^\top) = R_i^\top + 2\sigma t (A_x + A_y + A_z) u_c^\top$$

$+ O(\sigma t^3)$



Sparse matrix
 expensive to invert.

\Rightarrow approximate factorization.

$$\check{v} = (1-\alpha t A_x)(1-\alpha t A_y)(1-\alpha t A_z)(\hat{u}_c - \hat{u}_i) - \alpha t^2 A_x A_y (\hat{u}_c - \hat{u}_i) \theta(\alpha t)$$

$$\simeq (1-\alpha t A_x)(1-\alpha t A_y)(1-\alpha t A_z)(\hat{u}_c - \hat{u}_i) - \alpha t^2 A_x A_z (\text{---}) \theta(\alpha t)$$

$$- \alpha t^2 A_y A_z (\text{---}) \theta(\alpha t^3)$$

$$\Rightarrow (1-\alpha t A_x)(1-\alpha t A_y)(1-\alpha t A_z)(\hat{u}_c - \hat{u}_i) = R_c + 2\alpha t (A_x + A_y + A_z) \hat{u}_c$$

!.. neglect!

$$\equiv z_i$$

$$\rightarrow (1-\alpha t \underbrace{A_x}_{CD2}) z_i = R_c + 2\alpha t (A_x + A_y + A_z) \hat{u}_c$$

obtain z_i .

$w \bullet \rho \in [0 \parallel 0]$ tridiagonal matrix $\rightarrow \Theta(N)$
operations
no iteration is required.

$$(1-\sigma t A_y) \underbrace{(1-\sigma t A_z)(\hat{u}_i - \hat{u}_{i-1})}_{\equiv y_i} = z_i$$

$\rightarrow (1-\sigma t A_y) y_i = z_i \quad \rightarrow \text{obtain } y_i$
 CP2 \rightarrow tridiagonal matrix

$(1-\sigma t A_z)(\hat{u}_i - \hat{u}_{i-1}) = y_i \quad \Rightarrow \text{obtain } \hat{u}_i.$
 CD2 \rightarrow tridiagonal matrix

\Rightarrow Alternating directional implicit method
(ADI)
 saves a lot of CPU and memory.

Numerical stability

accuracy & stability.

We used a semi-implicit method
AB2 + CN

→ conditionally stable

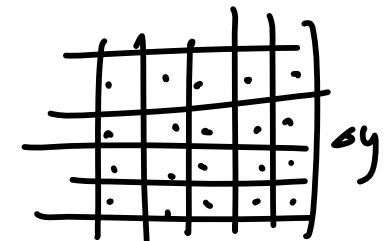
maximum
CFL number.

$$CFL = \left(\frac{|u|}{\delta x} + \frac{|v|}{\delta y} + \frac{|w|}{\delta z} \right) \delta t \leq 1 \text{ for AB2 + CN}$$

$$\Rightarrow \delta t \leq 1 / \left(\frac{|u|}{\delta x} + \frac{|v|}{\delta y} + \frac{|w|}{\delta z} \right)$$

$$\Delta t_{\max} = 1 / \left(\frac{|u|}{\delta x} + \frac{|v|}{\delta y} + \frac{|w|}{\delta z} \right)$$

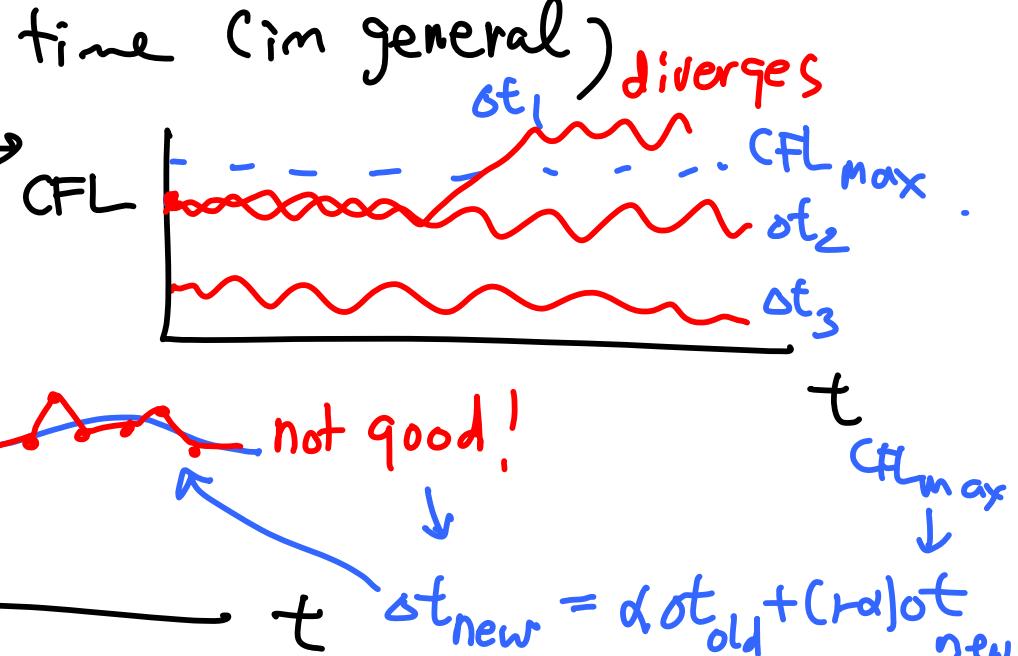
0.95 in practice



Search for
 δx
maximum value

Two ways of marching in time (in general)

- ① Fix Δt .
 - ② Fix CFL_{max} .
- Δt
Better in terms of CPU time.



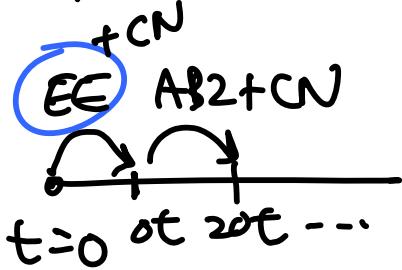
$$\Delta t_{new} = \alpha \Delta t_{old} + (1-\alpha) \Delta t_{new}$$

energy spectrum (ω) \leftarrow requires constant Δt . (\because FFT)
freq.

* AB2 + CN

$$n \rightarrow n+1$$

problems : ① not self-starting because of u_i^{n+1}



introduce another starting scheme @ 1st time step.
↓
determines α_t !

AB2 requires const. of $(\alpha_t, \alpha_i^n, \alpha_c^{nn})$

② Contains spurious root $(u_c^{n+1}, u_i^{n+1}, u_c^{nn})$
as $\alpha_t \rightarrow 0$
 u_c^{nn} has two roots.

③ store u_i^{n+1} as well as u_i^n .

=> either fully implicit method

or better semi-implicit method such as Runge-Kutta

method.

RK

Example : RK3 + CN $\rightarrow \Theta(\alpha t^2)$

$$\left[\begin{array}{l} \frac{\hat{u}_i^k - u_i^{k-1}}{\alpha t} = \frac{(\alpha_k + \beta_k) L_i(u_i^{k-1}) + \beta_k L_i(\hat{u}_i^k - u_i^{k-1})}{\alpha t} \\ \quad - \frac{\gamma_k N_i(u_i^{k-1}) - \zeta_k N_i(u_i^{k-2})}{\alpha t} - (\alpha_k + \beta_k) \frac{\partial p_i^k}{\partial x_i} \\ \frac{u_i^k - \hat{u}_i^{k-1}}{\alpha t} = -(\gamma_k + \zeta_k) \frac{\partial \phi^k}{\partial x_i} \\ (\gamma_k + \zeta_k) \nabla^2 \phi^k = \frac{1}{\alpha t} \frac{\partial \hat{u}_i^k}{\partial x_i} \end{array} \right] \quad \text{CN} \quad \text{RK3}$$

$k=0, 1, 2, 3$

L_i : diffusion
 N_i : convection

$k=1, 2, 3$

$u_i^0 (k=0) = \hat{u}_i^1$
 $u_i^3 (k=3) = \hat{u}_i^{n+1}$

$$\alpha_1 = \beta_1 = 4/15$$

$$\gamma_1 = 8/15$$

$$\zeta_1 = 0$$

$$\alpha_2 = \beta_2 = 1/15$$

$$\gamma_2 = 5/12$$

$$\zeta_2 = -19/60$$

$$\alpha_3 = \beta_3 = 1/6$$

$$\gamma_3 = 3/4$$

$$\zeta_3 = -5/12$$

$$CFL_{\max} = \sqrt{3}$$

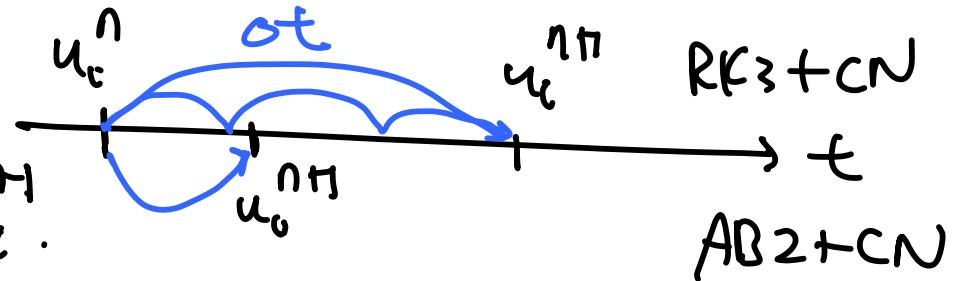
don't need to store u_i^{n+1} .

no spurious root

self starting

change Δt during computation based on CFL_{\max} .

but three sub marching per time step.



$$\hat{u}_c : \frac{\partial \hat{u}_c}{\partial t} = -N_i + \frac{1}{R_e} \nabla^2 \hat{u}_c + \text{error terms}$$

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2019-04-24

formula suggested by K & M (1985)

for this, we need b.c.'s for \hat{u}_i :

which may be different from those for u_i^{n+1} .

K & M (1985) $\rightarrow \hat{u}_i = u_i^{n+1} + \sigma \frac{\partial p^n}{\partial x_i}$ @ boundaries.

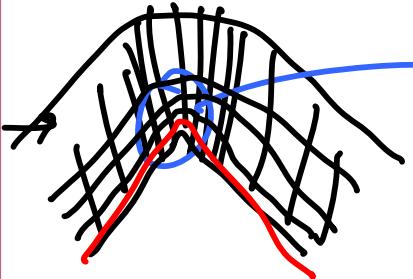
If we include the press. grad. such as $-\frac{\partial p^n}{\partial x_i}$ or $-\frac{\partial p^{k+1}}{\partial x_i}$.

in \hat{u}_i egs., then $\hat{u}_i = u_i^{n+1}$ @ boundaries.

$AB2 + CN$
 $RK3 + CN$

\rightarrow semi-implicit method $\Rightarrow \sigma t \leq$

Δx
to be stable



Δx or Δy becomes very small

→ Δt becomes small for semi-
implicit method.
⇒ fully implicit method → there is

choi & Moim (1994, JCP) : CN for all the terms.
551

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{1}{2} \frac{\partial}{\partial x_j} (u_i^{n+1} u_j^{n+1} + u_i^n u_j^n) = - \underbrace{\frac{\partial p^{n+1}}{\partial x_i}}_{\text{or}} + \frac{1}{2 Re} \nabla^2 (u_i^{n+1} + u_i^n)$$

can be
linearized

$$\text{or } - \frac{1}{2} \left(\frac{\partial p^{n+1}}{\partial x_i} + \frac{\partial p^n}{\partial x_i} \right)$$

FSM :

$$(1) \quad \frac{\hat{u}_i - u_i^n}{\Delta t} + \frac{1}{2} \frac{\partial}{\partial x_j} (\hat{u}_i \hat{u}_j + \hat{u}_i^n \hat{u}_j^n) = - \frac{\partial p^n}{\partial x_i} + \frac{1}{2 Re} \nabla^2 (\hat{u}_i + u_i^n)$$

iteration
↳ need iterative method such as Newton

$$\left\{ \begin{array}{l} \textcircled{2} \quad \frac{u_i^{*} - u_i^{\text{ref}}}{\Delta t} = \frac{\partial p^n}{\partial x_i} \\ \textcircled{3} \quad \nabla^2 p^{n+1} = \frac{1}{\Delta t} \frac{\partial u_i^{*}}{\partial x_i} \\ \textcircled{4} \quad \frac{u_i^{n+1} - u_i^*}{\Delta t} = - \frac{\partial p^{n+1}}{\partial x_i} \end{array} \right. \quad \left\{ \begin{array}{l} \text{no time step limit} \\ \text{self starting (no } u_i^{n+1} \text{)} \\ \text{need iterations at every time step.} \end{array} \right.$$

- FSM vs. SIMPLE for unsteady flow

↳ for steady flow

{ FSM [Solve Poisson eq. once - 3 times per time step.
[solve mtm eq
SIMPLE - solve mtm eqs
↓ ↑ ↓ ↑ ↓ ↓
Poisson eqs iteratively per time step

② stream function - vorticity method ($\psi - \omega$)

↳ continuity is identically satisfied automatically

no pressure

but 2D only

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

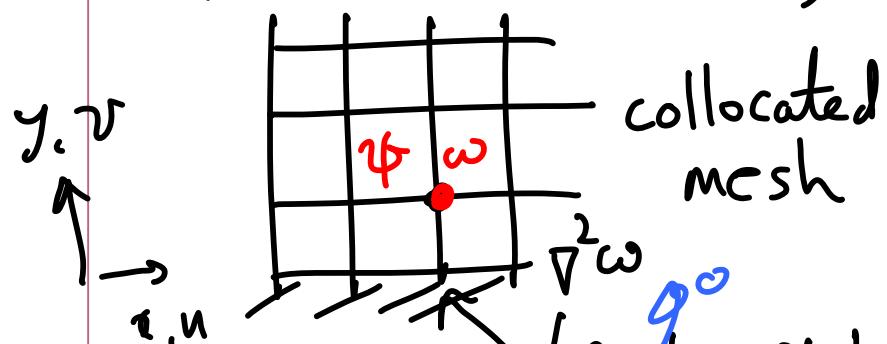
$$\omega = \frac{\partial u}{\partial z} - \frac{\partial v}{\partial y} \downarrow \rightarrow \boxed{\nabla^2 \psi = -\omega} \quad \text{--- (1)}$$

$$\begin{aligned} & \left| \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial t} + \dots \right) \right. \\ & \left. - \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial t} + \dots \right) \right. \Rightarrow \boxed{\rho \frac{\partial \omega}{\partial t} + \rho u \frac{\partial \omega}{\partial x} + \rho v \frac{\partial \omega}{\partial y} = \mu \nabla^2 \omega} \quad \text{--- (2)} \end{aligned}$$

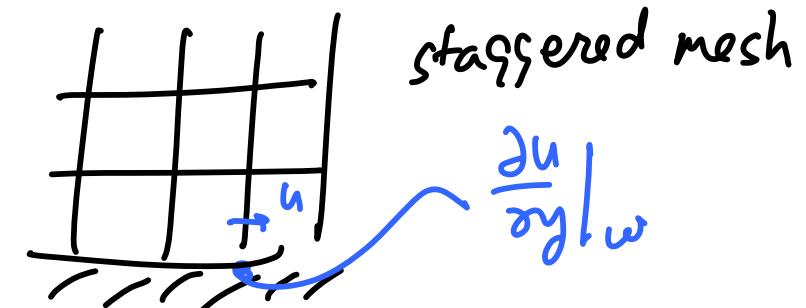
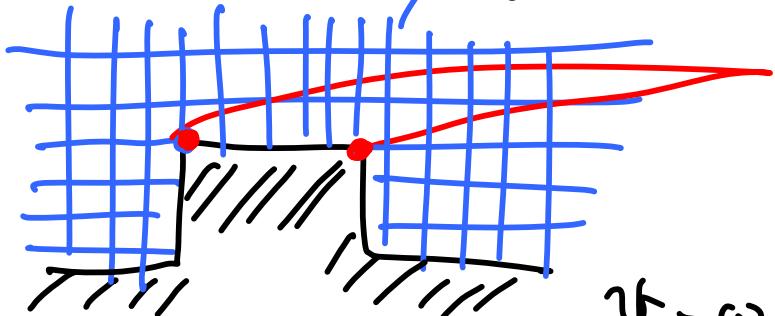
(u, v, ρ) 3 eqs. $\rightarrow (\psi, \omega)$ & 2 eqs.

Given initial $u_i^0 \rightarrow \omega^0 \rightarrow \begin{matrix} ① \psi \\ ② \omega \end{matrix} \rightarrow u_i^{n+1}$

problems: boundary condition for ω at the wall



$$\omega_w = \frac{\partial v}{\partial x} \Big|_w - \frac{\partial u}{\partial y} \Big|_w = \frac{u_i - u_o}{\Delta y} + O(\Delta) \quad \text{one-side difference}$$



ω ? singular at sharp corners.
2-D problems

ψ - ω approach is less popular nowadays.

③ Artificial compressibility method Chorin (1967, Jcp)

background $\alpha^2 = \frac{\partial P}{\partial \rho} \Big|_S = \gamma R T$ for perfect gas

$$P = \rho R T = \rho \frac{\alpha^2}{\gamma} \equiv \beta \rho \rightarrow \rho = \frac{1}{\beta} P$$

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0 \rightarrow \frac{1}{\beta} \frac{\partial P}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0$$

For incomp. flow, $\frac{\partial}{\partial x_i} (\rho u_i) = 0 \rightarrow \boxed{\frac{1}{\beta} \frac{\partial P}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0}$

+ N-S eq.

β : parameter to provide a prior;

large $\beta \rightarrow$ more like incompressible.

explicit method \rightarrow No. because of restriction in dt
use implicit method for large dt.

ACM \rightarrow good for steady flow.

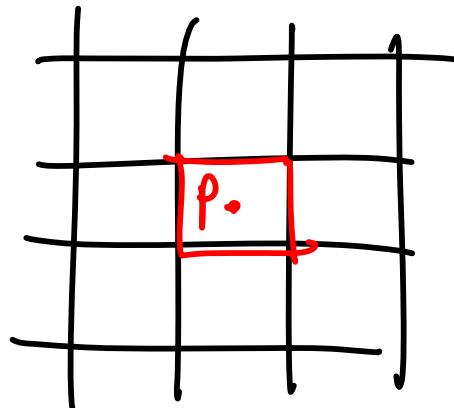
convergence depends on β .

If β is very large, ACM \sim SIMPLE Kwak et al.
(1986, AIAA J)

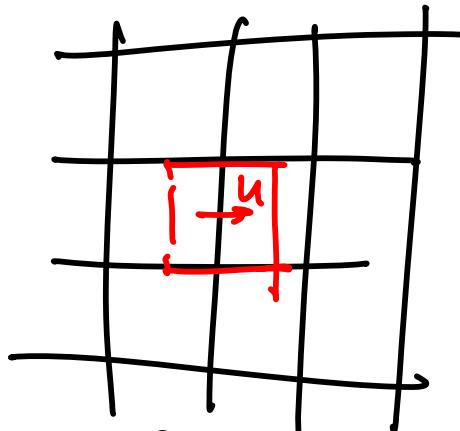
(May 7 6:30pm - 10pm . Lecture on FDM code)
#301 - 1st floor by Jaerim Kim.
computer Room

5. Solution methods for the NS eqs.

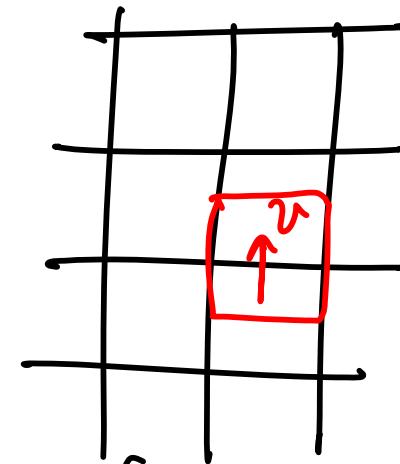
5.1 Implicit scheme using pressure correction and a staggered grid (CFVM).



for P
cell center



for u
cell face



for v
cell face

Conduct volume integration

5.2 Treatment of pressure for collocated variables

5.2 Treatment of pressure for collocated variables

노트 제목

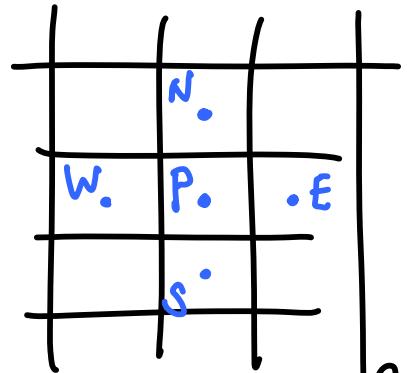
2019-04-29

$$\frac{\delta}{\delta x_i} \left(\frac{\delta p^n}{\delta x_i} \right) = \frac{\delta H_i^n}{\delta x_i}, \quad H_i^n = - \frac{\delta}{\delta x_j} (p u_i^n u_j^n) + \frac{\delta \tilde{L}_i^n}{\delta x_j}$$

gradient operator]
divergence operator [should be consistent for
energy conserving scheme

$$\left(\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_i u_j) = - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\mu \frac{\partial u_i}{\partial x_j} \right) \right)$$
$$\frac{\partial u_i}{\partial x_j} = \omega$$

For example, forward difference for gradient operator
backward " " " divergence " .



$$\begin{aligned}
 & \frac{1}{\Delta x} \left[\left(\frac{\delta \hat{P}}{\delta x} \right)_P - \left(\frac{\delta \hat{P}}{\delta x} \right)_W \right] + \frac{1}{\Delta y} \left[\left(\frac{\delta \hat{P}}{\delta y} \right)_P - \left(\frac{\delta \hat{P}}{\delta y} \right)_S \right] \\
 &= \frac{1}{\Delta x} \left[H_{x,P} - H_{x,W} \right] + \frac{1}{\Delta y} \left[H_{y,P} - H_{y,S} \right] = Q_P^n \\
 \rightarrow & \frac{1}{\Delta x} \cdot \frac{1}{\Delta x} \left[(P_E^n - P_F^n) - (P_P^n - P_W^n) \right] + \frac{1}{\Delta y} \cdot \frac{1}{\Delta y} \left[(P_N^n - P_F^n) - (P_P^n - P_S^n) \right] = Q_P^n \\
 \hookrightarrow & \frac{1}{\Delta x^2} (P_E^n - 2P_P^n + P_W^n) + \frac{1}{\Delta y^2} (P_N^n - 2P_P^n + P_S^n) = Q_P^n
 \end{aligned}$$

Same form as the one obtained on a staggered grid w/ central difference.

In N-S e.g. $-\frac{\partial P}{\partial x}$ \rightarrow forward difference $\rightarrow \underline{\text{O}(\Delta x)}$ approx.
order
 \Rightarrow 1st order approx. to the pressure grad. \Rightarrow Need higher

Now, CD2 for div. & grad operators.

$$\sum_i \left(\frac{\delta p^n}{\delta x_i} \right) = \frac{\delta H_c^n}{\delta x_c} \rightarrow \frac{1}{20x} \left(\cancel{\frac{\delta p}{\delta x_e}} - \frac{\delta p}{\delta x_w} \right)$$

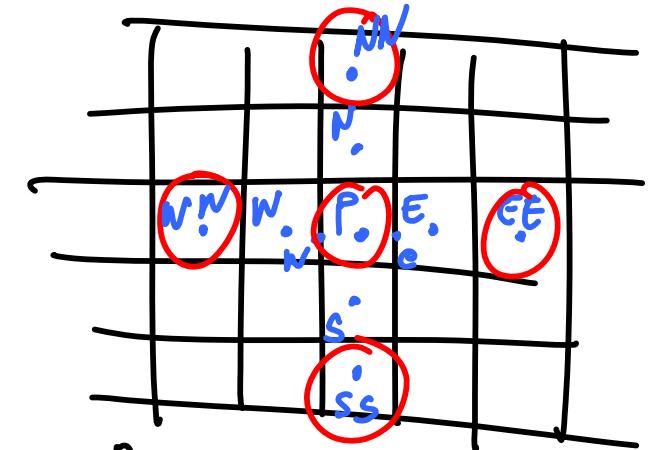
$$\rightarrow \frac{1}{20x} \left[\frac{\delta p^n}{\delta x_E} - \frac{\delta p^n}{\delta x_W} \right] + \frac{1}{20y} \left[\frac{\delta p^n}{\delta y_N} - \frac{\delta p^n}{\delta y_S} \right]$$

$$= \frac{1}{20x} \left[H_{x_E}^n - H_{x_W}^n \right] + \frac{1}{20y} \left[H_{y_N}^n - H_{y_S}^n \right] = Q_p^n$$

$$\rightarrow \frac{1}{20x} \frac{1}{20x} \left[(P_{EE}^n - P_p^n) - (P_p^n - P_{WW}^n) \right] + \frac{1}{20y} \frac{1}{20y} \left[(P_{NN}^n - P_p^n) - (P_p^n - P_{SS}^n) \right] = \vec{Q}_p^n$$

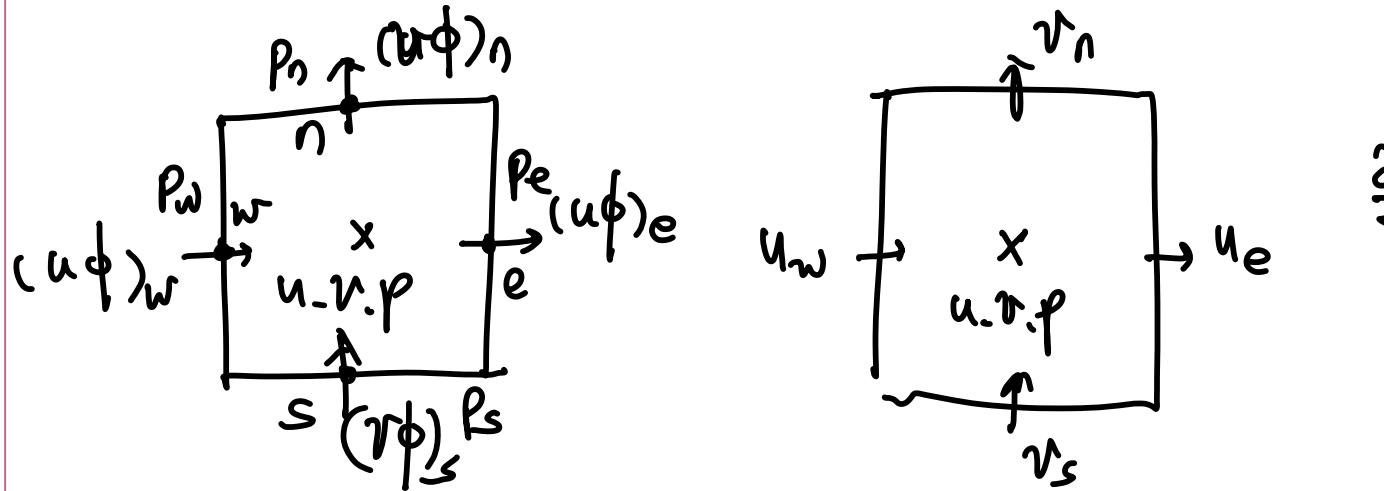
This eq. involves nodes which are $20x$ (& $20y$) apart!

\rightarrow may create a checkerboard press. distribution.



5.3

SIMPLZ for a collocated grid.



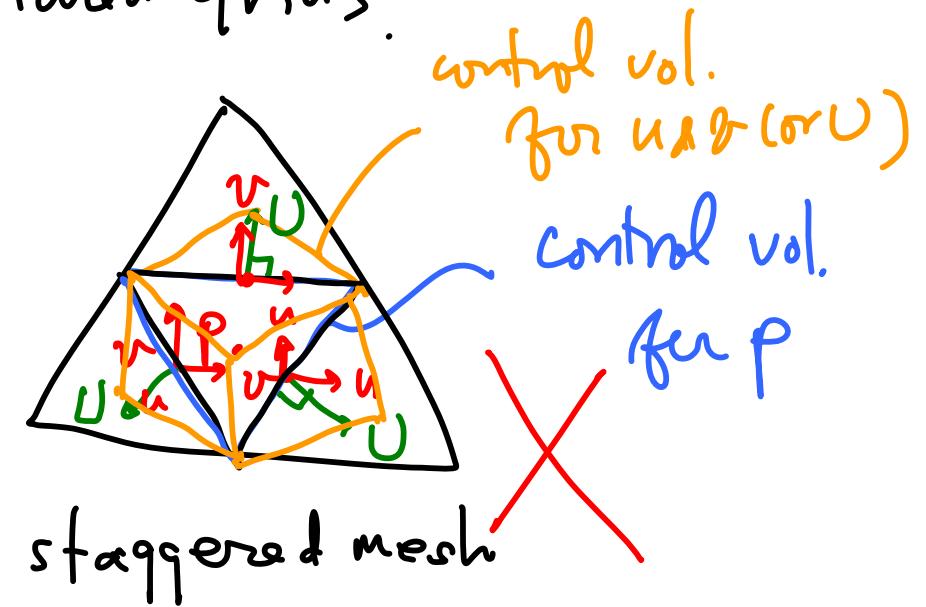
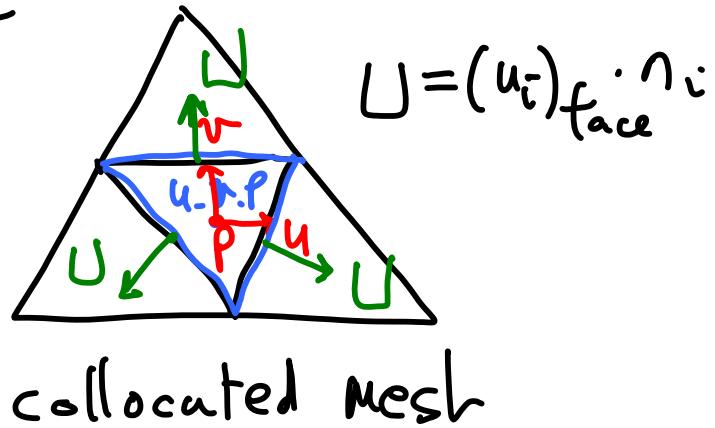
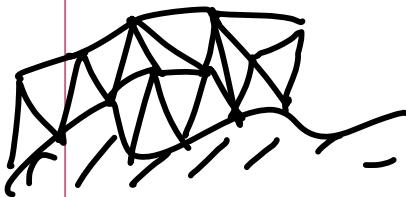
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

So, in mtm eq., it is important to use u & v at all faces
that satisfy the continuity.

→ use this concept, called "momentum interpolation method."
 this has been developed by

Rhie & Chow (AIAA J. 21, 1525 (1983)) for steady flow
 Zhang et al. (JCP, 114, 18 (1994)) for unsteady flow
 on structured curvilinear grids

Kim & Choi (JCP, 162, 411 (2000)) for unsteady flow
 on unstructured grids.



N-S eq (CN)

$$\frac{\underline{u}_c^{n+1} - \underline{u}_c^n}{\Delta t} + \frac{1}{2} \frac{\partial}{\partial x_j} (\underline{u}_c^{n+1} \underline{u}_j^{n+1} + \underline{u}_j^{n+1} \underline{u}_c^{n+1}) = -\frac{\partial p^{n+1}}{\partial x_i} + \frac{1}{2Re} \nabla^2 (\underline{u}_j^{n+1} + \underline{u}_c^{n+1}) + O(\Delta t^2)$$

$$= \underline{u}_c^{n+1} \underline{u}_j^n + \underline{u}_j^{n+1} \underline{u}_c^n - \underline{u}_c^n \underline{u}_j^n + O(\Delta t^2) : \text{linearization}$$

$$\rightarrow \frac{\underline{u}_c^{n+1} - \underline{u}_c^n}{\Delta t} + \frac{1}{2} \frac{\partial}{\partial x_j} (\underline{u}_i^{n+1} \underline{u}_j^n + \underline{u}_j^{n+1} \underline{u}_i^n) = -\frac{\partial p^{n+1}}{\partial x_i} + \frac{1}{2Re} \nabla^2 (\underline{u}_i^{n+1} + \underline{u}_c^n) + O(\Delta t^2),$$

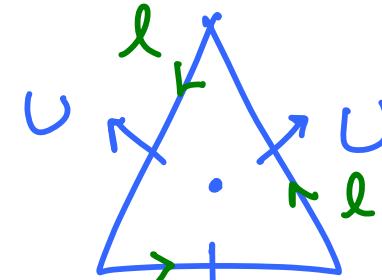
Apply FEM (Choi & Moin 1994)

$$\left\{ \begin{array}{l} \frac{\widehat{u}_c - \widehat{u}_c^n}{\Delta t} + \frac{1}{2} \frac{\partial}{\partial x_j} (\widehat{u}_i^n \widehat{u}_j^n + \widehat{u}_j^n \widehat{u}_i^n) = -\frac{\partial p^n}{\partial x_i} + \frac{1}{2Re} \nabla^2 (\widehat{u}_i^n + \widehat{u}_c^n) \\ \widehat{u}_c^n - \widehat{u}_c = \frac{\partial p^n}{\partial x_i} \end{array} \right.$$

↓
integrate this eq. over each cell and
apply the divergence theorem.

$$\left. \begin{aligned} \partial^2 p^{n+1} &= \frac{1}{\partial t} \frac{\partial u_i^*}{\partial x_i} \\ \frac{u_i^{n+1} - u_i^*}{\partial t} &= - \frac{\partial p^{n+1}}{\partial x_i} \end{aligned} \right.$$

$$\delta \hat{u}_i \equiv \hat{u}_i - u_i^*$$



$$\rightarrow \frac{\partial \hat{u}_i}{\partial t} \cdot A + \oint_{\Gamma} \frac{1}{2} (U^* \delta \hat{u}_i + u_i^* n_j \delta \hat{u}_j + 2u_i^* U^*) dl$$

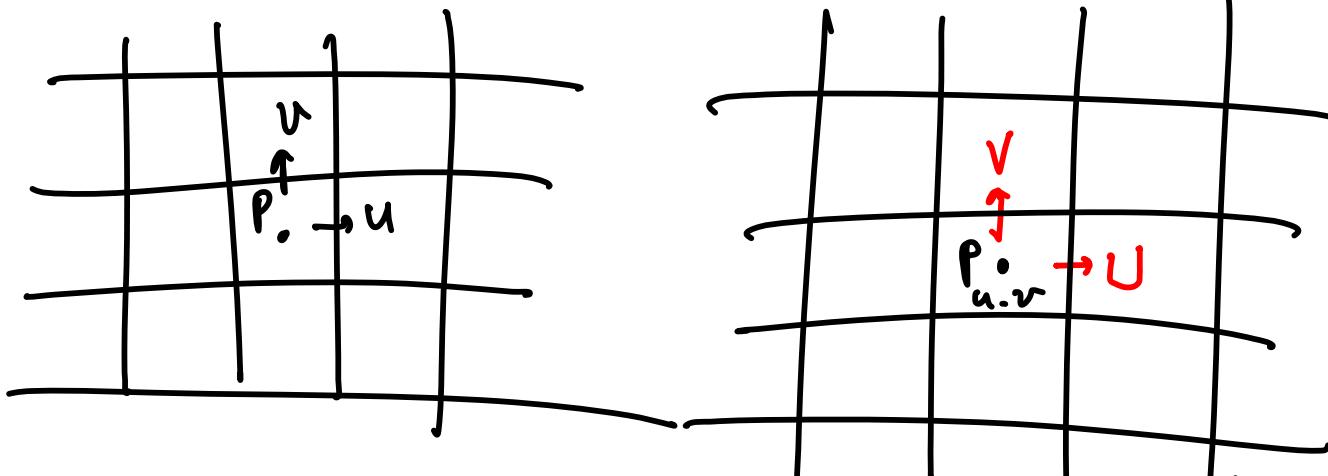
$$= - \int_A \frac{\partial p^n}{\partial x_i} dA + \oint_{\Gamma} \frac{1}{2k_e} \frac{\partial}{\partial n} (\delta \hat{u}_i + 2u_i^*) dl$$

$$\oint_{\Gamma} \frac{\partial p^n}{\partial n} dl = \frac{1}{\partial t} \oint_{\Gamma} U^* dl$$

obtained by interpolation
from adjacent cell-center velocities \hat{u}_i^*

$$u_i^{n+1} - u_i^* = - \partial t \frac{\partial p^{n+1}}{\partial x_i} \Rightarrow \frac{\partial u_i}{\partial x_i} \neq 0$$

$$U^{n+1} - U^* = -\alpha t \frac{\partial p^{n+1}}{\partial x_i} \Rightarrow \nabla \cdot U^{n+1} = 0$$



mtm. interpolation method,
on collocated mesh.

6. Note on pressure and incompressibility

To prove that press. is a math. quantity for the continuity.

$$\exists \underline{v}^* \text{ s.t. } \nabla \cdot \underline{v}^* \neq 0$$

Want to create \underline{v} s.t. $\nabla \cdot \underline{v} = 0$ & $\underline{v} \sim \underline{v}^*$.

$$\text{Set } \tilde{R} = \frac{1}{2} \int_{\Omega} [\underline{v}(r) - \underline{v}^*(r)]^2 d\Omega \quad \left. \begin{array}{l} \text{Find } \underline{v} \\ \text{minimizing } \tilde{R} \end{array} \right]$$

$$\nabla \cdot \underline{v} = 0$$

(Calculus of variation)

$$R = \frac{1}{2} \int_{\Omega} (\underline{v} - \underline{v}^*)^2 d\Omega - \int_{\Omega} \lambda(r) \nabla \cdot \underline{v}(r) d\Omega \quad \text{--- (x)}$$

Lagrange multiplier.

Suppose $\exists \underline{v}^+$ s.t. $R_{\min} = \frac{1}{2} \int_{\Omega} (\underline{v}^+ - \underline{v}^*)^2 d\Omega$ & $\nabla \cdot \underline{v}^+ = 0$

Let $\underline{v} = \underline{v}^+ + \delta \underline{v}$ $\Rightarrow \textcircled{x}$

$$\delta R = R - R_{\min}$$

$$= \int_{\Omega} \delta \underline{v}(r) \cdot [\underline{v}^+(r) - \underline{v}^*(r)] d\Omega - \int_{\Omega} \lambda(r) \cdot \nabla \delta \underline{v}(r) d\Omega$$

$$= \int_{\Omega} \delta \underline{v}(r) \cdot [\underline{v}^+(r) - \underline{v}^*(r) + \nabla \lambda(r)] d\Omega + \int_S \lambda(r) \delta \underline{v}(r) \cdot \underline{n} dS$$

then, for arbitrary $\delta \underline{v}$, we need $\delta R = 0$

$$\Rightarrow \underline{v}^+(r) - \underline{v}^*(r) + \nabla \lambda(r) = 0$$

$$\nabla \cdot ()$$

\downarrow
make it zero by
a natural way
or by changing λ .

$$\Rightarrow \nabla^2 \lambda(\underline{r}) = \nabla \cdot \underline{V}^P(\underline{r}) \quad \text{Poisson eq for } \lambda.$$

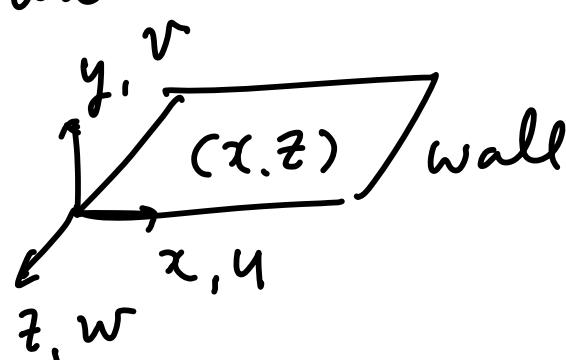
\therefore Lagrange multiplier plays the role of P and
the function of P is to allow continuity to be satisfied.

7. Boundary conditions for N-S eqs.

노트 제목

2019-05-01

(i) wall



$$\text{no slip : } \underline{u} = \underline{v} = \underline{w} = 0$$

$$(x, z) \quad (x, z) \quad (x, z)$$

$$\frac{\partial u}{\partial x} \Big|_w = 0$$

$$\frac{\partial w}{\partial z} \Big|_w = 0$$

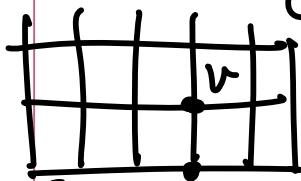
continuity

$$\frac{\partial v}{\partial y} \Big|_w = 0$$

$$v_w = 0$$

two b.c's for v .

One cannot satisfy both b.c.'s simultaneously.



wall

$$v_w = 0$$

$$\frac{\partial v}{\partial y} \Big|_w \neq 0$$

$$\frac{\partial v}{\partial y} \Big|_w = 0 \rightarrow v_w \neq 0$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \dots - \frac{\partial p}{\partial y} + \dots$$

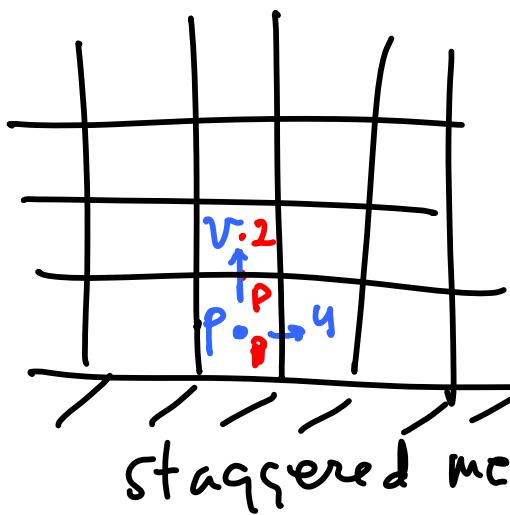
collocated mesh

p_w is required as b.c.

$$\rightarrow p_w \approx p_f \Rightarrow \frac{\partial p}{\partial y}|_w = 0$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mu \frac{\partial^2 \mathbf{v}}{\partial x^2} = - \frac{\partial p}{\partial y} + \mu \nabla^2 \mathbf{v}$$

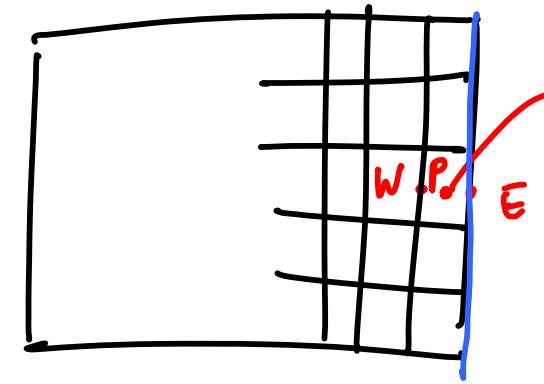
$$@ \text{wall}, \quad \frac{\partial p}{\partial y}|_w = \mu \nabla^2 \mathbf{v}|_w = \mu \frac{\partial^2 \mathbf{v}}{\partial y^2}|_w \neq 0$$



$$\rho \frac{\partial \mathbf{v}}{\partial t} = \dots - \frac{\partial p}{\partial y} + \dots$$

$$\frac{p_2 - p_1}{\Delta y}$$

p_w is not required.
Good!



collocated
mesh exit

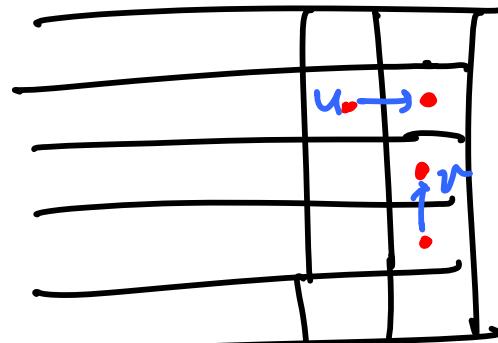
$$\rho \frac{\partial u}{\partial t} = \dots - \frac{\partial p}{\partial x} + \dots$$

||

$$- \frac{p_E - p_W}{\partial x}$$

is required
as b.c.

$$p_E \approx p_W \Rightarrow \frac{\partial p}{\partial x}|_{exit} = 0$$



staggered mesh

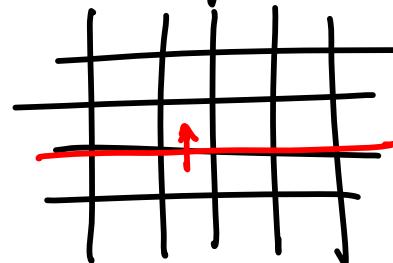
$$\rho \frac{\partial u}{\partial t} = \dots - \frac{\partial p}{\partial x} + \dots$$

$$\rho \frac{\partial v}{\partial t} = \dots - \frac{\partial p}{\partial y} + \dots$$

) no b.c. for p
is required
at exit .

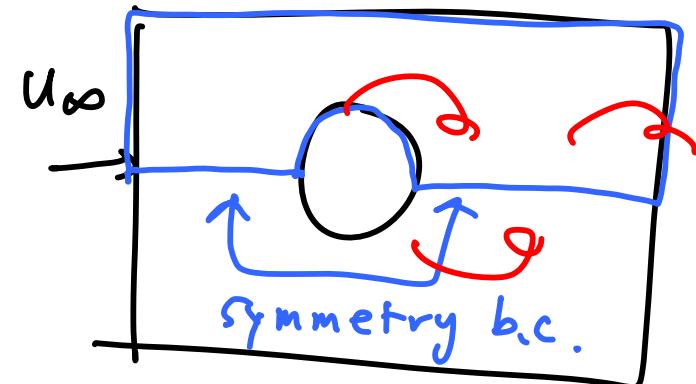
②

Symmetry



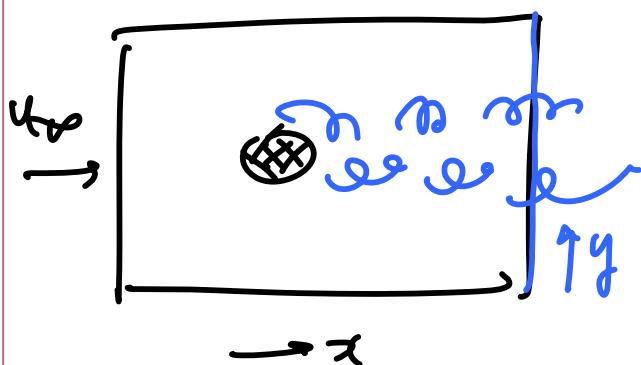
Symmetry

$$\begin{cases} v = 0 \\ \frac{\partial u}{\partial y} = 0 \\ \frac{\partial w}{\partial y} = 0 \end{cases}$$



③

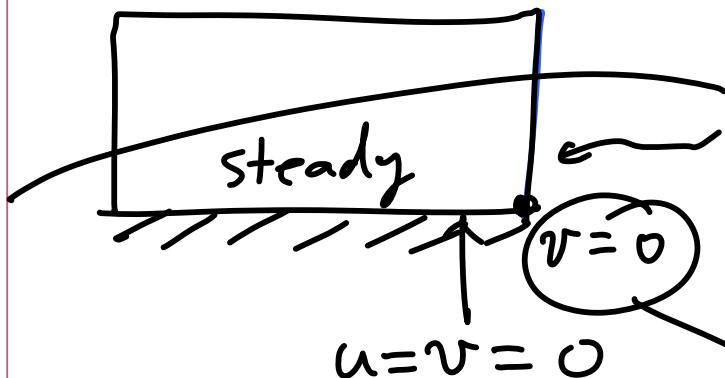
exit



$\frac{\partial u_i}{\partial x} = 0$ for steady flow Neumann b.c. but not accurate enough.

$\hookrightarrow \frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial x} = 0$

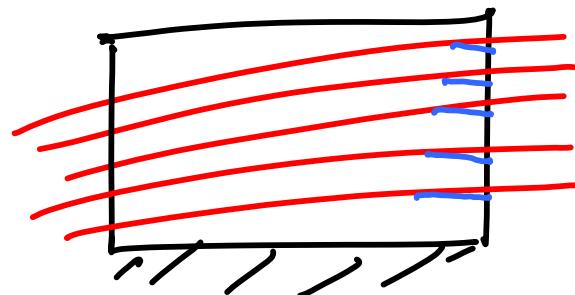
Continuity $\frac{\partial r}{\partial y} = 0 \rightarrow v \equiv \text{const} @ \text{exit}$



$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \quad \text{Neumann b.c.}$$

$$\frac{\partial v}{\partial y} = 0$$

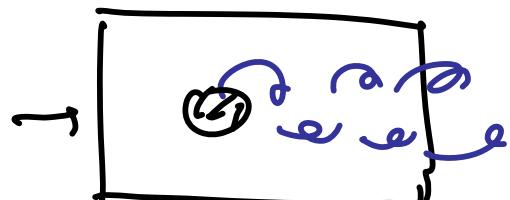
$\Rightarrow v \equiv 0 @ \text{exit}$.
inaccurate



streamlines

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} = 0 @ \text{exit.}$$

for unsteady flow



?

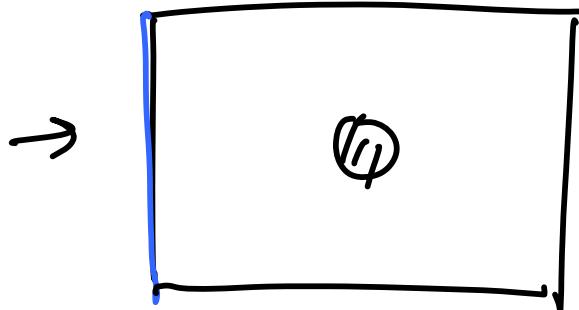
$$\frac{\partial u_i}{\partial t} + \left(\frac{\partial u_i}{\partial x} \right) = 0 : \text{convective outflow b.c.}$$

$\left(\rightarrow \frac{\partial u_i}{\partial x} = 0 \text{ for steady flow} \right)$

$$N-S \text{ eq.} : \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

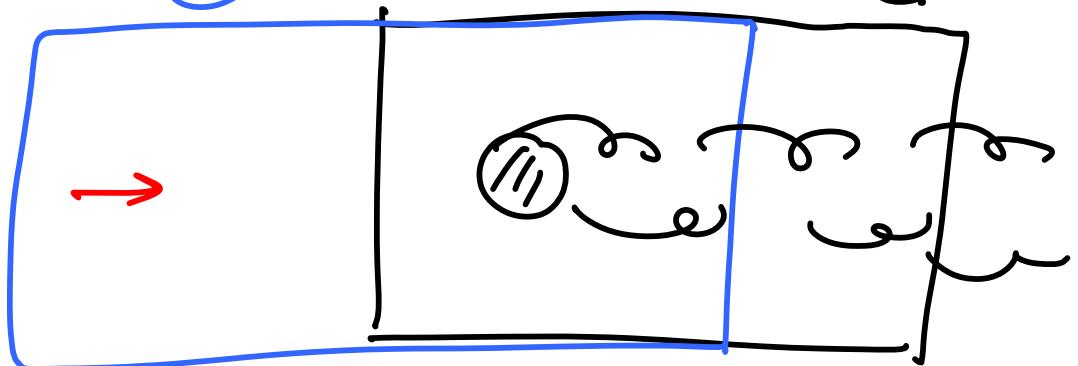
$$\begin{aligned} C_i &\equiv u_i(y, z, t) \\ C_{\infty} &\equiv \overline{u}|_{\text{exit}}(y, z, t) \leftarrow \text{better.} \end{aligned}$$

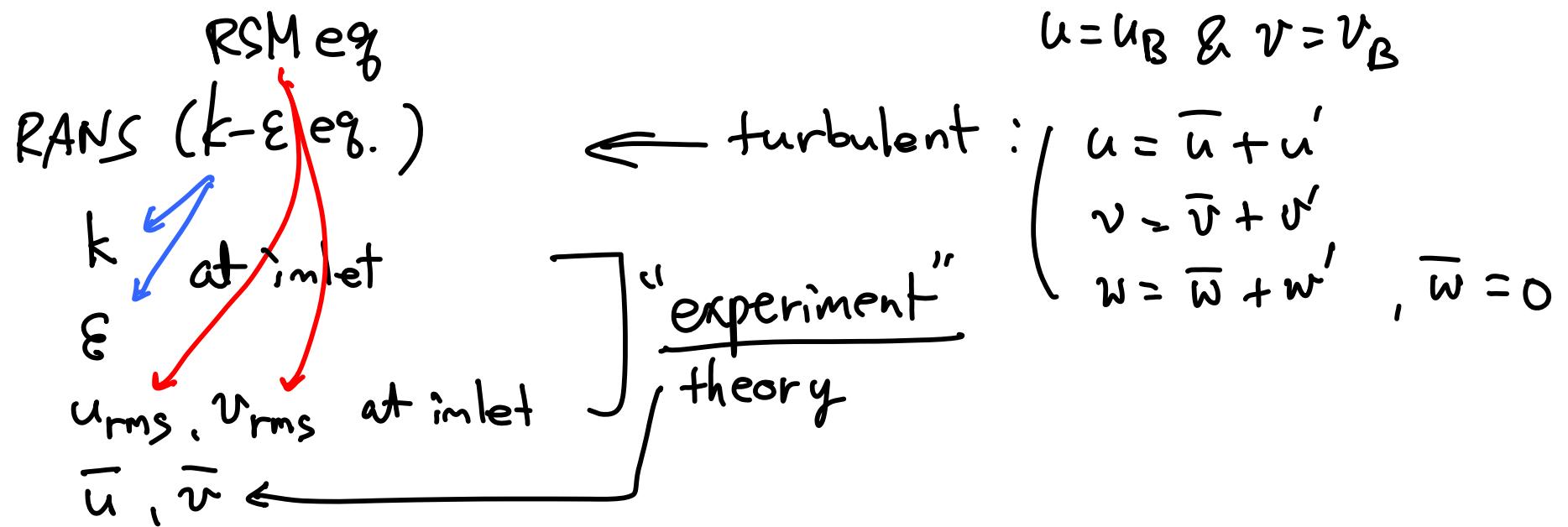
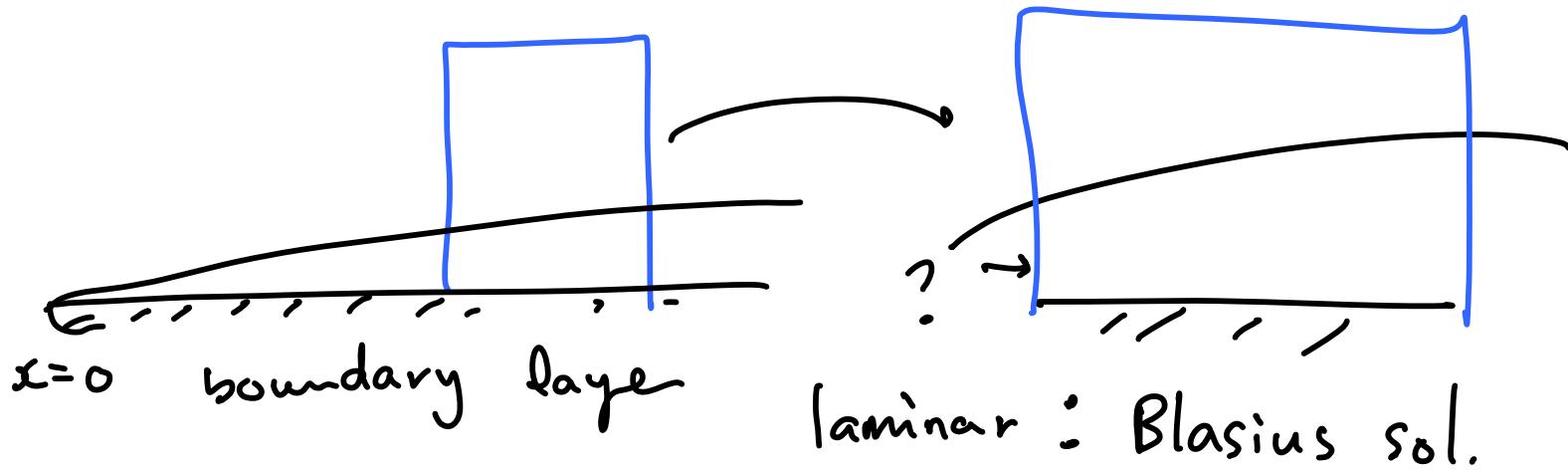
(4) inlet



$$\begin{aligned} u &= u_\infty \\ v &= 0 \end{aligned}$$

(B) is better than (A)

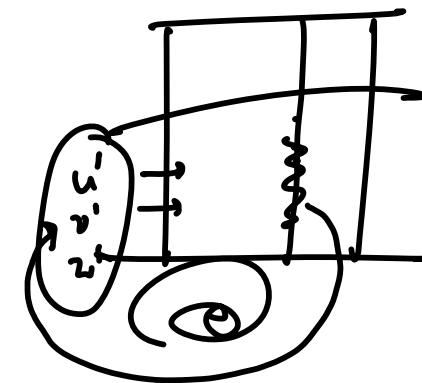




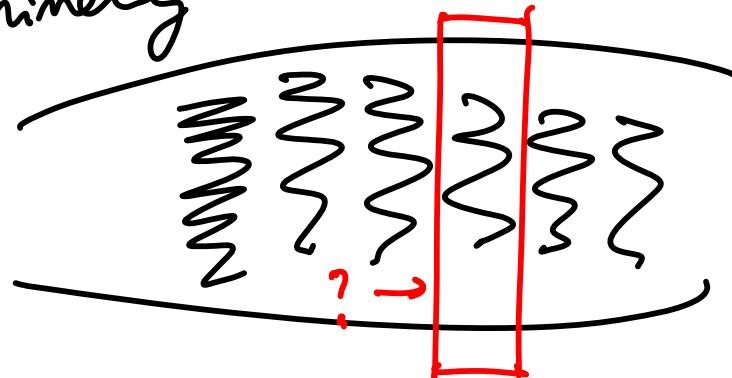
DNS & LES (direct numerical simulation, large eddy simulation)
↓
no turbulence model

$u', v', w' (y, z, t)$ @ inlet

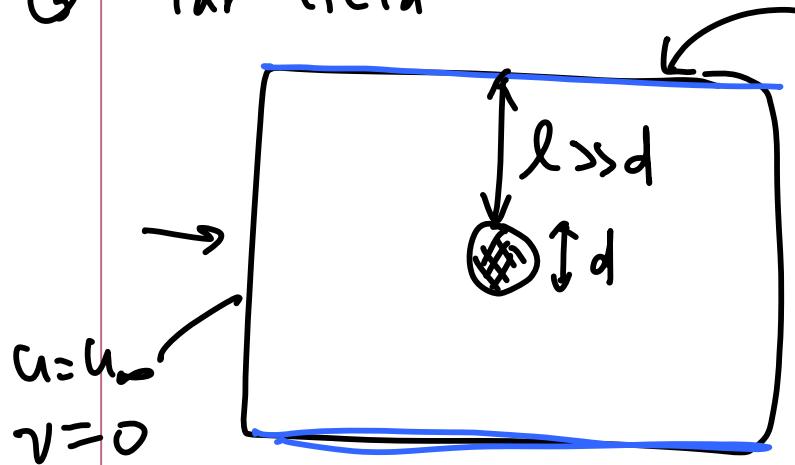
Lund et al. JCP (1998) #1335



turbomachinery



⑤ far-field



- $u = u_\infty \text{ & } v = 0$: Dirichlet b.c.

- $u = u_\infty \text{ & } \frac{\partial v}{\partial y} = 0 \rightarrow v \neq 0$

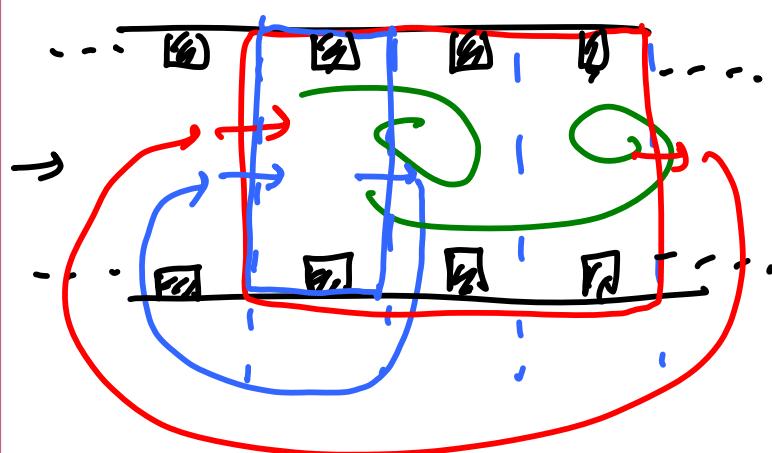
- $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} = 0$ Neumann b.c.

- $\frac{\partial u}{\partial y} = 0, \omega_2 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \rightarrow \frac{\partial v}{\partial x} = 0$

- $u = u_\infty, \omega_2 = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0 \rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$

- $\frac{\partial v}{\partial y} = 0, \omega_2 = 0 \rightarrow \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

⑥ periodic b.c. \rightarrow no b.c.

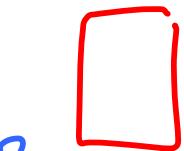


$$\phi_N = \phi_1 \quad \frac{\partial \phi}{\partial x}|_N = \frac{\partial \phi}{\partial x}|_1$$

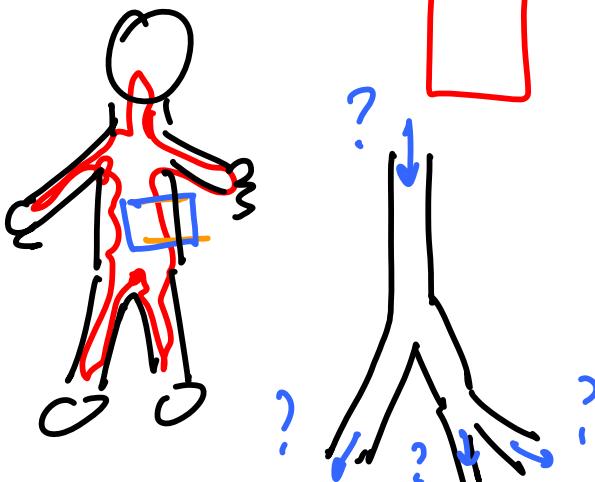
RANS ($k-\epsilon$ eq) - steady 2D flow
unsteady " "



DNS & LES - unsteady 3D flow

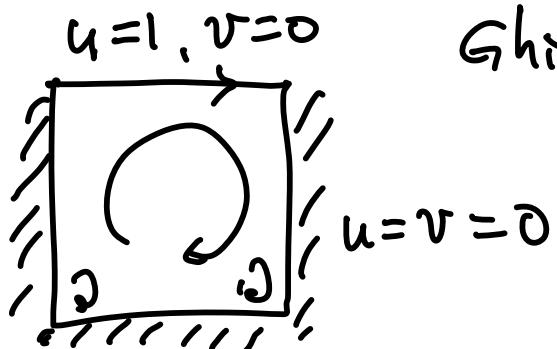


⑦ circulatory system



8. Examples

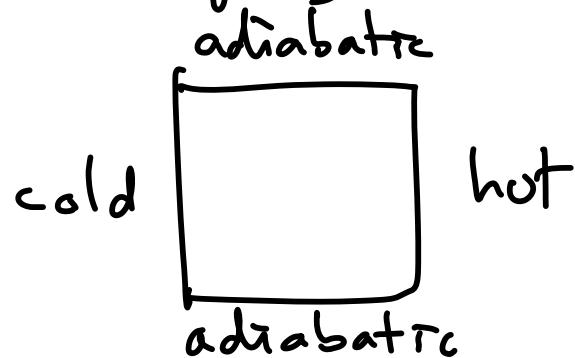
① Lid -driven cavity flow



Ghia, Ghia & Shim (1982, JCP, 48, 387)

Re #'s, # 4430

② Buoyancy -driven cavity flow



Hortmann, Peric & Scheuerer
(1990, Int'l J. Num. Meth. in Fluids,
11, 189)