# Ch. 9 Vector Differential Calculus.
# Grad, Div, Curl
# Ch. 9 벡터 미분법, 기울기, 발산, 회전

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# 9.1 Vectors in 2-Space and 3-Space

☑ **Scalar: A quantity that is determined by its magnitude**

Ex. Length, Voltage, Temperature

☑ **Vector: A quantity that is determined by both its magnitude and its direction (arrow or directed line segment).**

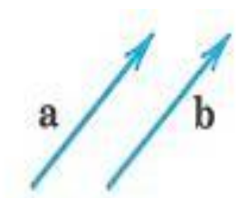
Ex. Force, Velocity (Giving the speed and direction of motion)

- Notation (표시): **a**, **b**, **v** (lowercase boldface letters) or $\vec{a},\ \vec{b}$
- |a| ( norm ): The length (or magnitude) of the vector **a**
- Unit Vector: A vector of length 1

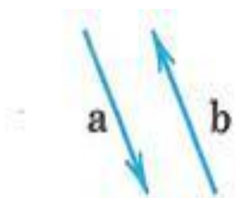
# 9.1 Vectors in 2-Space and 3-Space

## ☑ Definition Equality of Vectors (벡터의 상등)

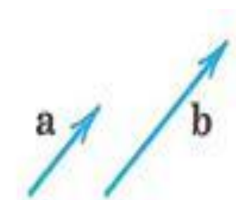
- Two vectors **a** and **b** are equal (**a** = **b**) if and only if they have the same length and the same direction.



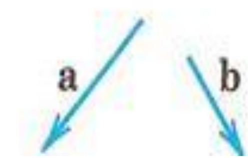

Equal vectors
**a** = **b**
(A)




Vectors having the same length but different directions
(B)




Vectors having the same direction but different length
(C)




Vectors having different length and different direction
(D)

**(A) Equal vectors. (B) ~ (D) Different vectors**

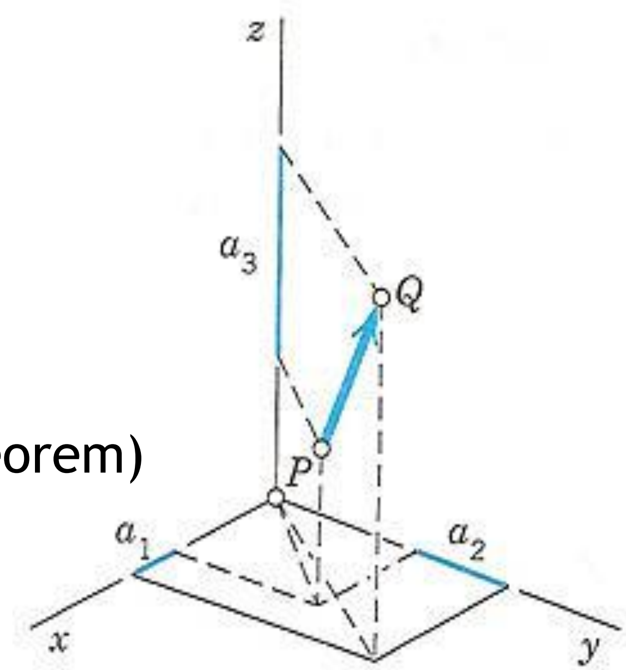
# 9.1 Vectors in 2-Space and 3-Space

## ☑ Components of a Vector

- Let **a** be a given vector with initial point P: $(x_1, y_1, z_1)$ and terminal point Q: $(x_2, y_2, z_2)$
- Components of **a**
  : The three coordinate differences

  $$a_1 = x_2 - x_1, \quad a_2 = y_2 - y_1, \quad a_3 = z_2 - z_1$$

- Notation: $\mathbf{a} = [a_1, a_2, a_3]$
- $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ (Use the Pythagorean theorem)



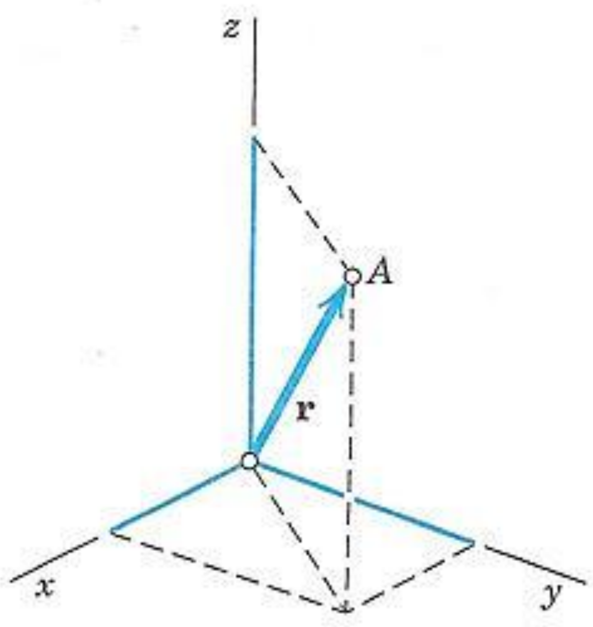

Components of a vector

# 9.1 Vectors in 2-Space and 3-Space

## ☑ Components of a Vector

- Position vector **r** of a point A (x, y, z)

  : The vector with the origin as the initial point and A the terminal point




**Position vector r of a point *A*: (*x*, *y*, *z*)**

# 9.1 Vectors in 2-Space and 3-Space

☑ **Theorem 1 Vectors as Ordered Triples of Real Numbers**

- A fixed Cartesian coordinate system being given, each vector is uniquely determined by its ordered triple (**순서를 갖는 삼중수**) of corresponding components.
- To each ordered triple of real numbers there corresponds precisely one vector, with (0 , 0 , 0) corresponding to the zero vector 0.

☑ **Zero vector: A vector which has length 0 and no direction.**

☑ $$\mathbf{a} = \mathbf{b} \quad \Leftrightarrow \quad a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$$

$$\left(\text{where } \mathbf{a} = [a_1,\ a_2,\ a_3],\ \mathbf{b} = [b_1,\ b_2,\ b_3]\right)$$

# 9.1 Vectors in 2-Space and 3-Space

## ☑ Definition Addition of Vectors

- The sum of two vectors $\mathbf{a} = [a_1,\ a_2,\ a_3]$, $\mathbf{b} = [b_1,\ b_2,\ b_3]$
  By adding the corresponding components:

$$\mathbf{a} + \mathbf{b} = [a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3]$$

- Geometrically: **a** + **b** is the vector drawn from the initial point of **a** to the terminal point of **b**

## ☑ Basic Properties of Vector Addition

(a) **a** + **b** = **b** + **a** (Commutativity, 교환법칙)

(b) (**u** + **v**) + **w** = **u** + (**v** + **w**) (Associativity, 결합법칙)

(c) **a** + 0 = 0 + **a** = **a**

(d) **a** + (-**a**) = 0

# 9.1 Vectors in 2-Space and 3-Space

## ☑ Definition Scalar Multiplication (Multiplication by Number)

: The product of any vector and any scalar *c* (real number *c*) by multiplying each component : $c\mathbf{a} = [ca_1,\ ca_2,\ ca_3]$

- Geometrically: $c\mathbf{a}$ with $c > 0$ has the direction of $\mathbf{a}$ and with $c < 0$ the direction opposite to $\mathbf{a}$.
- $|c\mathbf{a}| = |c|\ |\mathbf{a}|$
- $c\mathbf{a} = 0$ if and only if $\mathbf{a} = 0$ or $c = 0$

## ☑ Basic Properties of Scalar Multiplication

(a) $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$ (b) $(c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a}$

(c) $c(k\mathbf{a}) = (ck)\mathbf{a}$ (d) $1\mathbf{a} = \mathbf{a}$

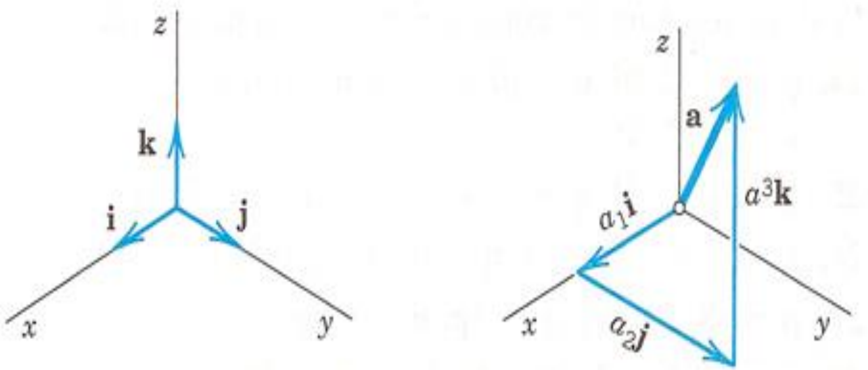
## ☑ Properties of Vector Addition and Scalar Multiplication

(a) $0\mathbf{a} = 0$ (b) $(-1)\mathbf{a} = -\mathbf{a}$

# 9.1 Vectors in 2-Space and 3-Space

## ☑ Unit Vectors i, j, k

- **i** = [ 1, 0, 0 ], **j** = [ 0, 1, 0 ], **k** = [ 0, 0, 1 ]




The unit vector **i**, **j**, **k** and the representation

- All the vectors **a**=[$a_1$, $a_2$, $a_3$] = $a_1$ **i**+ $a_2$ **j**+ $a_3$ **k** form the real vector space **R**$^3$ with two algebraic operations of vector addition and scalar multiplication as just defined.
- **R**$^3$ has dimension 3.
- The triple of vectors **i**, **j**, **k** is a standard basis (표준기저) of **R**$^3$.

# 9.2 Inner Product (Dot Product, 내적)

## ☑ Definition Inner Product (Dot Product) of Vectors

- Inner product or dot product
- The product of their lengths times the cosine of their angle.

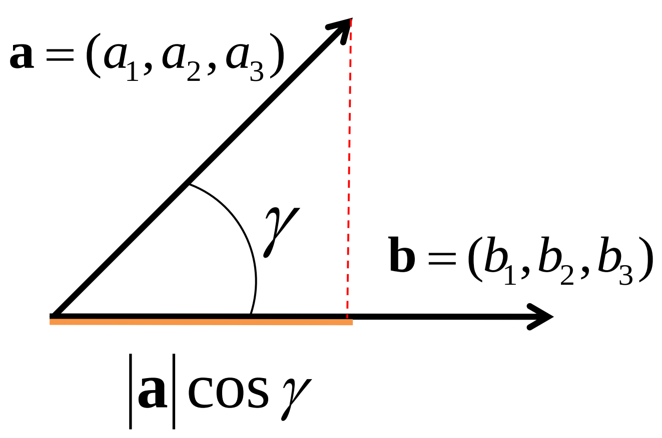
$$\mathbf{a} \bullet \mathbf{b} = \begin{cases} |\mathbf{a}||\mathbf{b}|\cos\gamma & \text{if} \quad \mathbf{a} \neq \mathbf{0},\ \mathbf{b} \neq \mathbf{0} \\ 0 & \text{if} \quad \mathbf{a} = \mathbf{0} \ \text{or} \ \mathbf{b} = \mathbf{0} \end{cases}$$

- The angle $\gamma$, $0 \le \gamma \le \pi$, between **a** and **b** is measured when the initial points of the vectors coincide.

$$\mathbf{a} = [a_1,\ a_2,\ a_3],\ \mathbf{b} = [b_1,\ b_2,\ b_3]$$

- In components.

$$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$



# 9.2 Inner Product (Dot Product)

## ☑ Theorem 1 Orthogonality (직교성)

- The inner product of two nonzero vectors is 0 if and only if these vectors are perpendicular.

## ☑ Length and Angle

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\gamma$$

$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$$

$$\cos\gamma = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\sqrt{\mathbf{a}\cdot\mathbf{a}}\sqrt{\mathbf{b}\cdot\mathbf{b}}}$$

# 9.2 Inner Product (Dot Product)

## ☑ Properties of the Inner Product

For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and scalars $q_1, q_2$

1. $[q_1\mathbf{a} + q_2\mathbf{b}] \bullet \mathbf{c} = q_1\mathbf{a} \bullet \mathbf{c} + q_2\mathbf{b} \bullet \mathbf{c}$ (Linearity)
2. $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ (Symmetry)
3. $\begin{cases} \mathbf{a} \bullet \mathbf{a} \geq 0 \\ \mathbf{a} \bullet \mathbf{a} = 0 \quad \text{if and only if } \mathbf{a} = \mathbf{0} \end{cases}$ (Positive definiteness)
4. $(\mathbf{a} + \mathbf{b}) \bullet \mathbf{c} = \mathbf{a} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{c}$ (Distributivity)
5. $|\mathbf{a} \bullet \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ (Cauchy-Schwarz inequality)
6. $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ (Triangle inequality)
7. $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2\left(|\mathbf{a}|^2 + |\mathbf{b}|^2\right)$ (Parallelogram equality)

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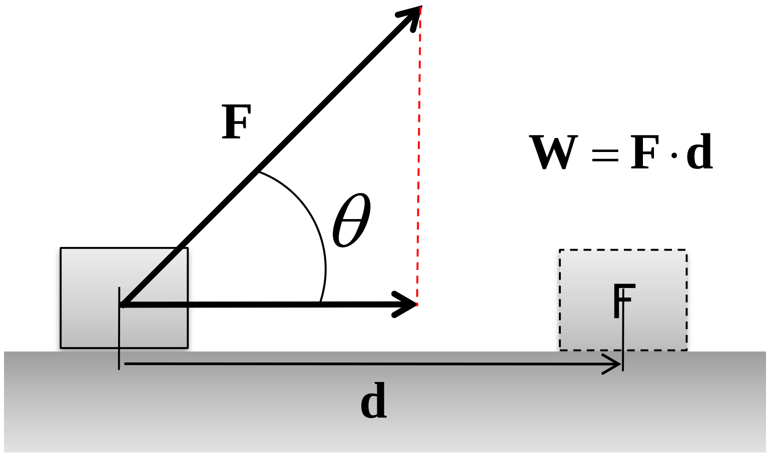
# 9.2 Inner Product (Dot Product)

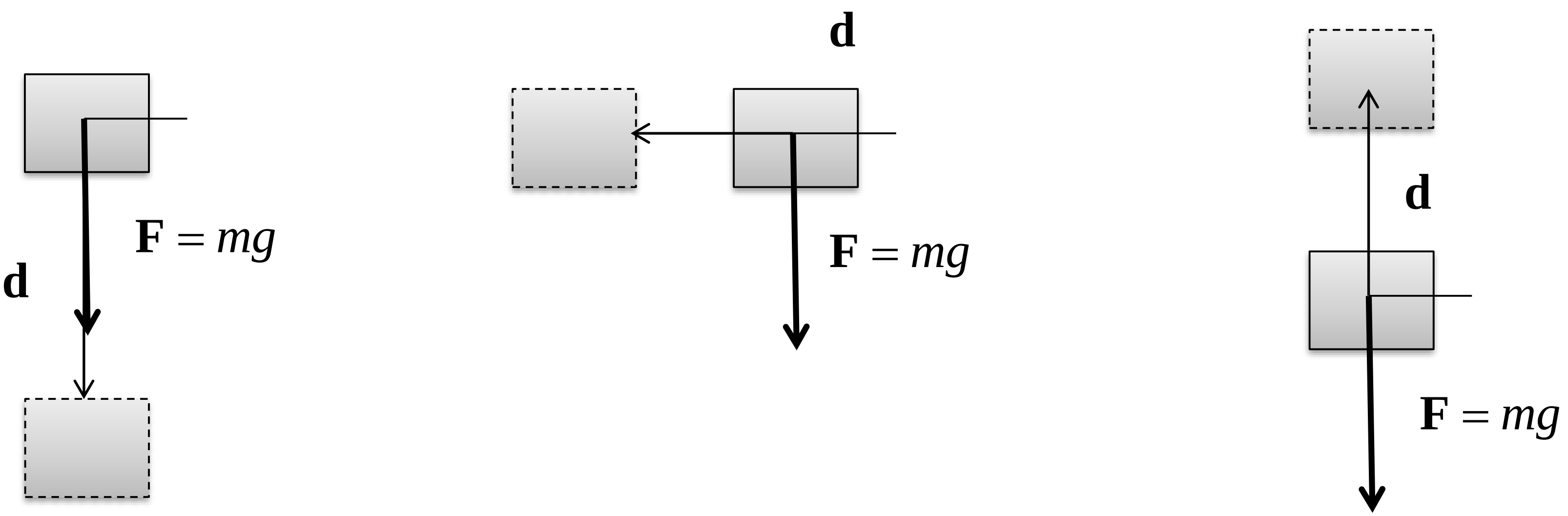
## ☑ Work done by force




$$\mathbf{W} = \mathbf{F} \cdot \mathbf{d}$$

$$W = |\mathbf{F}||\mathbf{d}|\cos\theta = \mathbf{F} \cdot \mathbf{d}$$

# 9.2 Inner Product (Dot Product)




$$W = |\mathbf{F}||\mathbf{d}|\cos 0 = |\mathbf{F}||\mathbf{d}| \qquad W = |\mathbf{F}||\mathbf{d}|\cos 90^\circ = 0 \qquad W = |\mathbf{F}||\mathbf{d}|\cos 180^\circ = -|\mathbf{F}||\mathbf{d}|$$

# 9.2 Inner Product (Dot Product)

☑ Component or projection of vector **a** in the direction of a vector **b**

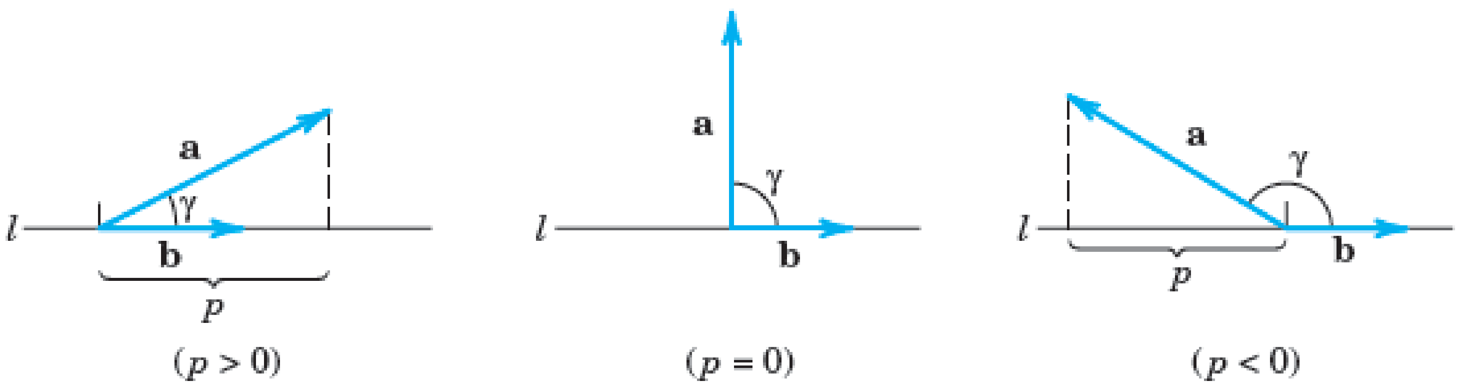



Fig. 181. Component of a vector **a** in the direction of a vector **b**

$$p = |\mathbf{a}|\cos\gamma$$

$$|\mathbf{b}| / |\mathbf{b}| = 1$$

$$p = |\mathbf{a}|\cos\gamma \frac{|\mathbf{b}|}{|\mathbf{b}|} = \frac{|\mathbf{a}||\mathbf{b}|\cos\gamma}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$$

# 9.2 Inner Product (Dot Product)

## ☑ Ex.6 Normal Vector to a Plane

- Find a unit vector perpendicular to the plane $4x + 2y + 4z = -7$

Q: $\mathbf{a} \bullet \mathbf{r} = a_1 x + a_2 y + a_3 z = 0$

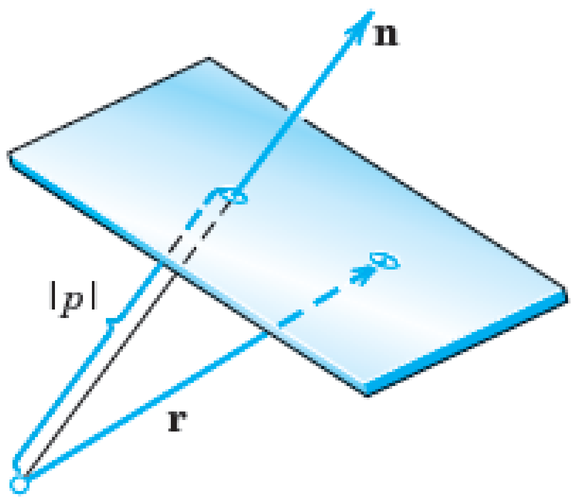



Fig. 184. Normal vector to a plane

We may write any plane in space as

$$\mathbf{a} \bullet \mathbf{r} = a_1 x + a_2 y + a_3 z = c \quad \ldots (1)$$

$$\mathbf{a} = [4, 2, 4], \quad c = -7$$

$$\mathbf{n} = \frac{1}{|\mathbf{a}|}\mathbf{a} \Rightarrow |\mathbf{a}| = 6, \; \mathbf{n} = \frac{1}{6}\mathbf{a}$$

Dividing Eq. (1) by $|\mathbf{a}|$

$$\mathbf{n}\Box\mathbf{r} = p \quad \ldots (2) \quad where, \; p = \frac{c}{|\mathbf{a}|}$$

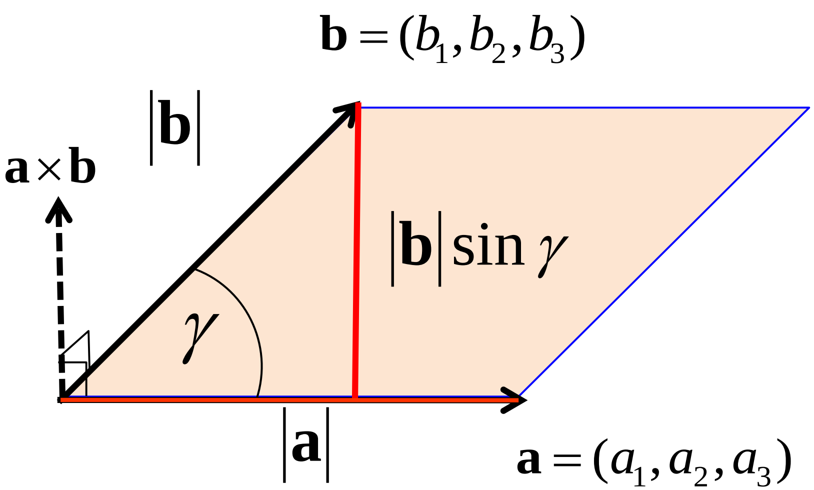
# 9.3 Vector Product (Cross Product, 외적)

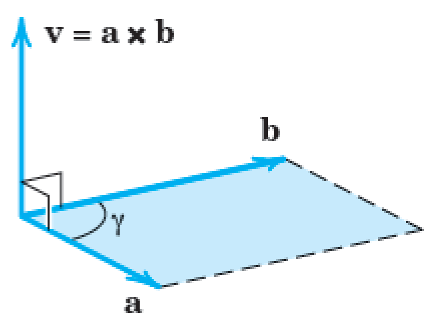
☑ **Definition Vector Product of Vectors**

- Vector product (Cross Product, Outer Product) $\mathbf{v} = \mathbf{a}\times\mathbf{b}$
- If $\mathbf{a}$ and $\mathbf{b}$ have the same or opposite direction, or if $\mathbf{a} = 0$ or $\mathbf{b} = 0$, then $\mathbf{a}\times\mathbf{b}=\mathbf{0}$

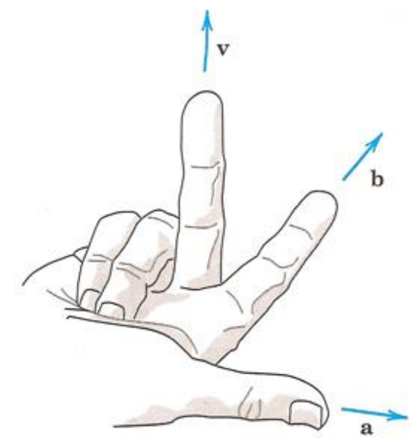
☑ Length of $|\mathbf{v}| = |\mathbf{a}||\mathbf{b}|\sin\gamma$ : the area of blue parallelogram

☑ Direction of $\mathbf{v}$: perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (form a right-handed triple)




Vector product




Right-handed triple of vectors **a, b, v**

# 9.3 Vector Product (Cross Product)

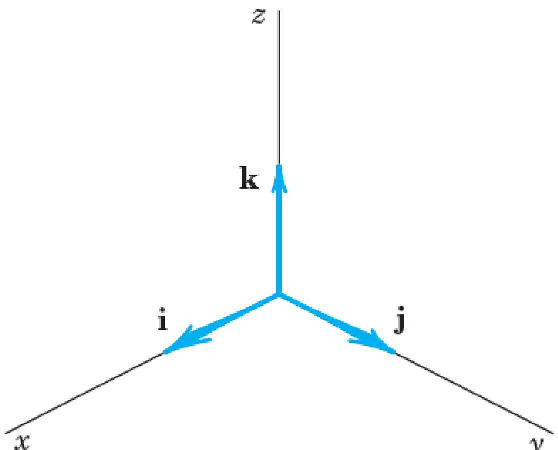
## ☑ In components

$$\mathbf{a} = [a_1,\ a_2,\ a_3],\ \mathbf{b} = [b_1,\ b_2,\ b_3]$$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

$$= [a_2b_3 - a_3b_2,\ \ a_3b_1 - a_1b_3,\ \ a_1b_2 - a_2b_1]$$

## ☑ Right-Handed Cartesian Coordinate System




Right-handed

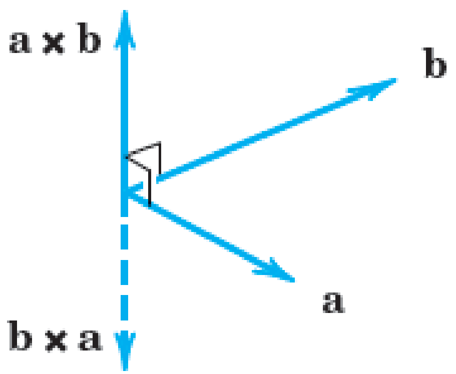
$$\mathbf{i}\times\mathbf{j} = \mathbf{k},\quad \mathbf{j}\times\mathbf{k} = \mathbf{i},\quad \mathbf{k}\times\mathbf{i} = \mathbf{j}$$

$$\mathbf{j}\times\mathbf{i} = -\mathbf{k},\quad \mathbf{k}\times\mathbf{j} = -\mathbf{i},\quad \mathbf{i}\times\mathbf{k} = -\mathbf{j}.$$

# 9.3 Vector Product (Cross Product)

## ☑ Theorem 1 General Properties of Vector Products

1. $(l\mathbf{a})\times\mathbf{b} = l(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(l\mathbf{b})$
2. $(a)\ \mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$
   $(b)\ (\mathbf{a}+\mathbf{b})\times\mathbf{c} = (\mathbf{a}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{c})$ (Distributive with respect to vector addition)
3. $\mathbf{b}\times\mathbf{a} = -(\mathbf{a}\times\mathbf{b})$ (Anticommutative, 반교환법칙)
4. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ (The parentheses cannot be omitted.)



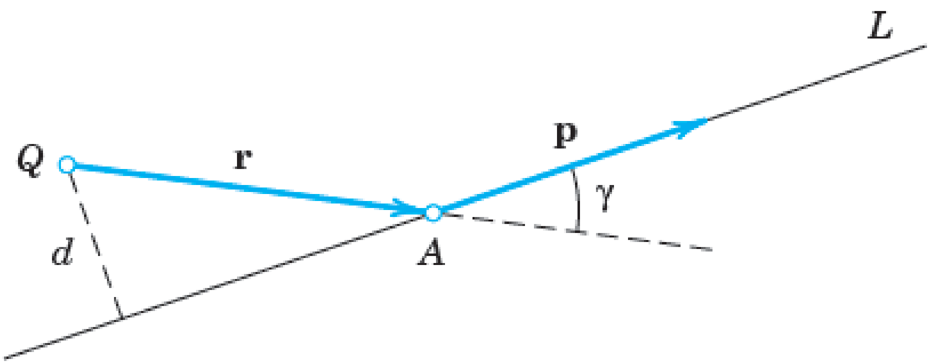

Anticommutativity

## ☑ Ex 3. Moment of a Force

- The moment $m$ of a force $\mathbf{p}$ about a point $Q$

$$m = |\mathbf{p}|\, d = |\mathbf{r}||\mathbf{p}|\sin\gamma = |\mathbf{r} \times \mathbf{p}|$$

$$Q\text{: } m = |\mathbf{p}|\, d = |\mathbf{p}||\mathbf{r}|\sin\gamma = \mathbf{p} \times \mathbf{r}\ ?$$



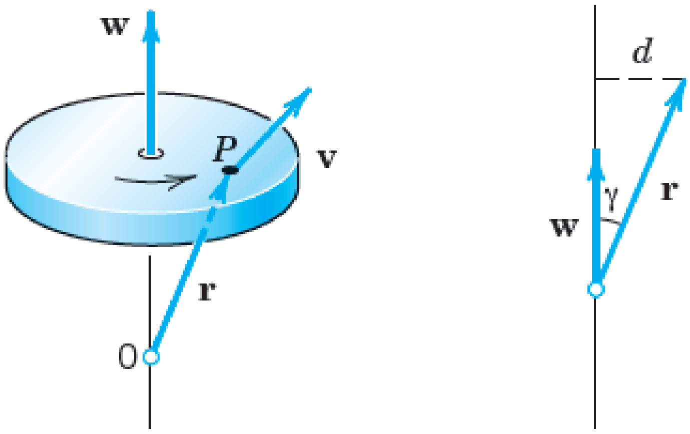
# 9.3 Vector Product (Cross Product)

## ☑ Ex 5. Velocity of a Rotating Body

- Rotation of a rigid body *B* in space can be simply and uniquely described by a vector **w**.
- The length of **w** = angular speed *ω* (각속력) of the rotation
- The speed (선속력) of P

$$\omega d = |\mathbf{w}||\mathbf{r}| \sin \gamma = |\mathbf{w} \times \mathbf{r}|$$

- V = **w** × **r**




Rotation of a rigid body

# 9.3 Vector Product (Cross Product)

## ☑ Scalar Triple Product (스칼라 삼중적)

- Scalar triple product of three vectors

$$\mathbf{a} = [a_1,\ a_2,\ a_3],\ \mathbf{b} = [b_1,\ b_2,\ b_3],\ \mathbf{c} = [c_1,\ c_2,\ c_3]$$

$$(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k}$$

# 9.3 Vector Product (Cross Product)

## ☑ Theorem 2 Properties and Applications of Scalar Triple Products

- The dot and cross can be interchanged:

$$(\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}) = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boxed{-}\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\boxed{=} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$$

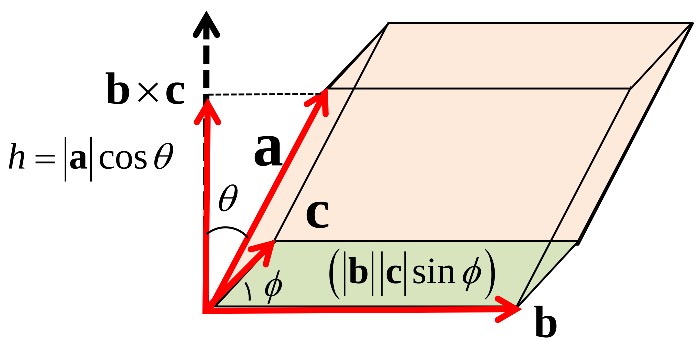
# 9.3 Vector Product (Cross Product)

## ☑ Theorem 2 Properties and Applications of Scalar Triple Products

- **Geometric Interpretation**
  The absolute value |( **a b c** )| is the volume of the parallelepiped (평행육면체) with **a**, **b**, **c** as edge vectors.
- **Linear Independence**
  Three vectors in $\mathbf{R}^3$ are linearly independent if and only if their scalar triple product is not zero.

The volume of the parallelepiped is

$$|\mathbf{b}\times\mathbf{c}|\,h = |\mathbf{a}|\,|\mathbf{b}\times\mathbf{c}|\,|\cos\theta| = |\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})|$$




**Geometric Interpretation**

Three nonzero vectors, whose initial points coincide, are linearly independent.

↕

The vectors do not lie in the same plane nor lie on the same straight line.

↕

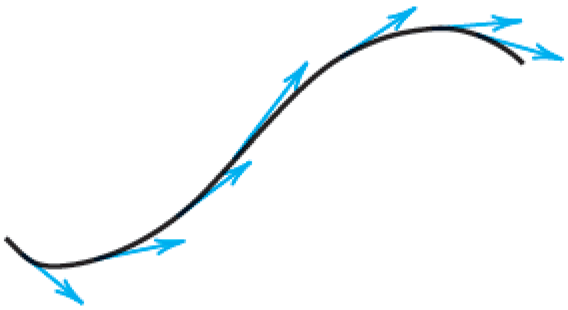
The triple product is not zero.

# 9.4 Vector and Scalar Functions and Fields. Vector Calculus: Derivatives

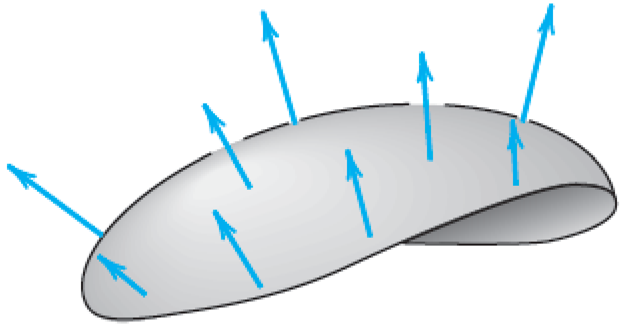
## ☑ Vector function (벡터 함수)

- Function whose values are vectors $\mathbf{v} = \mathbf{v}(P) = [v_1(P),\ v_2(P),\ v_3(P)]$ depending on the points P in space.
- A vector function defines a vector field.
- If we introduce Cartesian coordinates $x, y, z,$

$$\mathbf{v} = \mathbf{v}(x, y, z) = [v_1(x, y, z),\ v_2(x, y, z),\ v_3(x, y, z)]$$



Field of tangent vectors of a curve



Field of normal vectors of a surface

## ☑ Scalar function

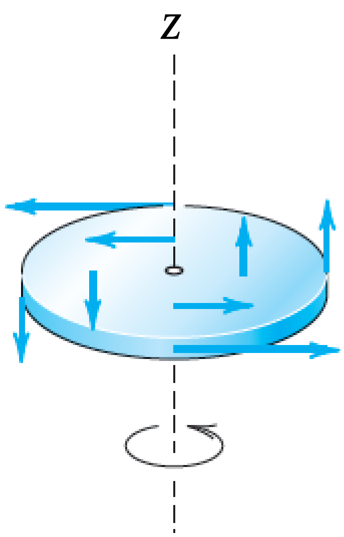
- Function whose values are scalars $f = f(P)$ depending in $P$
- A scalar function defines a scalar field.

  Ex. Temperature field in a body, Pressure field of the air in the earth's atmosphere

## ☑ Ex 2. Vector Field (Velocity Field)

$$\mathbf{v}(x, y, z) = \mathbf{w}\times\mathbf{r} = \mathbf{w}\times [x, y, z] = \underset{\mathbf{w} = \omega\mathbf{k}}{\underset{\uparrow}{\mathbf{w}}} \times(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \omega[-y, x, 0] = \omega(-yi + xj)$$




Velocity field of a rotation body

# 9.4 Vector and Scalar Functions and Fields. Vector Calculus: Derivatives

## ☑ Vector Calculus

- An infinite sequence (무한수열) of vectors $\mathbf{a}_{(n)}$ is said to converge if

$$\mathbf{a} = \lim_{n\to\infty} \mathbf{a}_{(n)} \quad \left(\mathbf{a}_{(n)}, n = 1, 2, \cdots \ \text{ converge to } \mathbf{a}\right)$$

$$\Leftrightarrow \ \text{There is } \mathbf{a} \text{ such that } \lim_{n\to\infty}\left|\mathbf{a}_{(n)} - \mathbf{a}\right| = 0$$

- $\mathbf{v}(t)$ is said to have limit $\boldsymbol{l}$ if $\lim_{t\to t_0} \mathbf{v}(t) = \mathbf{l} \quad \Leftrightarrow \quad \lim_{t\to t_0}\left|\mathbf{v}(t) - \mathbf{l}\right| = 0$

## ☑ Continuity

- $\mathbf{v}(t)$ is continuous at $t = t_0$ $\Leftrightarrow$ it is defined in some neighborhood of $t_0$ and $\lim_{t\to t_0} \mathbf{v}(t) = \mathbf{v}(t_0)$.
- $\mathbf{v}(t)=[v_1(t), v_2(t), v_3(t)]$ is continuous at $t_0$ $\Leftrightarrow$ its three components are continuous at $t_0$.

# 9.4 Vector and Scalar Functions and Fields. Vector Calculus: Derivatives

## ☑ Definition: Derivative of a Vector Function

- $\mathbf{v}(t)$ is differentiable at $t$ $\Leftrightarrow$ $\mathbf{v}'(t) = \lim_{\Delta t \to 0} \dfrac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$

- The derivative is obtained by differentiating each component separately.

$$\mathbf{v}'(t) = \left[ v_1'(t),\ \ v_2'(t),\ \ v_3'(t) \right]$$

- Properties of Derivative of a Vector Function

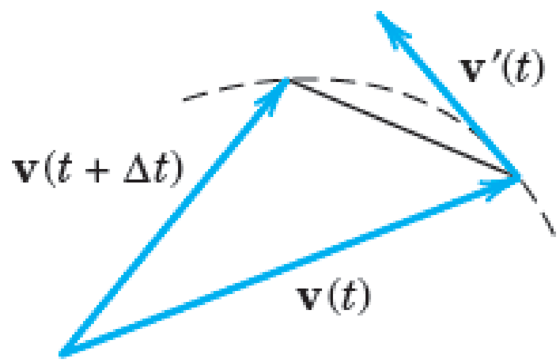
1. $(c\mathbf{v})' = c\mathbf{v}'$ $\qquad (c \text{ constant})$

2. $(\mathbf{u}+\mathbf{v})' = \mathbf{u}' + \mathbf{v}'$

3. $(\mathbf{u}\cdot\mathbf{v})' = \mathbf{u}'\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}'$

4. $(\mathbf{u}\times\mathbf{v})' = \mathbf{u}'\times\mathbf{v} + \mathbf{u}\times\mathbf{v}'$

5. $(\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w})' = (\mathbf{u}'\ \ \mathbf{v}\ \ \mathbf{w}) + (\mathbf{u}\ \ \mathbf{v}'\ \ \mathbf{w}) + (\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}')$




Derivative of a vector function

Q : Prove this.

# 9.4 Vector and Scalar Functions and Fields. Vector Calculus: Derivatives

## ☑ Partial Derivatives of a Vector Function

$$\mathbf{v} = [v_1,\ \ v_2,\ \ v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

are differentiable functions of $n$ variables $t_1, \ldots, t_n$

$$\frac{\partial \mathbf{v}}{\partial t_l} = \frac{\partial v_1}{\partial t_l}\mathbf{i} + \frac{\partial v_2}{\partial t_l}\mathbf{j} + \frac{\partial v_3}{\partial t_l}\mathbf{k}$$

$$\frac{\partial^2 \mathbf{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\mathbf{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\mathbf{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\mathbf{k}$$

# 9.5 Curves. Arc Length. Curvature. Torsion

☑ **Differential Geometry (미분 기하학)**

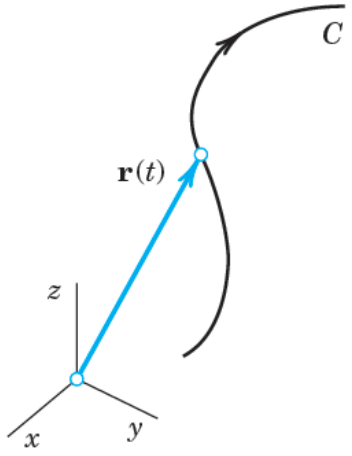
- A mathematical discipline that uses the methods of differential and integral calculus to study problems in geometry
- It plays a role in mechanics, computer-aided and traditional engineering design, geography (지리학), space travel, and relativity theory.

☑ **Parametric Representation:** Representation of the curve occurred as path of moving body

$$\mathbf{r}(t) = [x(t),\ \ y(t),\ \ z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$\rightarrow$ all three coordinates are dependent on $t$.

☑ Here $t$ is the parameter and $x$, $y$, $z$ are Cartesian coordinates.




Parametric representation of a curve

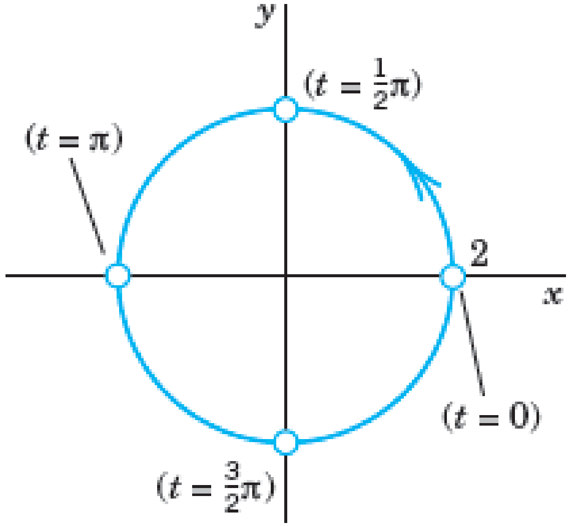
## ☑ Example 1

$$x^2 + y^2 = 4,\ z = 0$$

$$\mathbf{r}(t) = [2\cos t,\ \ 2\sin t,\ \ 0]$$
$$= 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$$

*Let* $t^* = -t \Rightarrow t = -t^*$

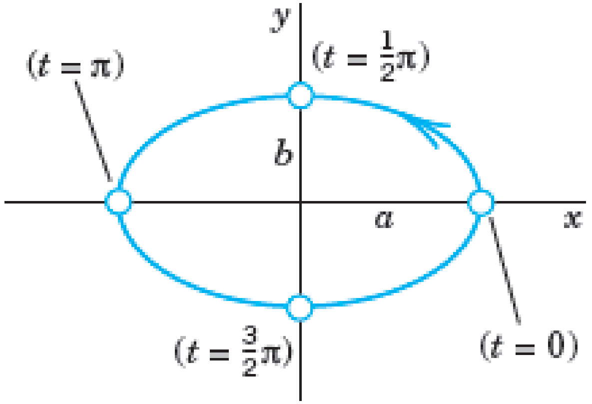
$$\mathbf{r}^*(t^*) = [2\cos(-t^*),\ \ 2\sin(-t^*),\ \ 0]$$



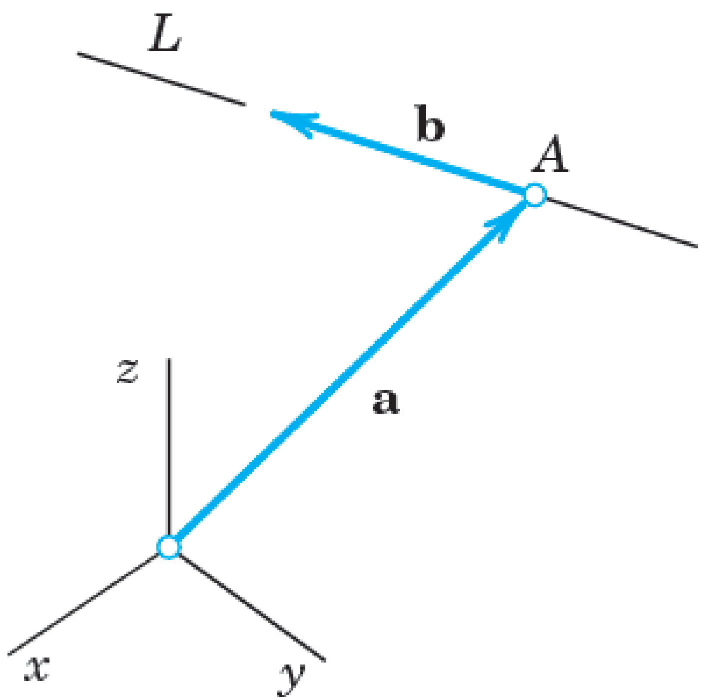

## ☑ Example 2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$$

$$\mathbf{r}(t) = [a\cos t,\ \ b\sin t,\ \ 0]$$
$$= a\cos t\mathbf{i} + b\sin t\mathbf{j}$$



## ☑ Example 3 Straight Line




Q: How to represent as a vector?

$$\mathbf{r}(t) = ?$$

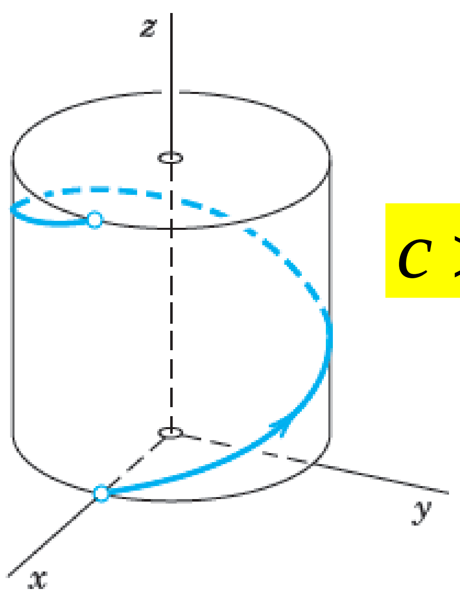
$$\mathbf{r}(t) = \mathbf{a} + t\mathbf{b}$$

Parametric representation of a straight line

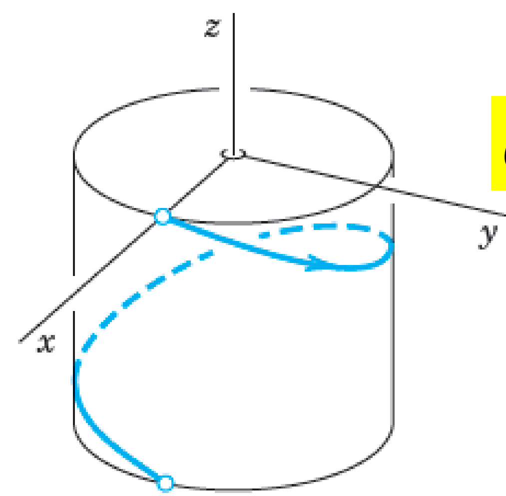
## ☑ Example 5 circular helix

$$\mathbf{r}(t) = [a\cos t,\ \ a\sin t,\ \ ct]$$

Q : How two helixes differ?




$c > 0$




$c < 0$

Right-handed circular helix

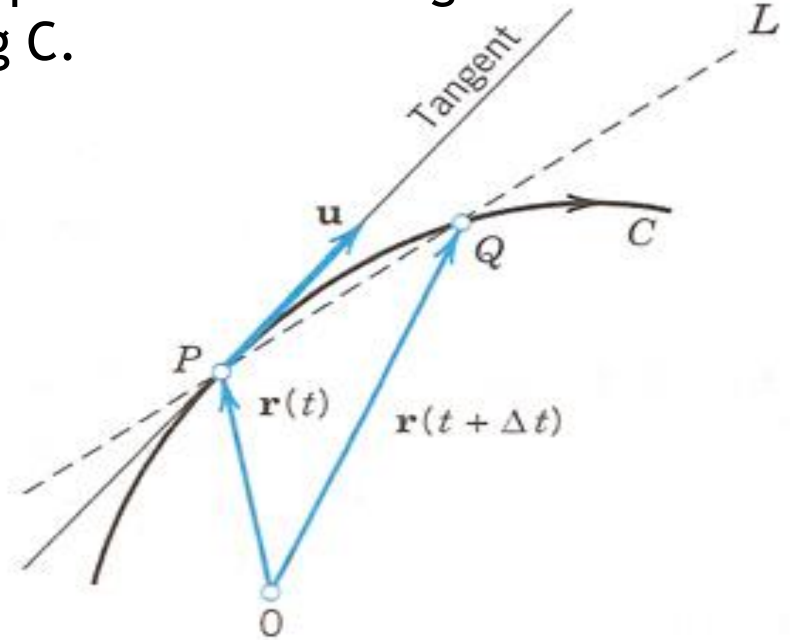
Left-handed circular helix

# 9.5 Curves. Arc Length. Curvature. Torsion

☑ **Tangent (접선) to a Curve**: The limiting position of a straight line L through P and Q as Q approaches P along C.

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)\right]$$

: Tangent vector of C at P



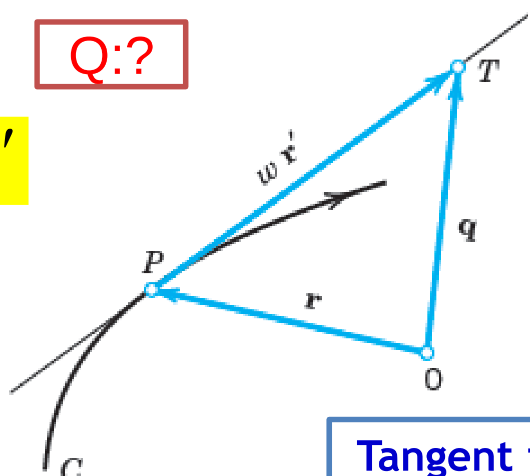

Tangent to a curve

☑ $\mathbf{u} = \dfrac{1}{|\mathbf{r}'|}\mathbf{r}'$ : Unit tangent vector

☑ Tangent to C at P

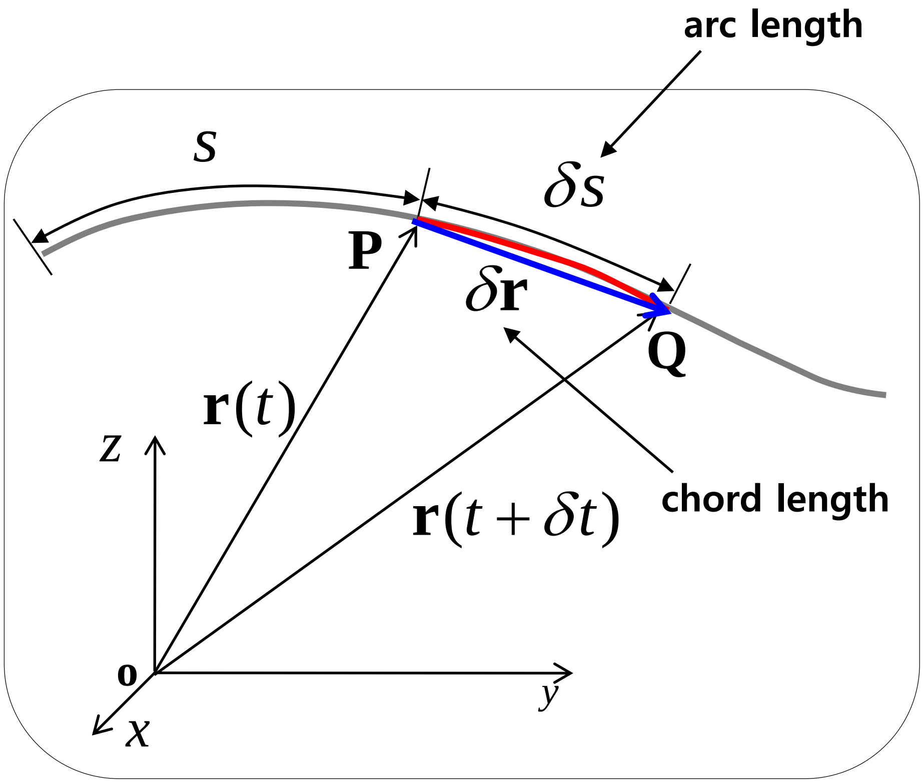
Q:?

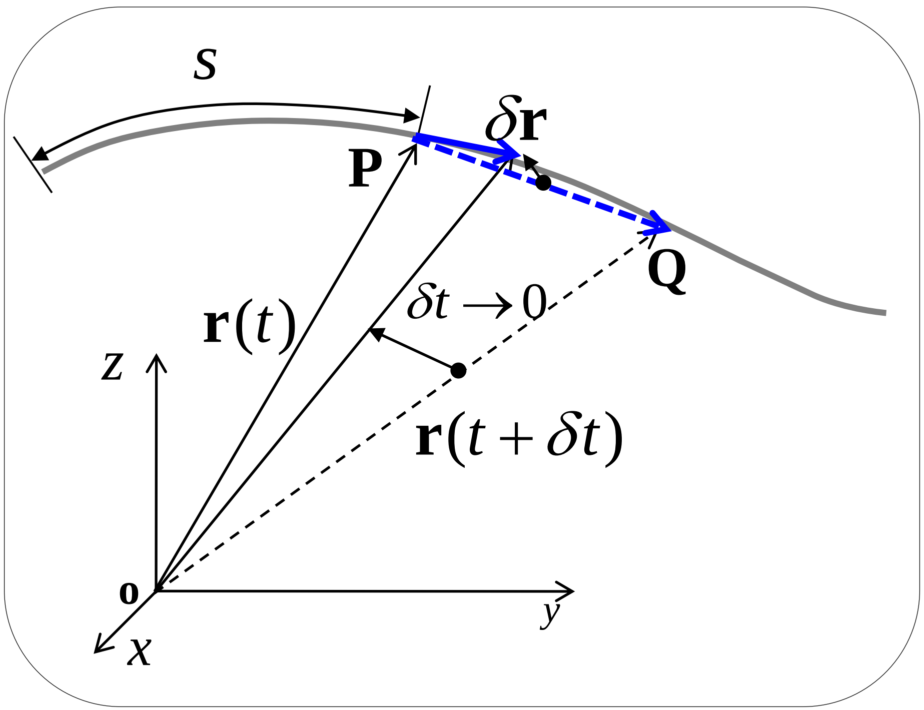
$$\mathbf{q}(w) = \mathbf{r} + w\mathbf{r}'$$




Tangent to C at P

# 9.5 Curves. Arc Length. Curvature. Torsion




**The chord PQ**

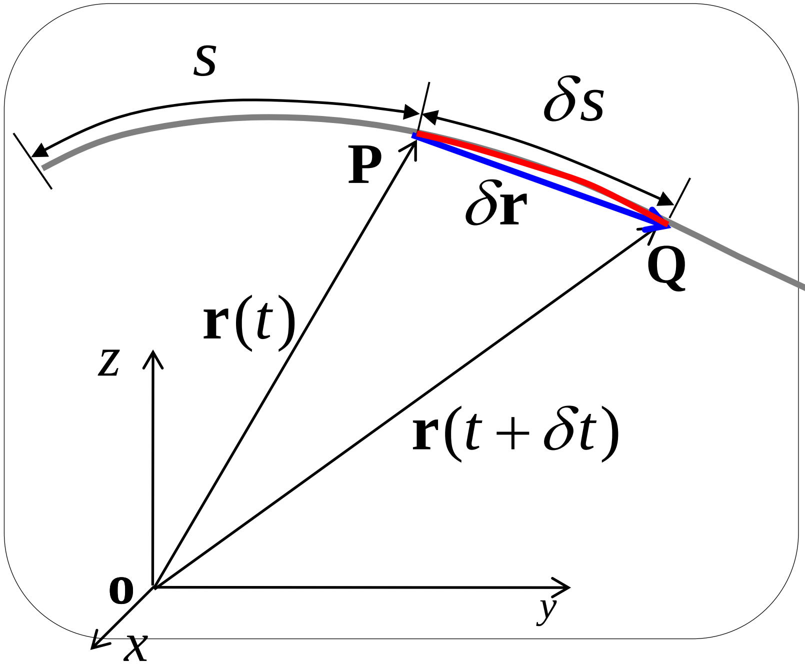
$$\delta\mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t)$$

**The tangent vector at P**

$$\lim_{\delta t \to 0} \frac{\delta\mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \mathbf{r}'$$

# 9.5 Curves. Arc Length. Curvature. Torsion

## ☑ Arc Length $s$ of a Curve




The chord PQ

**tangent vector at P**

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \mathbf{r}'$$

**Linear element of C: ds**

$$\lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \frac{ds}{dt} = \left| \frac{d\mathbf{r}}{dt} \right| = |\mathbf{r}'| \quad \Rightarrow \quad ds = |\mathbf{r}'|\, dt$$

$$s = \int_{t_0}^{t_1} ds = \int_{t_0}^{t_1} s'(t)\,dt = \int_{t_0}^{t_1} |\mathbf{r}'(t)|\, dt$$

$$= \int_{t_0}^{t_1} \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, dt$$

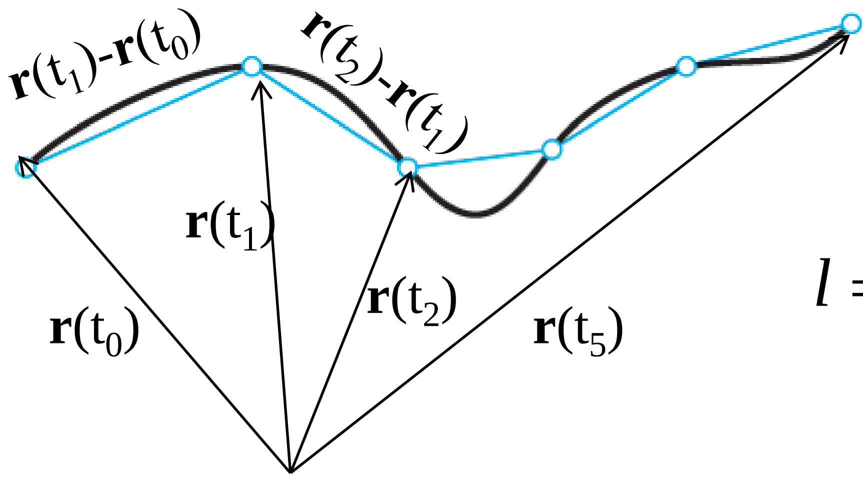
**Arc length of $s$**

$$s(t) = \int_{t_0}^{t} \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, d\tilde{t} \qquad \left( \mathbf{r}' = \frac{d\mathbf{r}}{d\tilde{t}} \right)$$

# 9.5 Curves. Arc Length. Curvature. Torsion

## ☑ Length of a Curve

$$t_0(=a),\, t_1, \ldots, t_{n-1},\, t_n(=b), \quad where \;\; t_0 < t_1 < \ldots, < t_n$$

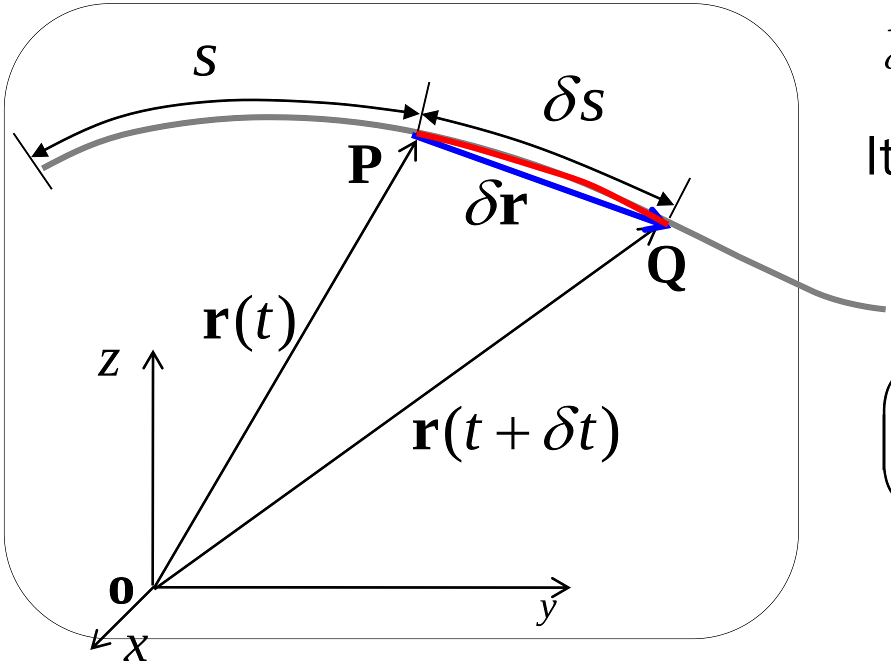



$$l = \int_a^b \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, dt \qquad \left( \mathbf{r}' = \frac{d\mathbf{r}}{dt} \right)$$

Length of a curve

# 9.5 Curves. Arc Length. Curvature. Torsion

## ☑ Linear Element *ds*




The chord PQ

$$\lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \frac{ds}{dt} = \left| \frac{d\mathbf{r}}{dt} \right| = |\mathbf{r}'|$$

It is customary to write

$$d\mathbf{r} = [dx, dy, dz]$$

$$\left( \frac{ds}{dt} \right)^2 = \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} = |\mathbf{r}'(t)|^2$$

$$= \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2$$

$$\Rightarrow \quad ds^2 = d\mathbf{r} \cdot d\mathbf{r} = dx^2 + dy^2 + dz^2$$

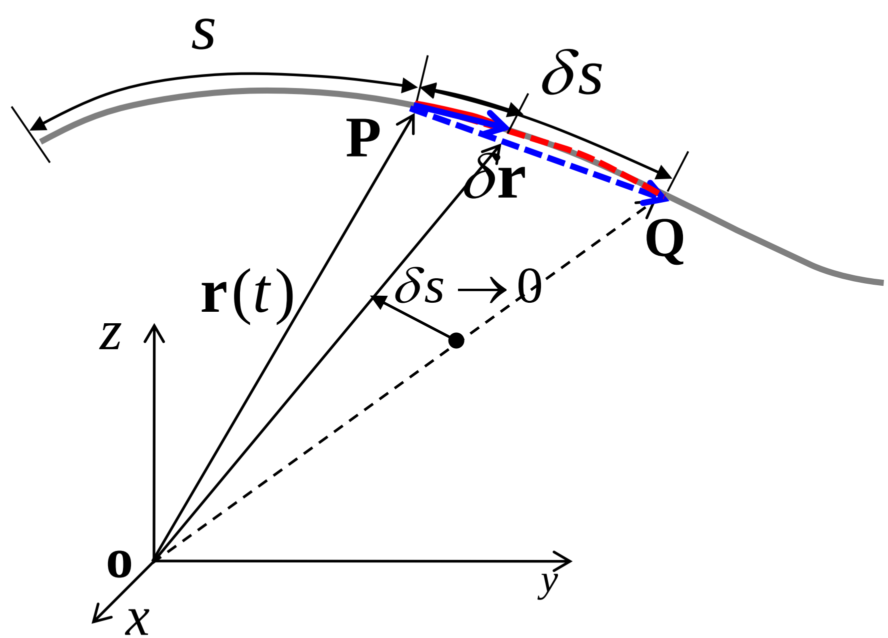
# 9.5 Curves. Arc Length. Curvature. Torsion

## ☑ Arc Length as Parameter

- Unit tangent vector

$$\mathbf{u}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t)$$

Q: **r**(t) vs **r**(s), **u**(t) vs **u**(s) ?




- **If choose $s$ as parameter, then chord length and arc length become equal in the limit.**

- **The unit tangent vector at P**

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{u}(s)$$

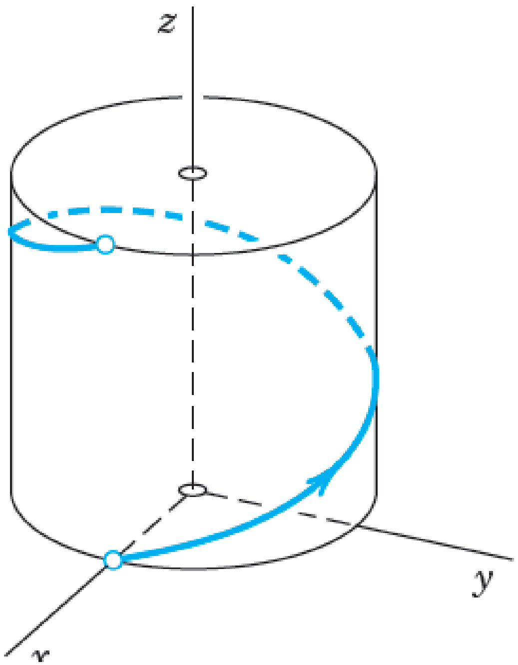
$$\therefore \mathbf{u}(s) = \frac{d\mathbf{r}(s)}{ds} = \mathbf{r}'(s)$$

$$|\mathbf{r}'(s)| = \lim_{\delta s \to 0}\left|\frac{\delta \mathbf{r}}{\delta s}\right| = \frac{ds}{ds} = 1$$

## ☑ Example 6 Circular helix

$$\mathbf{r}(t) = [a\cos t, \ \ a\sin t, \ \ ct]$$

Q: represent **r** with arc length $s$




$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = a^2 + c^2 = K^2$$

$$ds = K dt$$

$$s = Kt$$

$$t = s / K$$

$$\mathbf{r}^*(s) = \mathbf{r}\left(\frac{s}{K}\right) = \left[a\cos\frac{s}{K}, \quad a\sin\frac{s}{K}, \quad \frac{cs}{K}\right], \qquad K = \sqrt{a^2 + c^2}.$$

# 9.5 Curves. Arc Length. Curvature. Torsion

## ☑ Curves in Mechanics. Velocity. Acceleration

- Curves serve as path of moving body in mechanics
- Curve is represented by a parametric representation $\mathbf{r}(t)$ with $t$ as parameter.
- The tangent vector $\mathbf{r}'(t)$ of C is the velocity vector $\mathbf{v}(t)$
- The second derivative of $\mathbf{r}(t)$ is the acceleration vector $\mathbf{a}(t)$

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

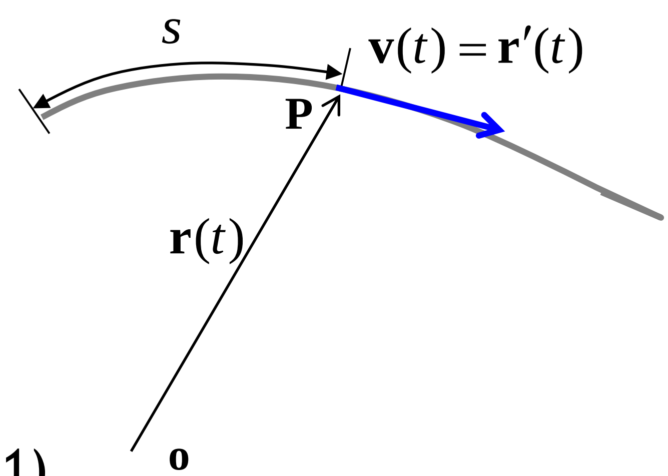
# 9.5 Curves. Arc Length. Curvature. Torsion

## ☑ Tangential and Normal Acceleration: $\mathbf{a} = \mathbf{a}_{\text{tan}} + \mathbf{a}_{\text{norm}}$

- Tangential velocity vector

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{u}(s)\frac{ds}{dt}$$

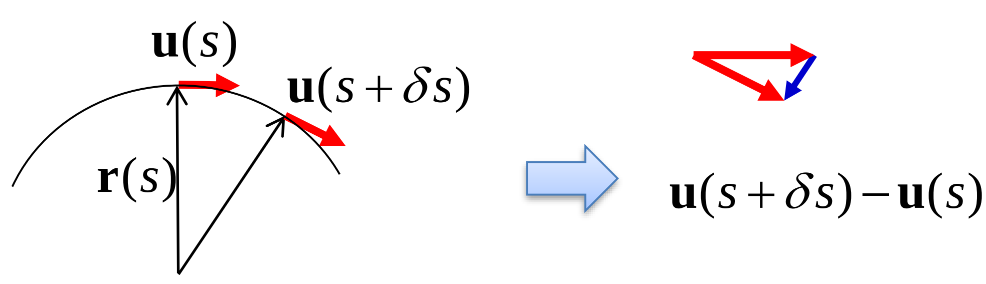
$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{u}(s)\frac{ds}{dt}\right) = \frac{d\mathbf{u}}{ds}\left(\frac{ds}{dt}\right)^2 + \mathbf{u}(s)\frac{d^2 s}{dt^2}$$



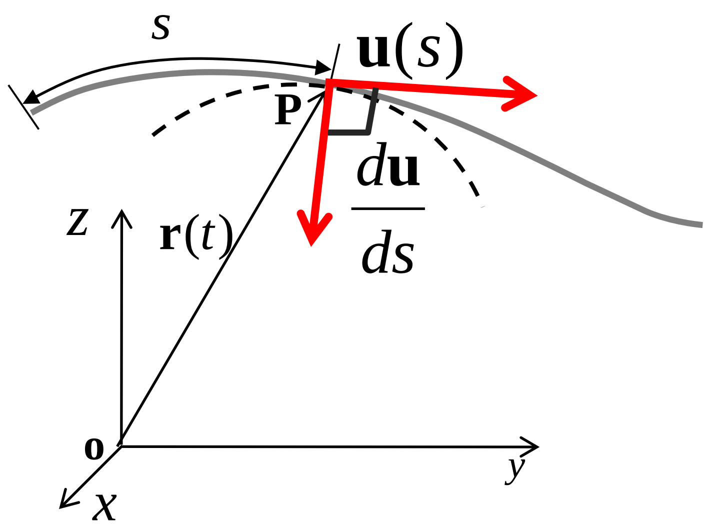

- Tangent vector $\mathbf{u}(s)$ has constant length (=1)

$\Rightarrow$ $\dfrac{d\mathbf{u}}{ds}$ is perpendicular to $\mathbf{u}(s)$

$$\frac{d\mathbf{u}}{ds} = \lim_{\delta s \to 0} \frac{\mathbf{u}(s+\delta s) - \mathbf{u}(s)}{\delta s}$$



## ☑ Derivative du/ds is perpendicular to u.




**Proof)**

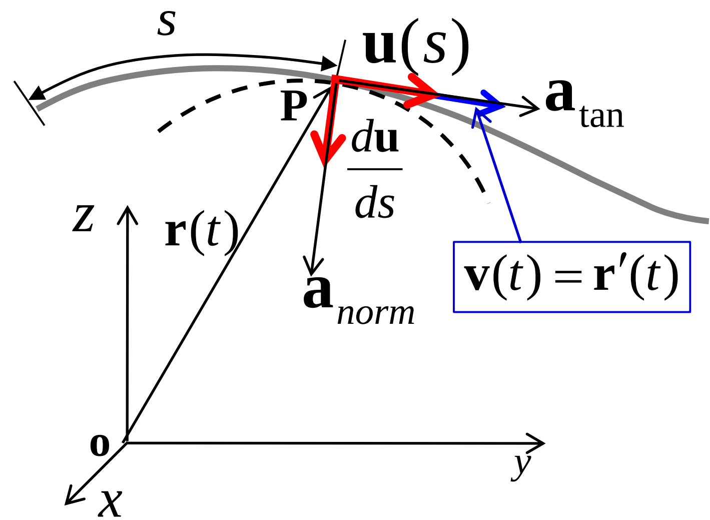
$$\mathbf{u}\cdot\mathbf{u}=1$$

$$\frac{d(\mathbf{u}\cdot\mathbf{u})}{ds}=\mathbf{u}\cdot\frac{d\mathbf{u}}{ds}+\frac{d\mathbf{u}}{ds}\cdot\mathbf{u}=2\frac{d\mathbf{u}}{ds}\cdot\mathbf{u}=0$$

$$\therefore\frac{d\mathbf{u}}{ds}\perp\mathbf{u}$$

## ☑ Tangential and Normal Acceleration: $\mathbf{a} = \mathbf{a}_{\tan} + \mathbf{a}_{\text{norm}}$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{u}(s)\frac{ds}{dt}\right) = \frac{d\mathbf{u}}{ds}\left(\frac{ds}{dt}\right)^2 + \mathbf{u}(s)\frac{d^2s}{dt^2}$$




- Normal acceleration vector (= $\mathbf{a}_{\text{norm}}$)

$$\mathbf{a}_{norm} = \frac{d\mathbf{u}}{ds}\left(\frac{ds}{dt}\right)^2$$

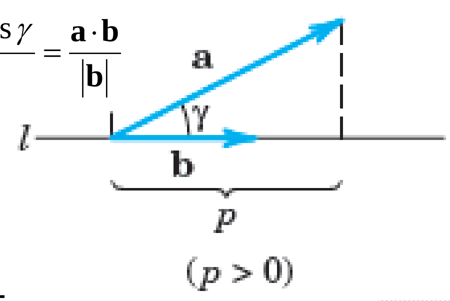
- Tangential acceleration vector (= $\mathbf{a}_{\tan}$)

$$\mathbf{a}_{\tan} = \mathbf{u}(s)\frac{d^2s}{dt^2}$$

- $|\mathbf{a}_{\tan}|$ : the absolute value of the projection of **a** in the direction of **V**

$$|\mathbf{a}_{\tan}| = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|}$$

$$p = |\mathbf{a}|\cos\gamma\frac{|\mathbf{b}|}{|\mathbf{b}|} = \frac{|\mathbf{a}||\mathbf{b}|\cos\gamma}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$$




$$\mathbf{a}_{\tan} = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|}\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}, \qquad \mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\tan}$$

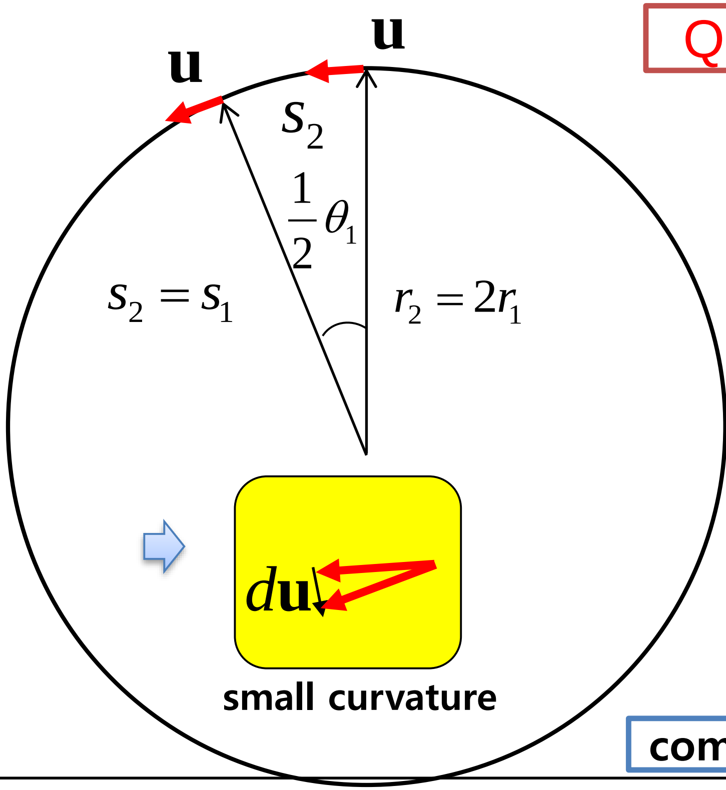
## ☑ Curvature (곡률)

- Curvature $\kappa(s)$ of a curve C ($\mathbf{r}(s)$) at P: The rate of change of the unit tangent vector $\mathrm{u}(s)$ at P

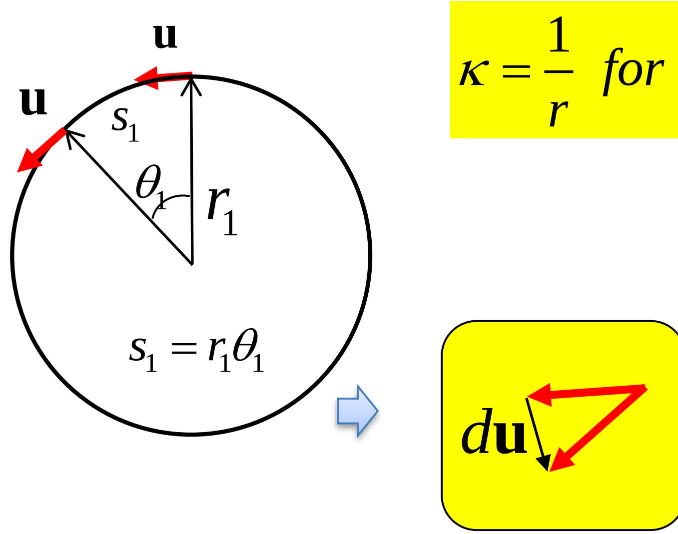
$$\kappa(s) = |\mathbf{u}'(s)| = |\mathbf{r}''(s)| \qquad \left( ' = \frac{d}{ds} \right)$$

$$\boxed{\therefore \mathbf{u}(s) = \mathbf{r}'(s)}$$

Q: which one has larger curvature?



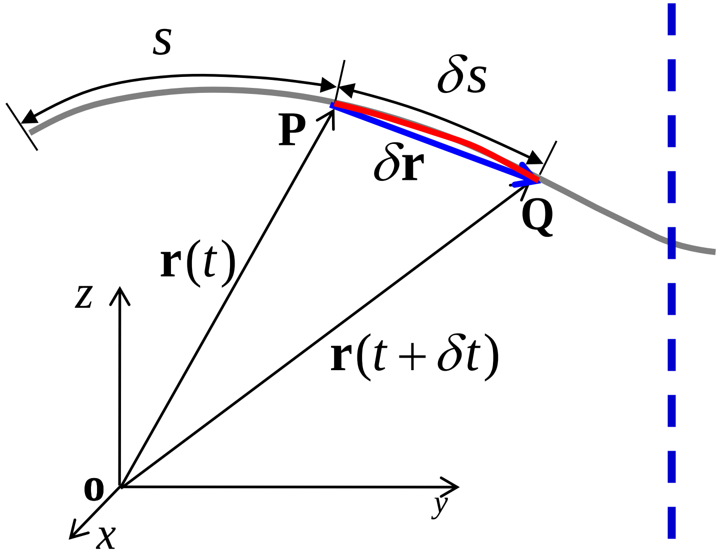

$$\kappa = \frac{1}{r} \textit{ for } \text{circle}$$

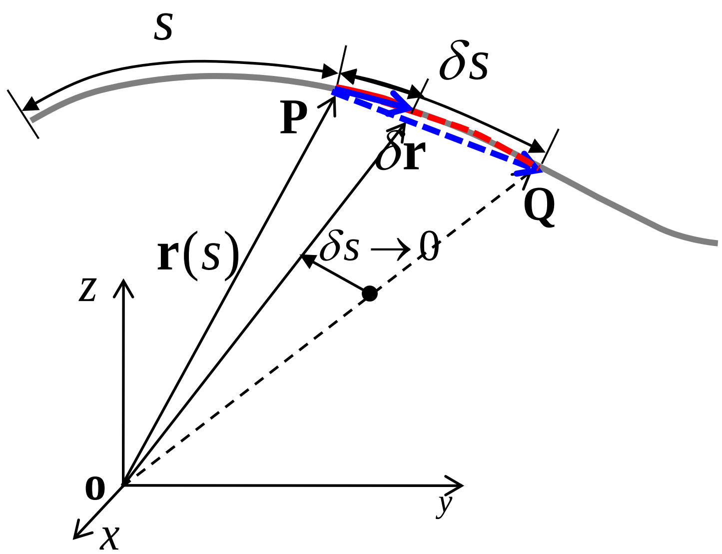



compare with the same arc length

# 9.5 Curves. Arc Length. Curvature. Torsion

## ☑ Summary




$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right|$$

Unit tangent vector

$$\mathbf{u}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t)$$

$$\mathbf{u}(s) = \frac{d\mathbf{r}(s)}{ds} = \mathbf{r}'$$

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{u}(s)\frac{ds}{dt}$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{u}(s)\frac{ds}{dt}\right) = \frac{d\mathbf{u}}{ds}\left(\frac{ds}{dt}\right)^2 + \mathbf{u}(s)\frac{d^2 s}{dt^2}$$

# 9.5 Curves. Arc Length. Curvature. Torsion

## ☑ Example 7. Centripetal acceleration (구심가속도). Centrifugal force (원심력)

Q: prove acceleration is toward center.

$$\mathbf{r}(t) = [R\cos\omega t,\ \ R\sin\omega t] = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j}$$

$$\mathbf{v} = \mathbf{r}' = [-R\omega\sin\omega t,\ \ R\omega\cos\omega t] = -R\omega\sin\omega t\,\mathbf{i} + R\omega\cos\omega t\,\mathbf{j}$$

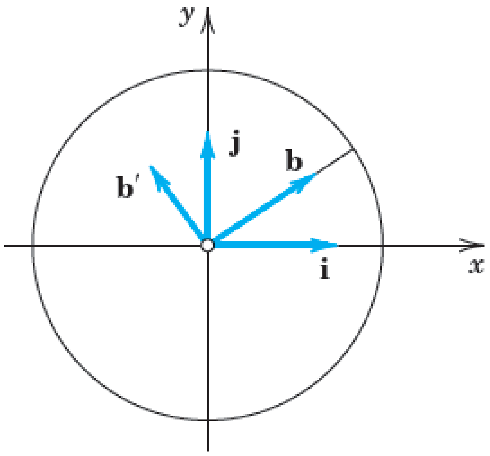
$$|\mathbf{v}| = |\mathbf{r}'| = \sqrt{\mathbf{r}'\cdot\mathbf{r}'} = R\omega$$

$\mathbf{v}$ is tangent to C

$$\mathbf{a} = \mathbf{v}' = \left[-R\omega^2\cos\omega t,\ \ -R\omega^2\sin\omega t\right] = -R\omega^2\cos\omega t\,\mathbf{i} - R\omega^2\sin\omega t\,\mathbf{j}$$

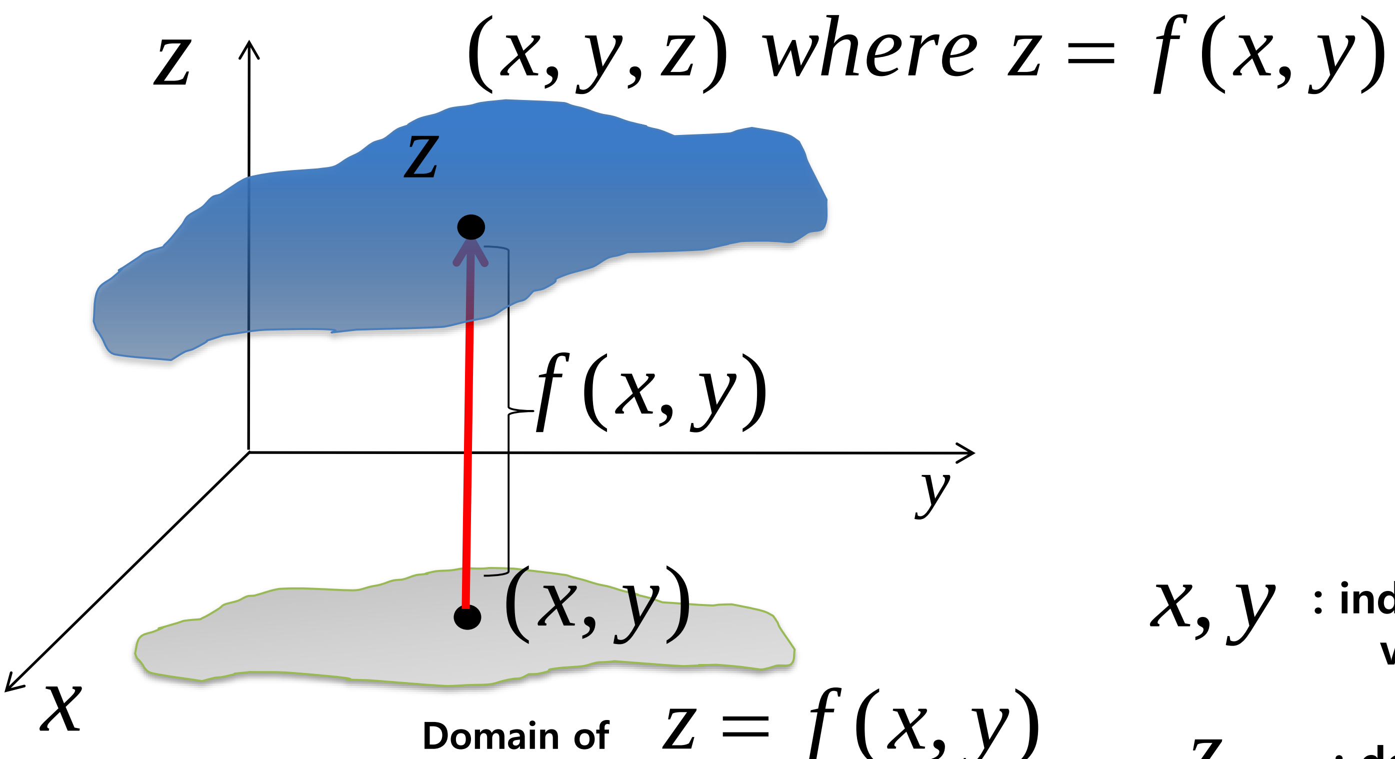
$$\mathbf{a} = -\omega^2\mathbf{r}$$

$\mathbf{a}$ : toward center, centrifugal force
$m\mathbf{a}$ : centripetal force (구심력)
$-m\mathbf{a}$ : centrifugal force (원심력)



# 9.6 Calculus Review: Functions of Several Variables

## ☑Partial Derivatives




$x, y$ : independent variables

$z$ : dependent variable

# 9.6 Calculus Review: Functions of Several Variables

**Ordinary Derivatives**

$$y = f(x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

**Partial Derivatives**

$$z = f(x, y)$$

$$\frac{\partial z}{\boxed{\partial x}} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\boxed{\Delta x}} = \frac{\partial f}{\partial x} = z_x = f_x$$

**partial derivative with respect to** $x$ **treating** $y$ **as a constant**

$$\frac{\partial z}{\boxed{\partial y}} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\boxed{\Delta y}} = \frac{\partial f}{\partial y} = z_y = f_y$$

**partial derivative with respect to** $y$ **treating** $x$ **as a constant**

## ☑ Example 1
## Partial Derivatives

If $z=4x^3y^2-4x^2+y^6+1$, find $\partial z/\partial x$ and $\partial z/\partial y$.

$$\frac{\partial z}{\partial x} = 12x^2y^2 - 8x$$

$$\frac{\partial z}{\partial y} = 8x^3y^2 + 6y^5$$

## ☑ Example 2
## Partial Derivatives

If $F(x,y,t)=e^{-3\pi t}\cos 4x \sin 6y$, then the partial derivatives with respect to $x$, $y$, and $t$ are, in turn,

$$F_x(x,y,t) = -4e^{-3\pi t}\sin 4x \sin 6y$$

$$F_y(x,y,t) = 6e^{-3\pi t}\cos 4x \cos 6y$$

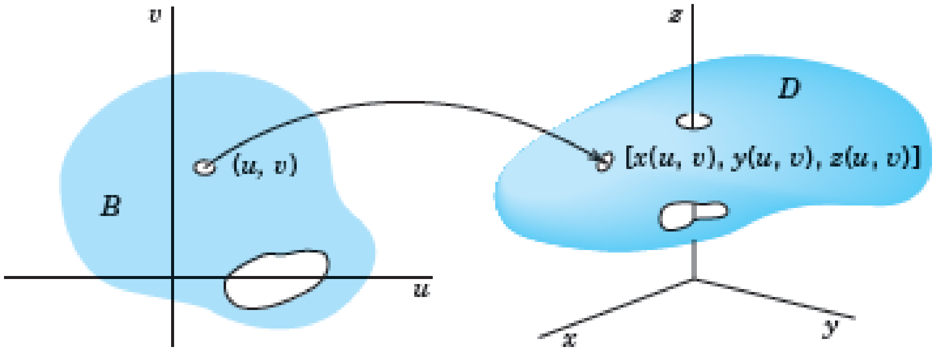
$$F_t(x,y,t) = -3\pi e^{-3\pi t}\cos 4x \sin 6y$$

## ☑ Chain Rules

- *Let $w=f(x,y,z)$ be continuous and have continuous first partial derivatives in a domain D in xyz-space. Let $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$ be functions that are continuous and have first partial derivatives in a domain B in the uv-plane*

$$w=f\left(x(u,v),y(u,v),z(u,v)\right) \quad \Rightarrow \quad \frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$




**Notations in Theorem 1**

# 9.6 Calculus Review: Functions of Several Variables

## ☑ Special Cases of Practical Interest

$$w = f\left(x(u,v), y(u,v)\right) \quad \Rightarrow \quad \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$

$$w = f\left(x(t), y(t), z(t)\right) \quad \Rightarrow \quad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$w = f\left(x(t), y(t)\right) \quad \Rightarrow \quad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

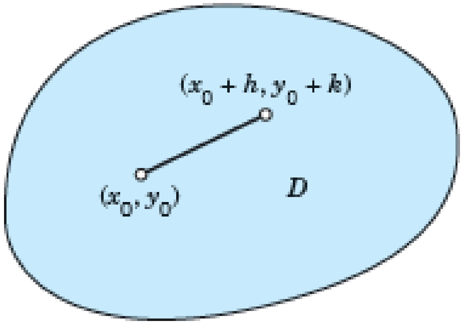
$$w = f\left(x(t)\right) \quad \Rightarrow \quad \frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

# 9.6 Calculus Review: Functions of Several Variables

## ☑ Theorem 2 Mean Value Theorem (평균값 정리)

- Let $f(x, y, z)$ be continuous and have continuous first partial derivatives in a domain D in xyz-space.
- Let $P_0 : (x_0, y_0, z_0)$ and $P : (x_0+h, y_0+k, z_0+l)$ be points in D such that the straight line segment $P_0P$ joining these points lies entirely in D.

  Then $$\boxed{f(x_0+h, y_0+k, z_0+l) - f(x_0, y_0, z_0) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z}}$$
- the partial derivatives being evaluated at a suitable point of that segment.



# 9.6 Calculus Review: Functions of Several Variables

## ☑ Special Cases of Practical Interest

- For a function $f(x, y)$ of two variables,

$$f\left(x_0+h, y_0+k\right)-f\left(x_0, y_0\right)=h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}$$

- For a function $f(x)$ of a single variable,

$$f\left(x_0+h\right)-f\left(x_0\right)=h\frac{\partial f}{\partial x}$$

$$\frac{f\left(x_0+h\right)-f\left(x_0\right)}{h}=\left.\frac{\partial f}{\partial x}\right|_{\text{a point between } x_0 \text{ and } x_0+h}$$

# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Definition 1 Gradient (기울기)

**Differentiable function**

$$z = f(x, y)$$

$$w = F(x, y, z)$$

**Vector differential operator**

**Gradients of function**

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$\nabla F(x, y, z) = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$\nabla$ : "del" or "nabla"

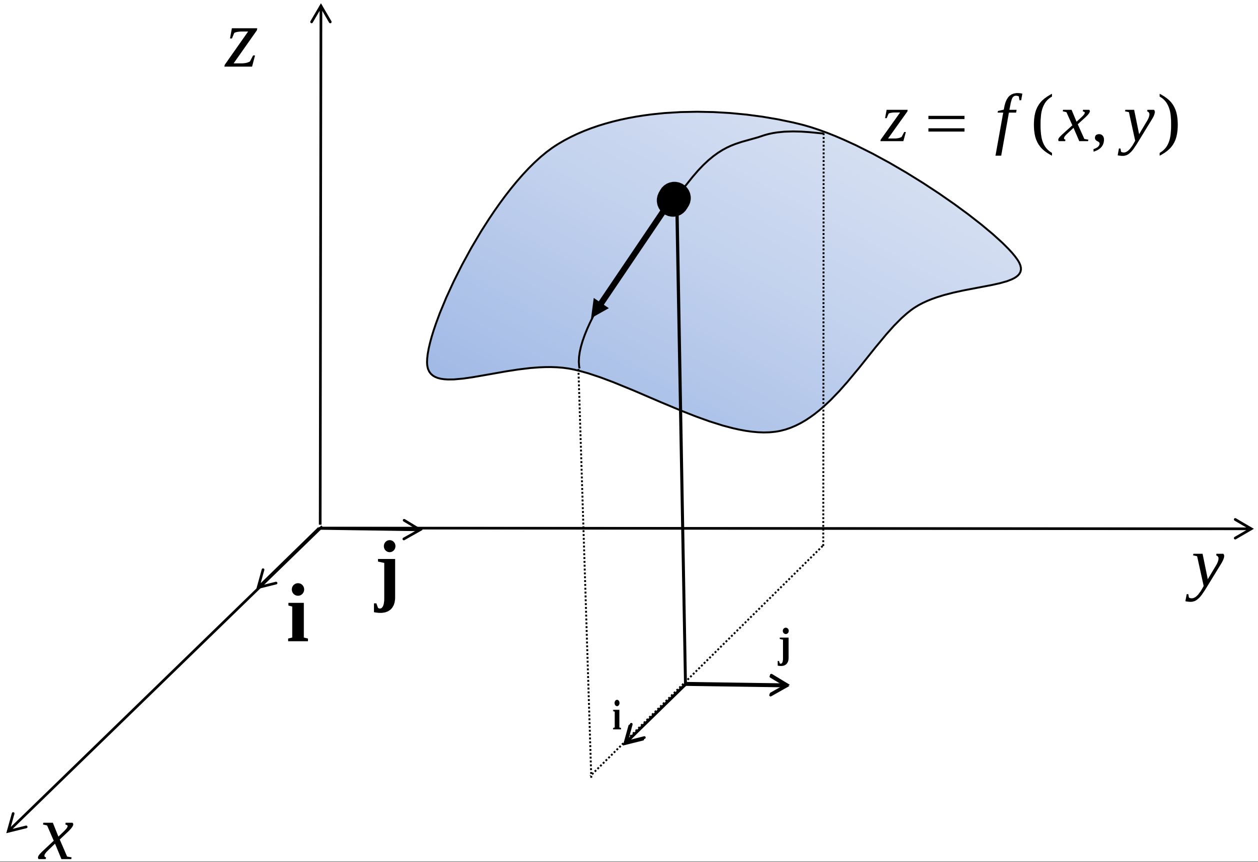
$\nabla f$ : "grad $f$"

*** Gradient: scalar field => vector field**
*** Divergence: vector field => scalar field**
*** Curl: vector field => new vector field**

# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Directional Derivatives (방향도함수)

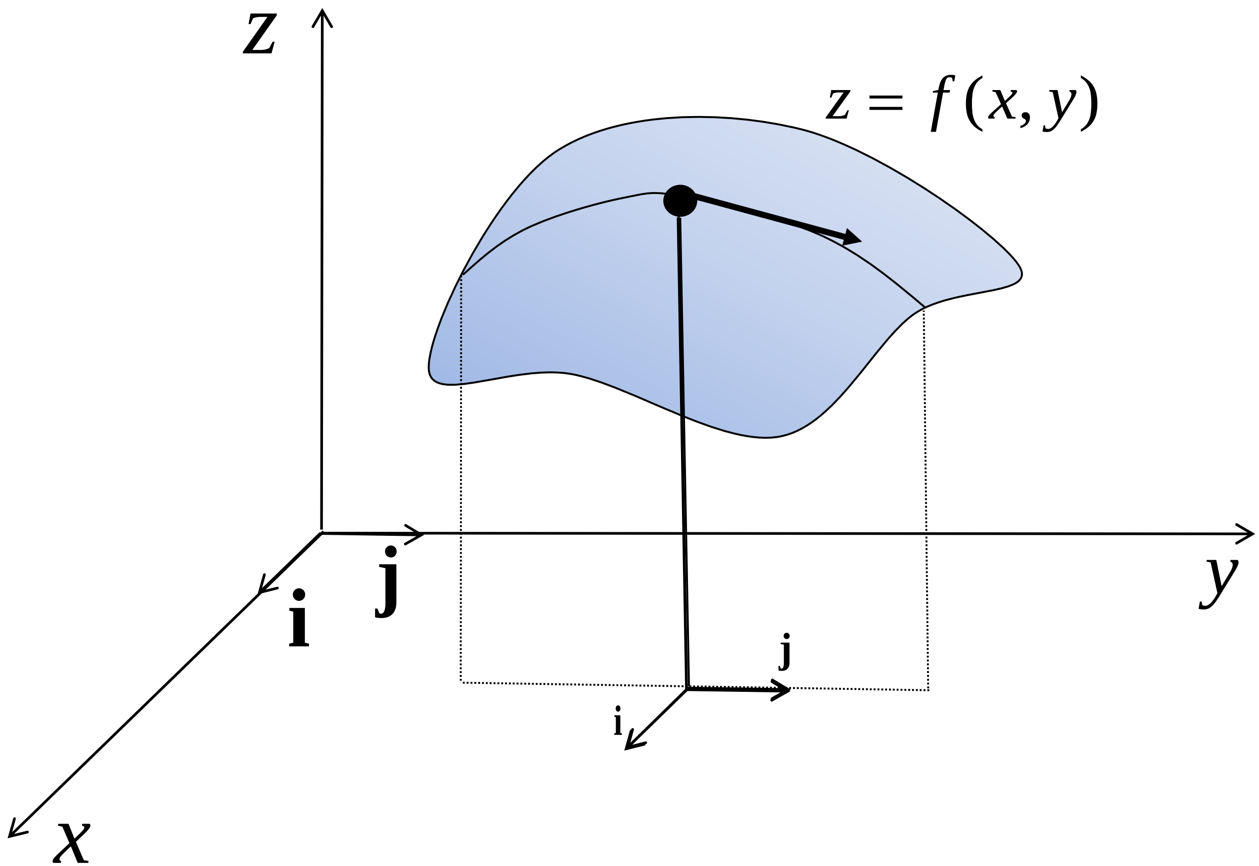
$\dfrac{\partial f}{\partial x}$ :Rate of change of $f$ in the i -direction



# 9.7 Gradient of a Scalar Field. Directional Derivative

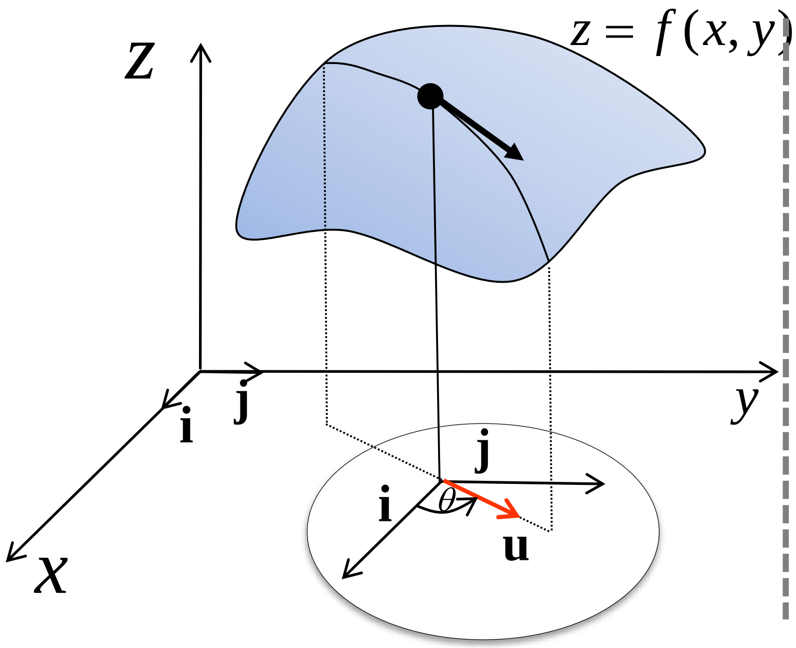
## ☑ Directional Derivatives (방향도함수)

$\dfrac{\partial f}{\partial y}$ :Rate of change of $f$ in the j-direction



# 9.7 Gradient of a Scalar Field. Directional Derivative

☑ **The rate of change of $f$ in the direction given by the vector u : $D_{\mathbf{u}}f$**




The rate of change of $f$ in $\mathbf{x}$ : $\dfrac{\partial f}{\partial x}$

$\Rightarrow$ The component of $\dfrac{\partial f}{\partial x}$ in $\mathbf{u}$ : $\dfrac{\partial f}{\partial x}\cos\theta$

The rate of change of $f$ in $\mathbf{y}$ : $\dfrac{\partial f}{\partial y}$

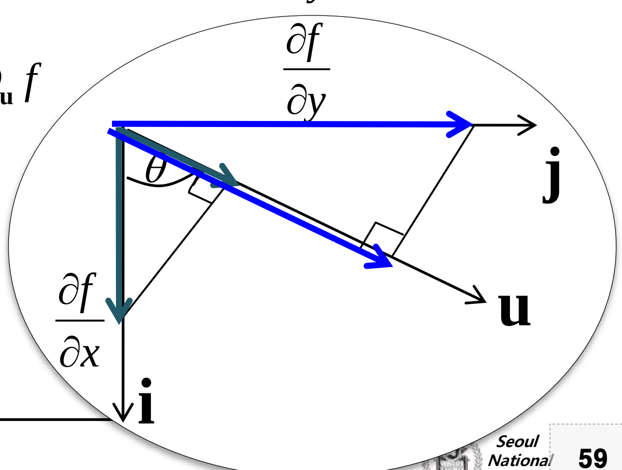
$\Rightarrow$ The component of $\dfrac{\partial f}{\partial y}$ in $\mathbf{u}$ : $\dfrac{\partial f}{\partial y}\cos(\dfrac{\pi}{2}-\theta)$

$$= \frac{\partial f}{\partial y}\sin\theta$$

The rate of change of $f$ in the direction given by the vector $\mathbf{u}$ : $D_{\mathbf{u}}f$

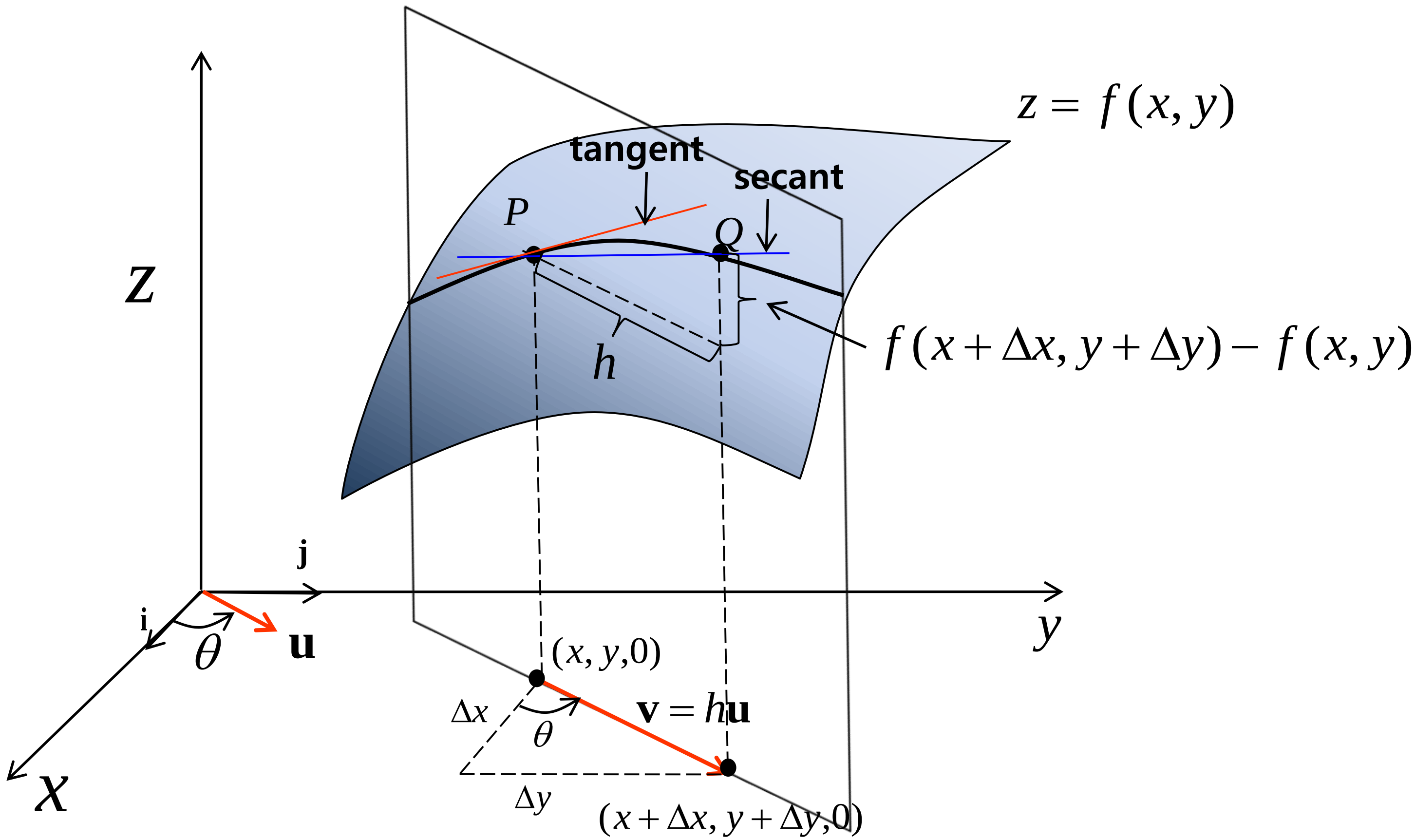
$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$= (\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j})\cdot(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = \nabla f\cdot\mathbf{u}$$



# 9.7 Gradient of a Scalar Field. Directional Derivative

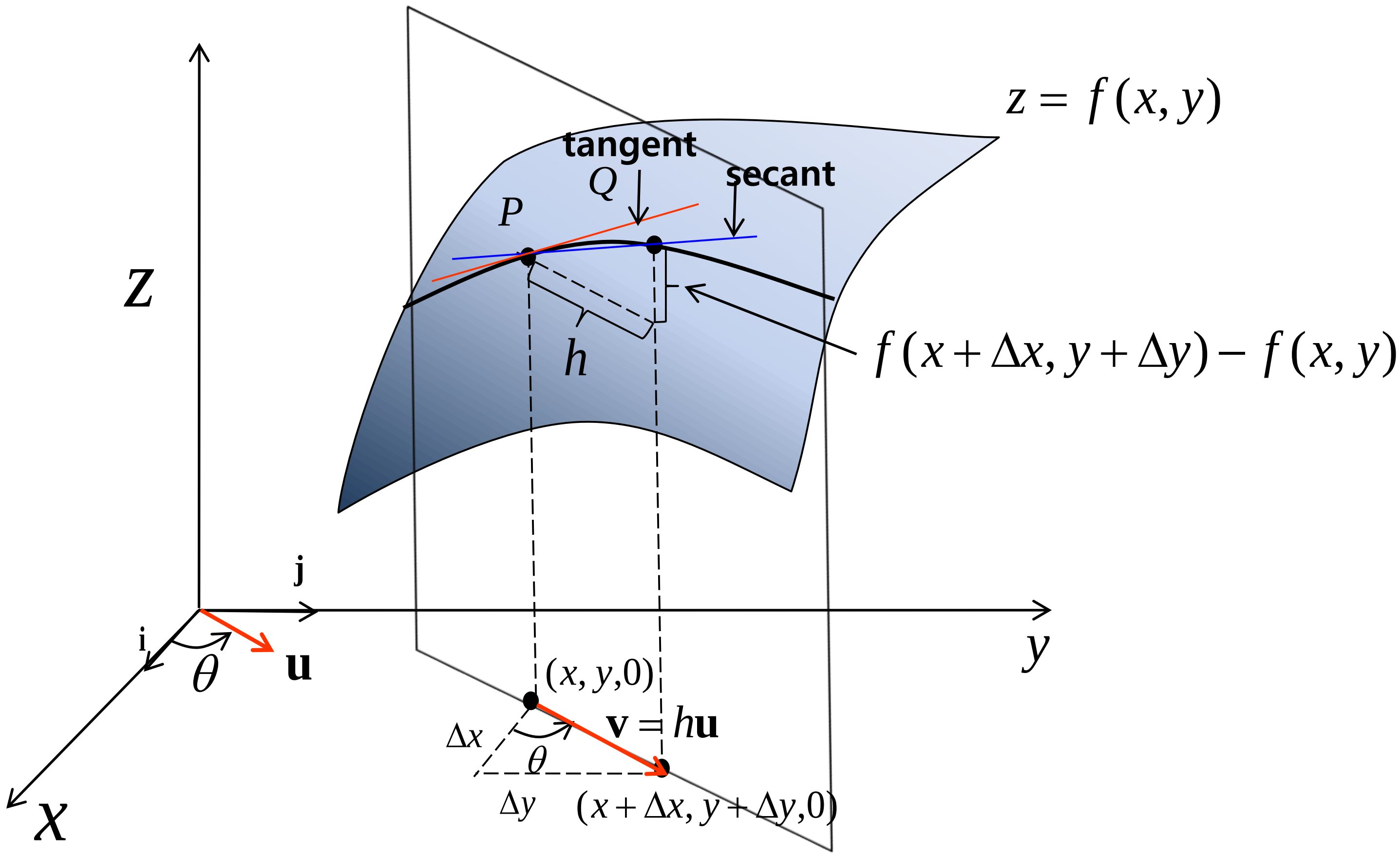
## ☑ Directional Derivatives (방향도함수)




**Slope of indicated secant line is**

$$\frac{f(x+\Delta x, y+\Delta y) - f(x, y)}{h} = \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x, y)}{h}$$

## ☑ Directional Derivatives (방향도함수)

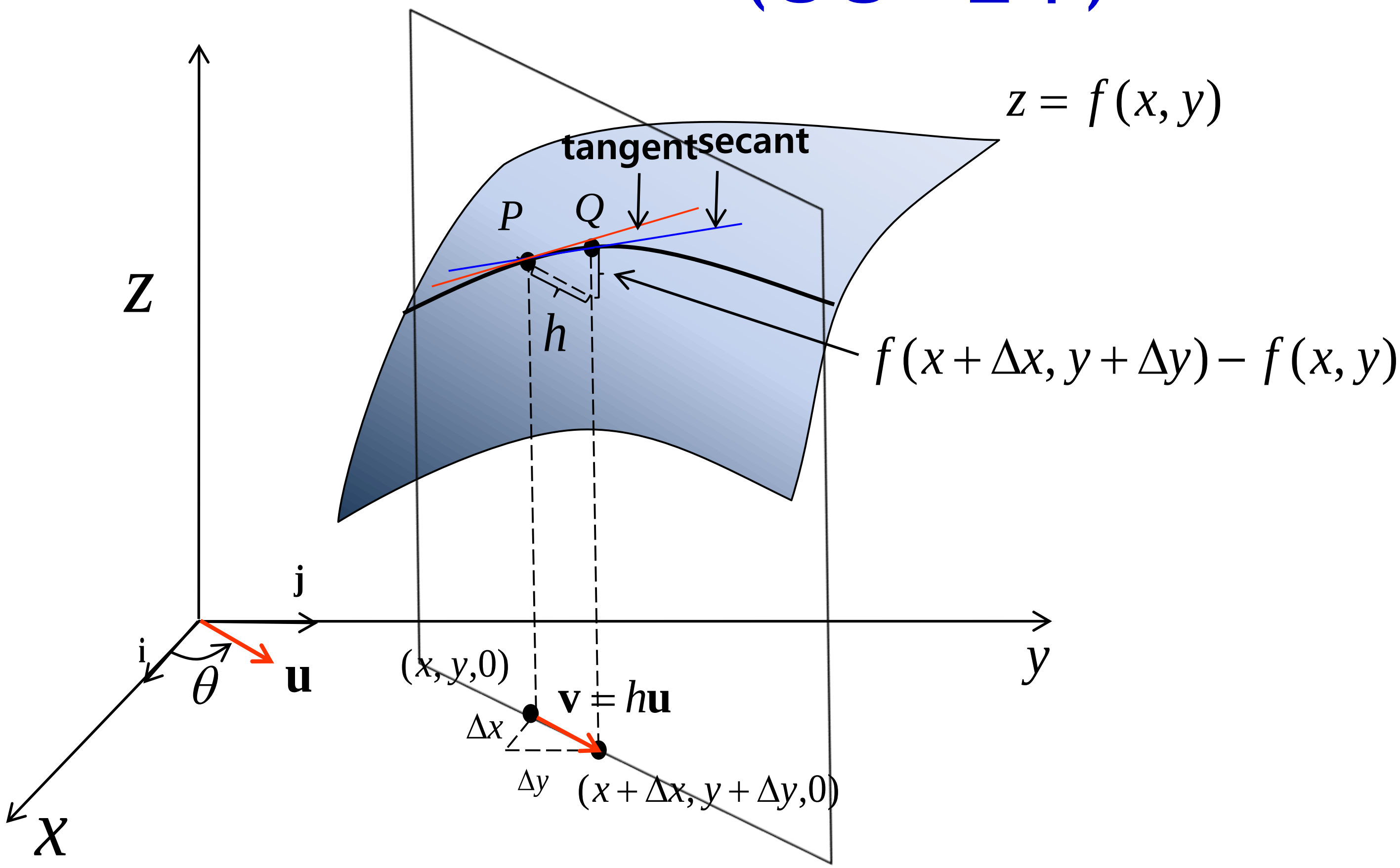



**Slope of indicated secant line is**

$$\frac{f(x+\Delta x, y+\Delta y)-f(x,y)}{h}=\frac{f(x+h\cos\theta, y+h\sin\theta)-f(x,y)}{h}$$

# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Directional Derivatives (방향도함수)




**As $h \to 0$ we expect the slope of secant line would be tangent line**

$$\frac{f(x+\Delta x, y+\Delta y) - f(x, y)}{h} = \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x, y)}{h}$$

# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Directional Derivative

The directional derivative of $z = f(x, y)$ in the direction of a unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is

$$D_{\mathbf{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h}$$

**provided the limit exists.**

$$\theta = 0 \text{ implies } D_{\mathbf{i}} f(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial z}{\partial x}$$

$$\theta = \frac{\pi}{2} \text{ implies } D_{\mathbf{j}} f(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} = \frac{\partial z}{\partial y}$$

# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Directional Derivative

**If** $z = f(x, y)$ **is differentiable function of x and y and** $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ **then,**

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

**Proof)**

$$g(t) = f(x + t\cos\theta, y + t\sin\theta)$$

by definition of a derivative

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h} = D_{\mathbf{u}} f(x, y)$$

by the Chain Rule

$$g'(t) = f_x(x + t\cos\theta, y + t\sin\theta)\frac{d}{dt}(x + t\cos\theta) + f_y(x + t\cos\theta, y + t\sin\theta)\frac{d}{dt}(y + t\sin\theta)$$

$$= f_x(x + t\cos\theta, y + t\sin\theta)\cos\theta + f_y(x + t\cos\theta, y + t\sin\theta)\sin\theta$$

$$\therefore g'(0) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Directional Derivative

**If** $z = f(x, y)$ **is differentiable function of x and y and** $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ **then,**

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

**Proof)**

$$g(t) = f(x + t\cos\theta, y + t\sin\theta)$$

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}} f(x, y) = g'(0) &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \\ &= [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \\ &= \nabla f(x, y) \cdot \mathbf{u} \end{aligned}$$

# 9.7 Gradient of a Scalar Field. Directional Derivative

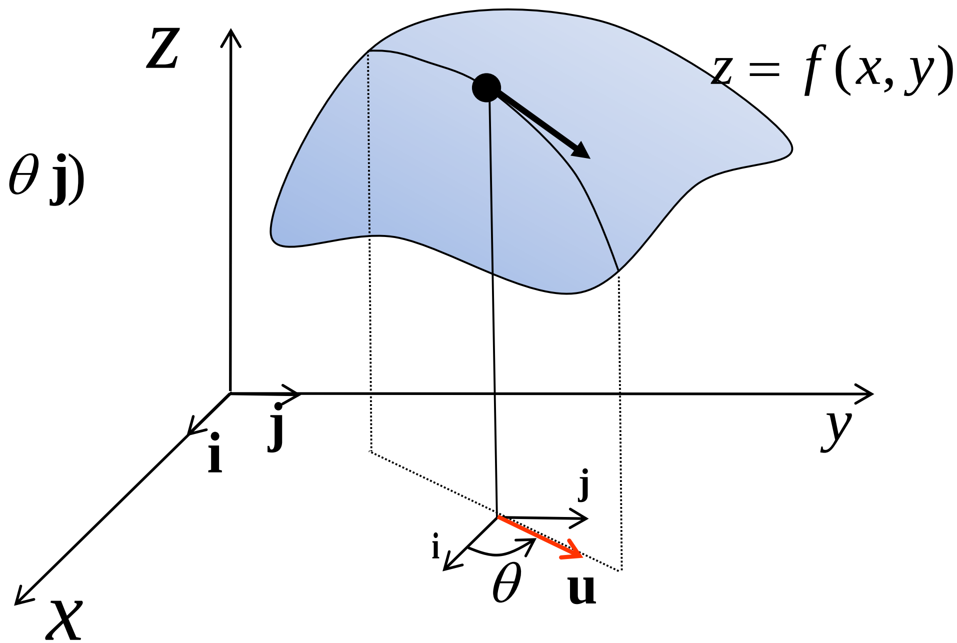
## ☑ Directional Derivative

If $z = f(x, y)$ is differentiable function of x and y and $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ then,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \\ &= \nabla f(x, y) \cdot \mathbf{u} \end{aligned}$$

(scalar value)

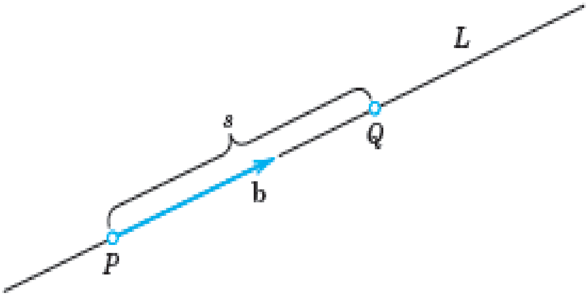


# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Definition 2 Directional Derivative

- The directional derivative of $f(x, y, z)$ at *P* in the direction of **b**:

$$D_{\mathbf{b}} f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}$$




- Here *Q* is a variable point on the straight line L in the direction of **b** and |s| is the distance between *P* and *Q*

$$L\ :\ \mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k} = \mathbf{p}_0 + s\mathbf{b}$$

$$\left(|\mathbf{b}| = 1, \quad \text{where } \mathbf{p}_0 \text{ the position vector of } P\right)$$

$$\mathbf{r}'(s) = x'(s)\mathbf{i} + y'(s)\mathbf{j} + z'(s)\mathbf{k} = \mathbf{b}$$

$$\Rightarrow \quad D_{\mathbf{b}} f = \frac{df}{ds} = \frac{\partial f}{\partial x} \underset{\uparrow \atop \frac{\partial x}{\partial s}}{x'} + \frac{\partial f}{\partial y} \underset{\uparrow \atop \frac{\partial y}{\partial s}}{y'} + \frac{\partial f}{\partial z} \underset{\uparrow \atop \frac{\partial z}{\partial s}}{z'} = \mathbf{b} \cdot \operatorname{grad} f$$

for a vector **a** of any length

$$D_{\mathbf{a}} f = \frac{1}{|\mathbf{a}|} \mathbf{a} \cdot \operatorname{grad} f$$

# ☑ Example 1
## Directional Derivative

Find the directional derivative of $f(x,y)=2x^2y^3+6xy$ at $(1,1)$ in the direction of a unit vector whose angle with the positive x-axis is $\pi/6$.

Solution)

$$\nabla f(x,y)=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}$$

$$=(4xy^3+6y)\mathbf{i}+(6x^2y^2+6x)\mathbf{j}$$

$$\nabla f(1,1)=10\mathbf{i}+12\mathbf{j}$$

$$\mathbf{u}=\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}=\cos\frac{\pi}{6}\mathbf{i}+\sin\frac{\pi}{6}\mathbf{j}$$

$$D_{\mathbf{u}}f(1,1)=\nabla f(1,1)\cdot\mathbf{u}$$

$$=(10\mathbf{i}+12\mathbf{j})\cdot\left(\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)$$

$$=5\sqrt{3}+6$$

## ☑ Example 2
## Directional Derivative

- Consider the plane that is perpendicular to the $xy$-plane.
- The plane passes through the points $P(2,1)$ and $Q(3,2)$.
- What is the slope of the tangent line to the curve on intersection of this plane with the surface $f(x,y)=4x^2+y^2$ at $(2,1,17)$ in the direction of Q?

Solution)

$$f(x,y) = 4x^2 + y^2$$

$$\nabla f(x,y) = 8x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla f(2,1) = 16\mathbf{i} + 2\mathbf{j}$$

$$\overrightarrow{PQ} = \mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}} f(2,1) &= \nabla f(2,1) \bullet \mathbf{u} \\ &= (16\mathbf{i} + 2\mathbf{j}) \bullet \left( \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \right) \\ &= 9\sqrt{2} \end{aligned}$$

# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Theorem 1 Gradient is a Vector. Maximum Increase

- Let $f(P) = f(x, y, z)$ be a scalar function having continuous first partial derivatives in some domain *B* in space.
- Then **grad** $f$ exists in *B* and is a vector, that is, its length and direction are independent of the particular choice of Cartesian coordinates.
- If $\text{grad}\, f(P) \neq 0$ at some point *P*, it has the direction of maximum increase of $f$ at *P*.

*** Gradient: scalar field => vector field**
*** Divergence: vector field => scalar field**
*** Curl: vector field => new vector field**

# 9.7 Gradient of a Scalar Field. Directional Derivative

$$D_{\mathbf{b}} f(x, y) = \nabla f(x, y) \cdot \mathbf{b}, \quad |\mathbf{b}| = 1$$

## ☑ Proof

$$D_{\mathbf{b}} f = |\nabla f||\mathbf{b}|\cos\phi = |\nabla f|\cos\phi, \quad \phi\text{:angle between } \nabla f \text{ and } \mathbf{b}$$

$$-1 \le \cos\phi \le 1$$

**The maximum value of** $D_{\mathbf{b}} f$ ⇨ $D_{\mathbf{b}} f = |\nabla f|$ **, When** $\cos\phi = 1, \ \phi = 0$

⇩

**b** has the same direction of $\nabla f$

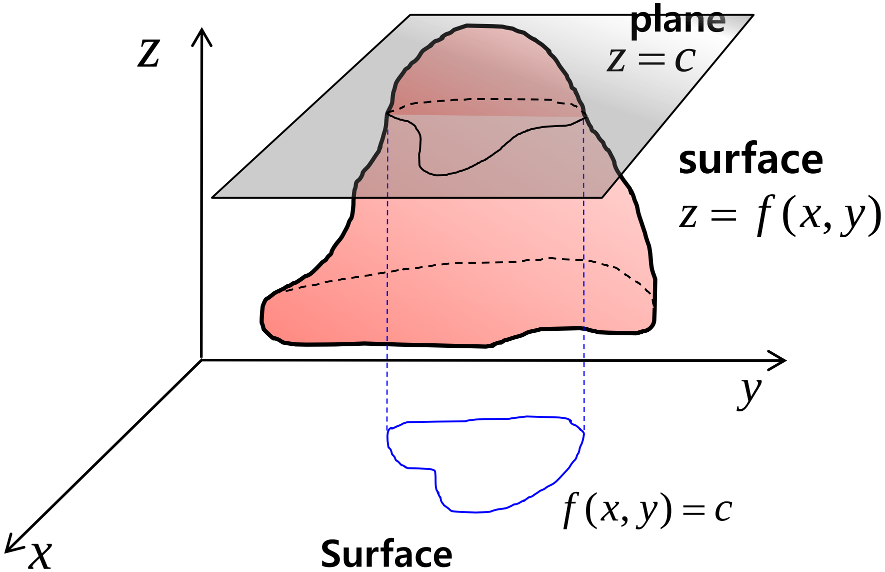
⇨ $\nabla f$ **is the direction of maximum increase of** $f$ **at P**

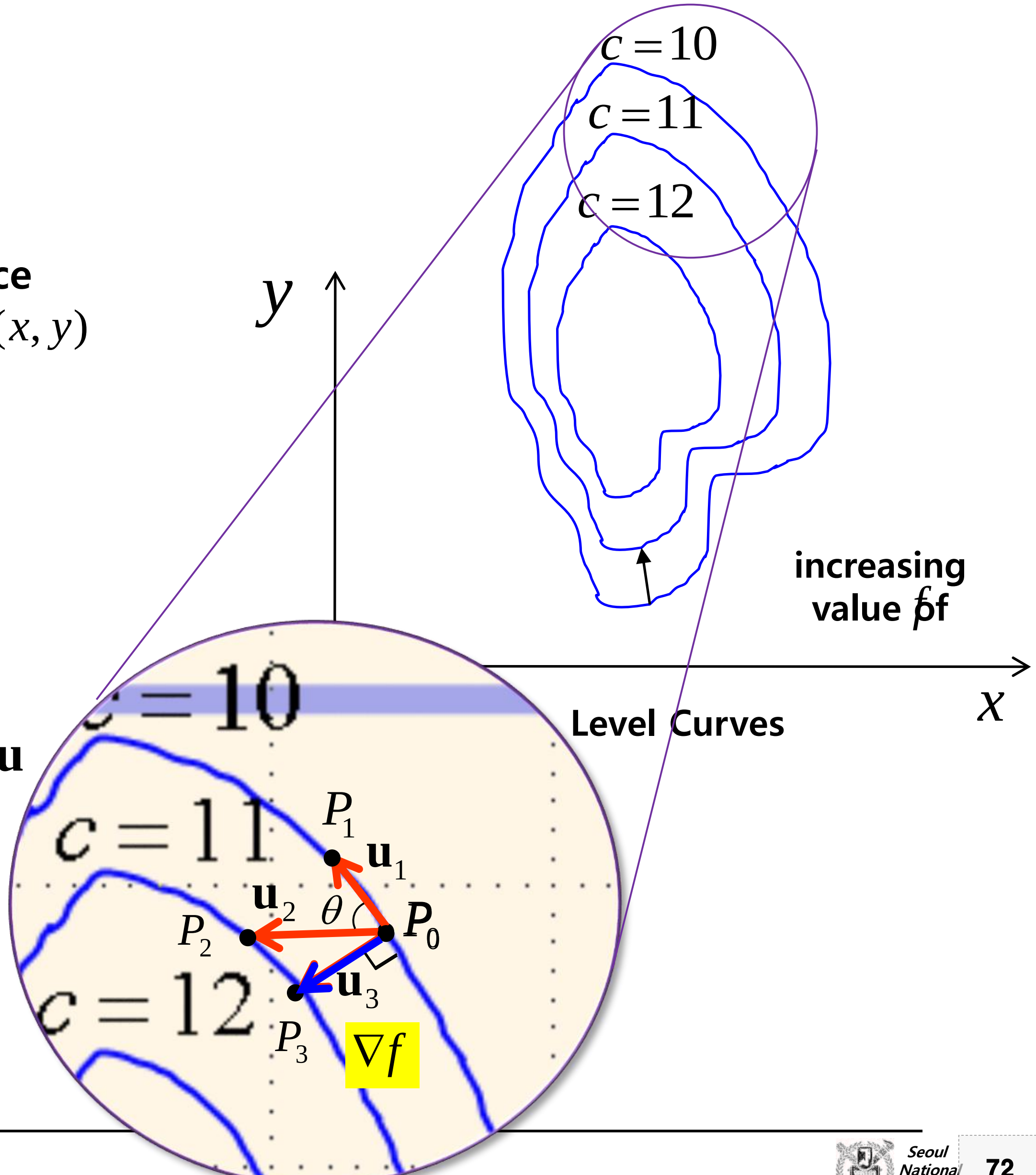
$-\nabla f$ **is the direction of maximum decrease of** $f$ **at P**

즉, $\nabla f$ ($f$ 의 기울기)가 $f$ 증가시키는 가장 큰 방향인지 모르니 b를 바꾸어 가며 가장 큰 값을 찾아보니, $\nabla f$ 와 일치

# 9.7 Gradient of a Scalar Field. Directional Derivative

**Level Curves**




The rate of change of $f$ in the direction $\mathbf{u}$ given by the vector : $D_{\mathbf{u}} f(x, y)$

**$\nabla f$ is the direction of maximum increase of $f$ at $P_0$**

# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Gradient as Curve Normal Vector

$f\big(x(t), y(t)\big) = c$ **: A curve passes through a specified point P($x_0$,$y_0$).**

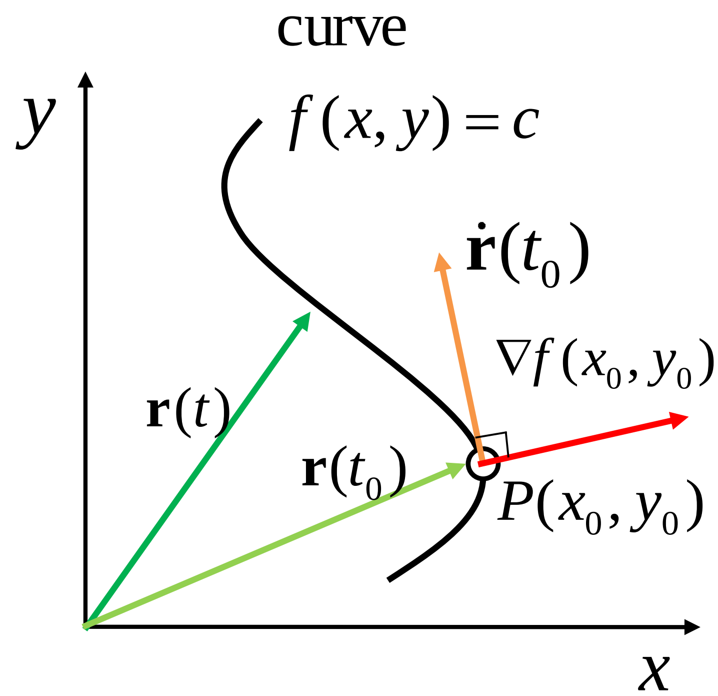
$\mathbf{r} = \big(x(t), y(t)\big)$**: A position vector of a point on the curve.**

$$f\big(x(t), y(t)\big) = c \quad \Rightarrow \frac{df\big(x(t), y(t)\big)}{dt} = 0$$

$$\text{LHS}: \frac{df\big(x(t), y(t)\big)}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{= \nabla f(x, y)} \bullet \underbrace{\left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right)}_{= \dot{\mathbf{r}}(t)}$$

**Tangent vector**




$$\therefore \nabla f(x, y) \cdot \dot{\mathbf{r}}(\mathbf{t}) = 0$$

$\nabla f(x, y)$ is normal to the curve at the point P.

## ☑ Example 1
## Gradient at a Point

Find the level curve of $f(x,y)=-x^2+y^2$ passing through $(2,3)$. Graph the gradient at the point.

Solution)

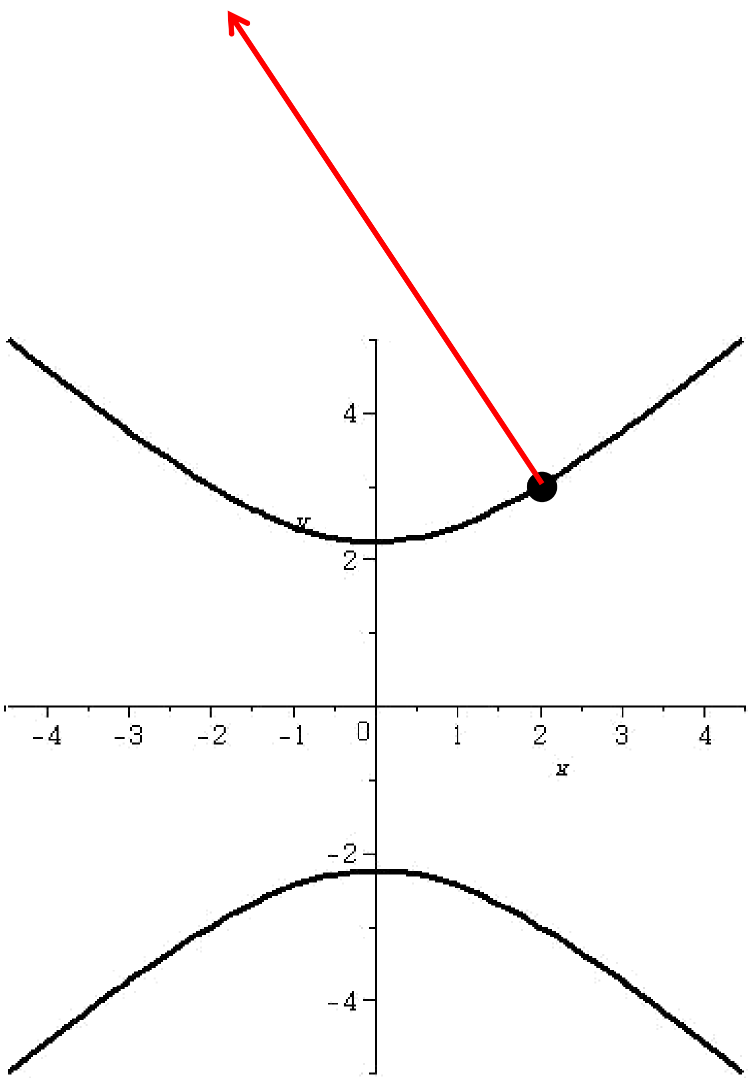
$$f(2,3) = -2^2 + 3^2 = 5$$

$$\text{Level curve : } -x^2 + y^2 = 5$$

$$\nabla f(x,y) = -2x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla f(2,3) = -2\cdot 2\mathbf{i} + 2\cdot 3\mathbf{j}$$

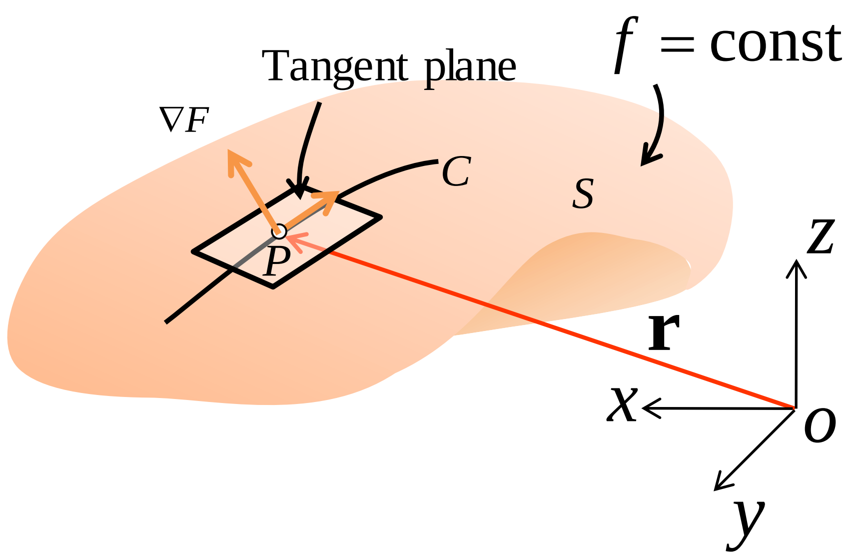
$$= -4\mathbf{i} + 6\mathbf{j}$$



## ☑ Gradient as Surface Normal Vector

- Level Surface of $f$: A surface represented by $f(x, y, z) = c = \text{const}$
- Tangent Plane of S at P

  : A plane which is formed by the tangent vectors of all curves on S passing through P
- Curve C on the surface $\mathrm{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
- Surface Normal to S at P: The straight line through P perpendicular to the tangent plane



# 9.7 Gradient of a Scalar Field. Directional Derivative

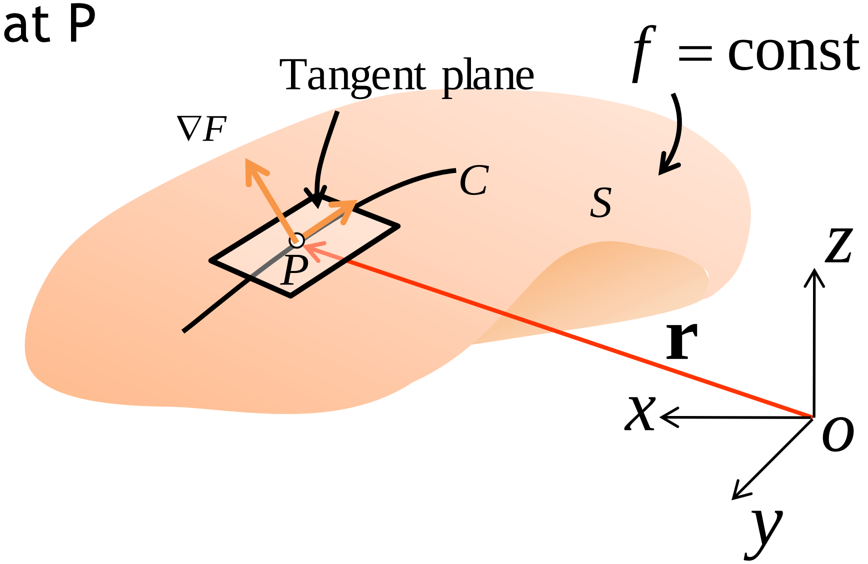
## ☑ Gradient as Surface Normal Vector

- A Surface Normal Vector of S at P: A vector in the direction of the surface normal

$$f\left(x(t), y(t), z(t)\right) = c \xrightarrow{\text{The chain rule}} \frac{df}{dx}x' + \frac{df}{dy}y' + \frac{df}{dz}z' = \left(\text{grad } f\right)\cdot\mathbf{r}' = 0$$

→ grad $f$ is orthogonal to all the vectors **r**′ in the tangent plane

→ grad $f$ is a normal vector of S at P



## ☑ Theorem 2 Gradient as Surface Normal Vector

- Let $f$ be a differentiable scalar function in space. Let $f(x, y, z) = c = \text{const}$ represent a surface $S$.
- Then if the gradient of $f$ at a point P of $S$ is not the zero vector, it is a normal vector of S at P.

## ☑ Example 2

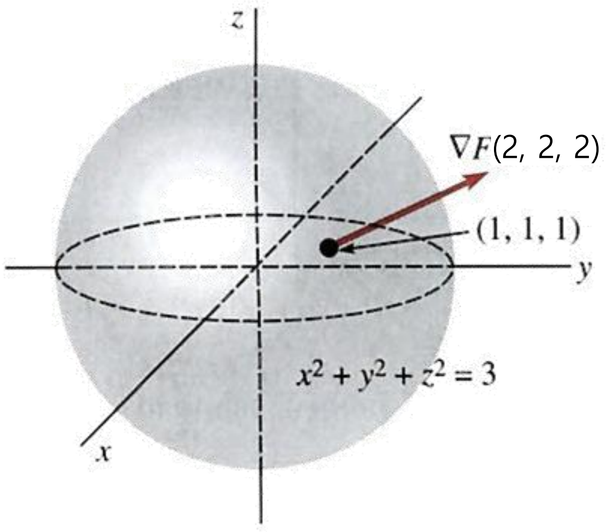
Find the level surface of $F(x,y,z)=x^2+y^2+z^2$ passing through $(1,1,1)$. Graph the gradient at the point.

Solution)

$$F(1,1,1) = 1^2 + 1^2 + 1^2 = 3$$

Level surface : $x^2 + y^2 + z^2 = 3$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$




$$\nabla F(1,1,1) = 2 \cdot 1\mathbf{i} + 2 \cdot 1\mathbf{j} + 2 \cdot 1\mathbf{k}$$

# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Vector Fields That Are Gradients of Scalar Fields ("Potentials")

- $f(P)$ is potential function of $\mathbf{v}(P)$: $\mathbf{v}(P) = \text{grad}\, f(P)$
- Vector field is conservative: energy is conserved in a vector field

## ☑ Theorem 3 Gravitational Field (인력장). Laplace's Equation

- The force of attraction (인력)

$$\mathbf{p} = -\frac{c}{r^3}\mathbf{r} = -c\left[\frac{x-x_0}{r^3}, \frac{y-y_0}{r^3}, \frac{z-z_0}{r^3}\right]$$

  between two particles at points $P_0$: $(x_0, y_0, z_0)$ and $P$: $(x, y, z)$ has the potential $f(x, y, z) = c/r$, where $r\ (> 0)$ is the distance between $P_0$ and $P$.

- Thus $\mathbf{p} = \text{grad}\, f = \text{grad}\,(c/r)$. This potential $f$ is a solution of Laplace's equation.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

# 9.7 Gradient of a Scalar Field. Directional Derivative

## ☑ Proof

$$r = ((x-x_0)^2 + (y-y_0)^2 + (y-y_0)^2)^{1/2}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-2(x-x_0)}{2\left[(x-x_0)^2 + (y-y_0)^2 + (y-y_0)^2\right]^{3/2}} = -\frac{x-x_0}{r^3} \Rightarrow \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(x-x_0)^2}{r^5}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y-y_0}{r^3} \Rightarrow \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(y-y_0)^2}{r^5}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z-z_0}{r^3} \Rightarrow \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(z-z_0)^2}{r^5}$$

$$3(x-x_0)^2 + 3(y-y_0)^2 + 3(y-y_0)^2 = 3r^2$$

$$\boxed{\therefore \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{3}{r^3} + \frac{3r^2}{r^5} = 0}$$

**p**는 스칼라 함수 $f$의 기울기(**p**$=\nabla f$)
즉, $f$는 **p**의 포텐셜
$f$는 Laplace's Equation을 만족함

# ❖ Gradient, Divergence, Curl in Fluid Mechanics

## ☑ Velocity Potential

$$\mathbf{V} = \mathrm{grad}\,\phi$$

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

"Gradient is used here"

$\mathbf{V} \rightarrow \phi$

**(Vector, 3 unknown components)** **(1 unknown)**

**We can reduce the unknowns from 3 to 1.**

# 9.8 Divergence (발산) of a Vector Field

☑ **Divergence of v :** $\operatorname{div}\mathbf{v} = \dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} + \dfrac{\partial v_3}{\partial z}$, (scalar field)

$v_1, v_2, v_3$ the components of $\mathbf{v}$

☑ **Example** $\mathbf{v} = [3xz, 2xy, -yz^2] = 3xz\mathbf{i} + 2xy\mathbf{j} - yz^2\mathbf{k}$

$\Rightarrow \operatorname{div}\mathbf{v} = 3z + 2x - 2yz$

*** Gradient: scalar field => vector field**
*** Divergence: vector field => scalar field**
*** Curl: vector field => new vector field**

☑ **Common notation**

$$\operatorname{div}\mathbf{v} = \nabla \bullet \mathbf{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \bullet [v_1, v_2, v_3]$$

$$= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \bullet (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

# 9.8 Divergence of a Vector Field

## ☑ Theorem 1 Invariance (불변성) of the Divergence

- The divergence div **v** is a scalar function, that is, its values depend only on the points in space but not on the choice of the coordinates.

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1{}^*}{\partial x^*} + \frac{\partial v_2{}^*}{\partial y^*} + \frac{\partial v_3{}^*}{\partial z^*}$$

with respect to other Cartesian coordinates $x^*, y^*, z^*$ and corresponding components $v_1^*$, $v_2^*$, $v_3^*$ of **v.** (to be proved in Sec. 10.7)

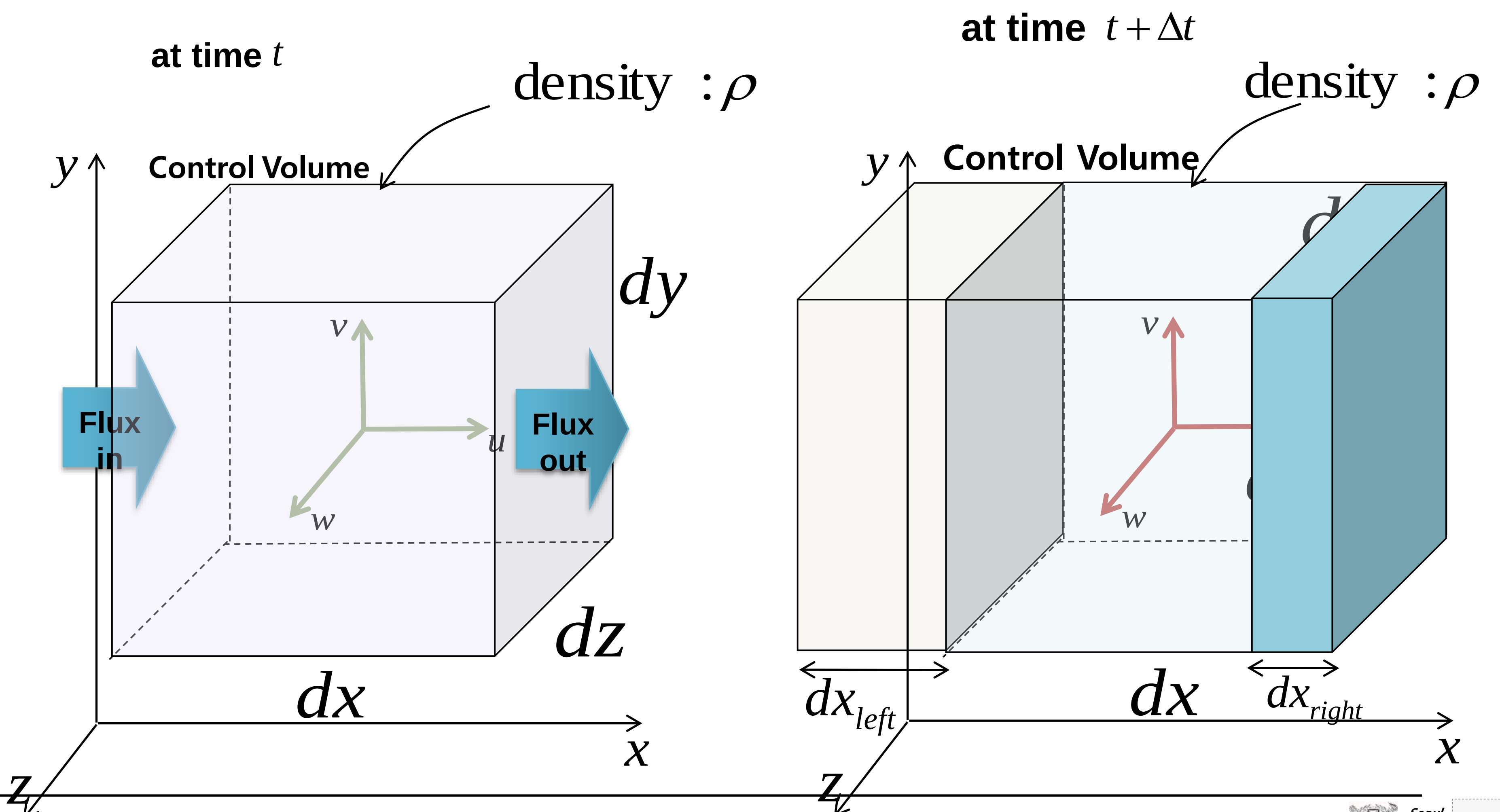
- Let $f(x, y, z)$ be a twice differentiable scalar function.

$$\mathbf{v} = \operatorname{grad} f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$$

$$\operatorname{div}(\mathbf{v}) = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

**(기울기에 대한 발산은 Laplace 방정식)**

## ☑ Ex.2 Flow of a Compressible Fluid. Physical meaning of the Divergence



# 9.8 Divergence of a Vector Field

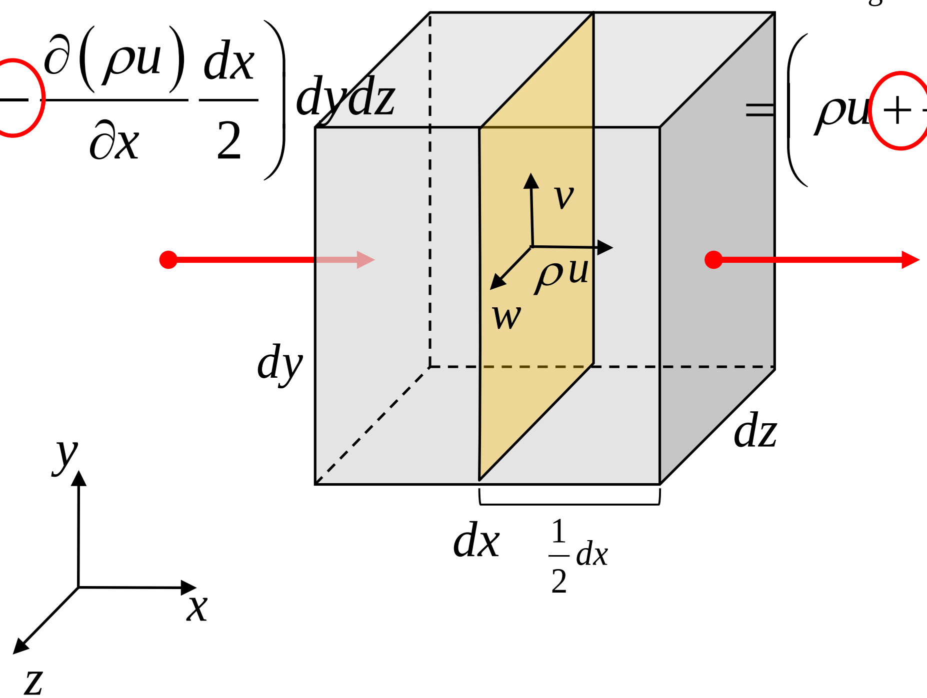
## ☑ Ex.2 Flow of a Compressible Fluid.

단위 시간당 왼쪽 면을 통해
들어온 유체의 질량

$$(\rho u)_{\text{left}}\, dydz \cong \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dydz$$

단위 시간당 오른쪽 면을 통해
빠져나간 유체의 질량

$$(\rho u)_{\text{right}}\, dydz = \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dydz$$



# Ex.2 Flow of a Compressible Fluid.

✓ 오른쪽 면을 통해 검사체적으로부터 빠져나간 유체의 부피

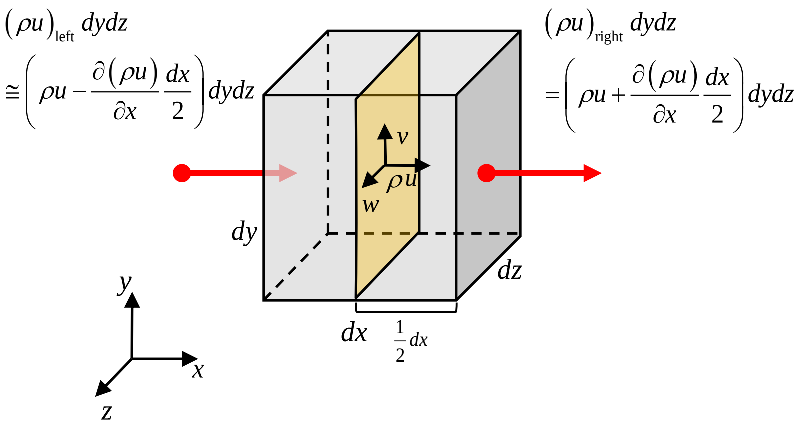
$$dV_{right} = (dydz)dx_{right} = (dydz)u_{right}dt$$

✓ 오른쪽 면을 통해 검사체적으로부터 빠져나간 유체의 질량

$$\rho_{\mathrm{right}} dV_{\mathrm{right}} = \rho_{\mathrm{right}}(dydz)u_{\mathrm{right}}dt = (\rho u)_{\mathrm{right}} dydz \cdot dt$$

✓ 단위 시간당 오른쪽 면을 통해 빠져나간 유체의 질량

$$\rho_{\mathrm{right}} \frac{dV_{\mathrm{right}}}{dt} = \rho_{right}(dydz)u_{\mathrm{right}} = (\rho u)_{\mathrm{right}} dydz$$




✓ $(\rho u)_{right}$ 를 Taylor Series로 전개하면,

$$(\rho u)_{right} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2 + \cdots$$

$$\cong \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$$ **(1차항까지만 선택)**

✓ 단위 시간당 오른쪽 면을 통해 빠져나간 유체의 질량

$$(\rho u)_{\mathrm{right}} dydz = \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right)dydz$$

# Ex.2 Flow of a Compressible Fluid.

✓ 단위 시간당 각 면을 통해 빠져나간 유체의 질량

**Right(+x):** $(\rho u)_{\text{right}}\, dydz \cong \left(\rho u + \dfrac{\partial(\rho u)}{\partial x}\dfrac{dx}{2}\right)dydz$

**Top(+y):** $(\rho v)_{\text{top}}\, dxdz \cong \left(\rho v + \dfrac{\partial(\rho v)}{\partial y}\dfrac{dy}{2}\right)dxdz$

**Front(+z):** $(\rho w)_{\text{front}}\, dxdy \cong \left(\rho w + \dfrac{\partial(\rho w)}{\partial z}\dfrac{dz}{2}\right)dxdy$

✓ 단위 시간당 각 면을 통해 들어온 유체의 질량

**left(-x):** $(\rho u)_{\text{left}}\, dydz \cong \left(\rho u - \dfrac{\partial(\rho u)}{\partial x}\dfrac{dx}{2}\right)dydz$

**Bottom(-y):** $(\rho v)_{\text{bottom}}\, dxdz \cong \left(\rho v - \dfrac{\partial(\rho v)}{\partial y}\dfrac{dy}{2}\right)dxdz$

**Rear(-z):** $(\rho w)_{\text{rear}}\, dxdy \cong \left(\rho w - \dfrac{\partial(\rho w)}{\partial z}\dfrac{dz}{2}\right)dxdy$

⇓

✓ 단위 시간당 검사 체적을 통과한 유체의 유입량 (들어온 양을 +로 봄)

$$\sum \dot{m} = \sum_{in} \dot{m} - \sum_{out} \dot{m} = \left[\left((\rho u) - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right)dydz + \left((\rho v) - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right)dxdz + \left((\rho w) - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right)dxdy\right]$$

$$-\left[\left((\rho u) + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right)dydz + \left((\rho v) + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right)dxdz + \left((\rho w) + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right)dxdy\right]$$

$$= -\frac{\partial(\rho u)}{\partial x}dxdydz - \frac{\partial(\rho v)}{\partial y}dxdydz - \frac{\partial(\rho w)}{\partial z}dxdydz$$

# Ex.2 Flow of a Compressible Fluid.

## ❖ Mass conservation

✓ 단위 시간당 검사 체적을 통과한 유체의 유입량 (들어온 양을 +로 봄)

$$= -\frac{\partial(\rho u)}{\partial x}dxdydz - \frac{\partial(\rho v)}{\partial y}dxdydz - \frac{\partial(\rho w)}{\partial z}dxdydz \text{ ---------- ①}$$

✓ (검사체적 내부의 질량 변화율)$= \frac{\partial \rho}{\partial t}dxdydz$ ---------- ②

① = ② (mass conservation)

$$\frac{\partial \rho}{\partial t}dxdydz = -\frac{\partial(\rho u)}{\partial x}dxdydz - \frac{\partial(\rho v)}{\partial y}dxdydz - \frac{\partial(\rho w)}{\partial z}dxdydz$$

$dxdydz$ **으로 양변을 나누면,**

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{V}) = 0}$$

=> Continuity Equation

# Ex.2 Flow of a Compressible Fluid.

**Continuity Equation (압축성 유체흐름의 연속성 방정식)**

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{V}) = 0$$

**비압축성 유체**(Incompressible fluid)**라고 가정하면,**

$$(\rho = const)$$

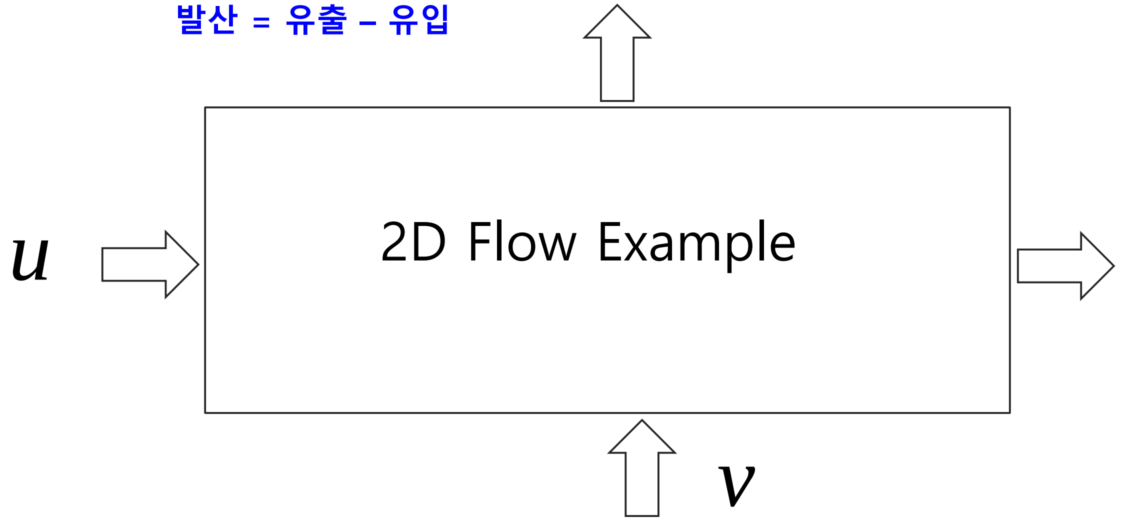
$$\nabla \bullet \mathbf{V} = 0 \quad \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right)$$

**(비압축성의 조건)**

**Divergence**

**주어진 체적에 대해 임의의 시간에서의 유입과 유출의 차가 0인 평형 상태**

**발산 = 유출 – 유입**



# ❖ Gradient, Divergence, Curl in Fluid Mechanics

## ☑ Velocity Potential

$$\mathbf{V} = \mathrm{grad}\,\phi \qquad \mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

"Gradient is used here"

$$+$$

$$\nabla \bullet \mathbf{V} = 0 \quad \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right)$$

**"divergence is used here"**

$$||$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

**"Laplace Equation"**

# 9.9 Curl of a Vector Field

## ☑ Curl (회전):

- Let $\mathbf{v}(x, y, z) = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a differentiable vector function of Cartesian coordinates $x, y, z$.
- Then the curl of the vector function $\mathbf{v}$ or of the vector field given by $\mathbf{v}$ is defined by the "symbolic" determinant.

$$\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

*** Gradient: scalar field => vector field**
*** Divergence: vector field => scalar field**
*** Curl: vector field => new vector field**

# 9.9 Curl of a Vector Field

## ☑ Example

Let $\mathbf{v} = [yz, 3zx, z] = yz\mathbf{i} + 3zx\mathbf{j} + z\,\mathbf{k}$ with right-handed $x, y, z$.
Find $curl\ \mathbf{v}$.

**Solution)**

$$\operatorname{curl}\mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

$$= \left(\frac{\partial z}{\partial y} - \frac{\partial 3zx}{\partial z}\right)\mathbf{i} + \left(\frac{\partial yz}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial 3zx}{\partial x} - \frac{\partial yz}{\partial y}\right)\mathbf{k}$$

$$= -3x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$$

# 9.9 Curl of a Vector Field

## ☑ Theorem 1 Rotating Body and Curl

- The curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation,
- and its magnitude equals twice the angular speed of the rotation.

## ☑ Theorem 2 Grad, Div, Curl

- Gradient fields are irrotational.

$$\operatorname{curl}(\operatorname{grad} f) = 0$$

- The divergence of the curl of a function is zero.

$$\operatorname{div}(\operatorname{curl} \mathbf{v}) = 0$$

$$\operatorname{curl} \mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

Q: Prove them!

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \qquad \operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

# 9.9 Curl of a Vector Field

## ☑ Theorem 3 Invariance of the Curl

- curl **v** is a vector. That is, it has a length and direction that are independent of the particular choice of a Cartesian coordinate system in space.

# 9.9 Curl of a Vector Field

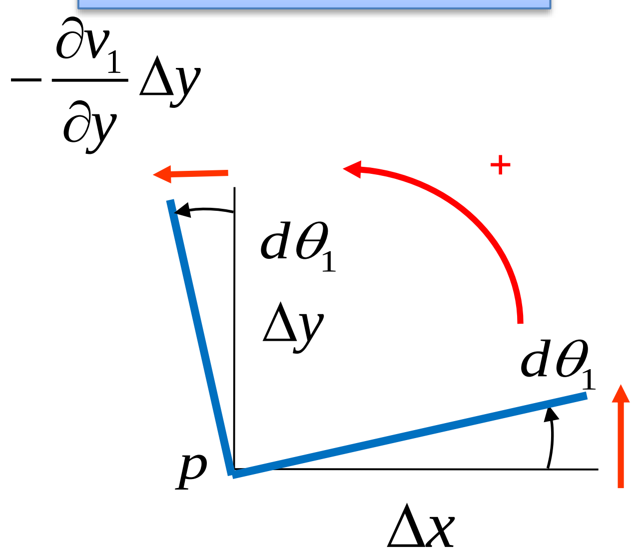
## ☑ Physical Meaning of Curl (회전)

- Focusing on xy-plane

$\mathbf{v}=[v_1, v_2, v_3]$

$$\operatorname{curl}\mathbf{v}=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

**순수 회전 (rotation)**




$\frac{\partial v_2}{\partial x}\Delta x$

$|v_2|=|v_1|$  $\tan(d\theta_1)$

두 각속도의 평균

$$d\theta=\frac{1}{2}\left(\boxed{\frac{\partial v_2}{\partial x}}dt-\frac{\partial v_1}{\partial y}dt\right)=\frac{1}{2}2\frac{\partial v_2}{\partial x}dt=d\theta_1$$

속도 가속도 시간

$$\Rightarrow\quad \omega_3=\frac{d\theta}{dt}=\frac{\partial v_2}{\partial x}$$

# 9.9 Curl of a Vector Field

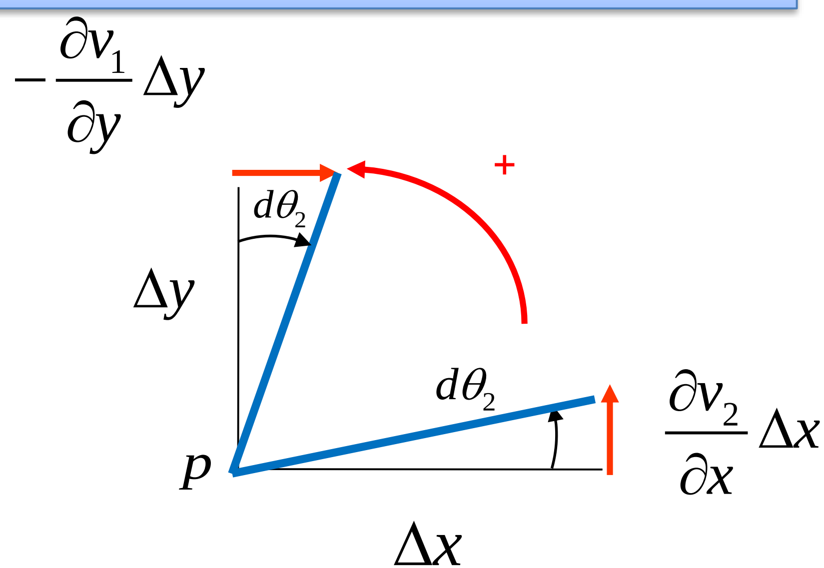
## ☑ Physical Meaning of Curl (회전)

- Focusing on xy-plane

$\mathbf{v}=[v_1, v_2, v_3]$

$$\operatorname{curl}\mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

**순수 전단 변형** (shear deformation )




$$|v_2| = |v_1|$$

$$d\theta = \frac{1}{2}\left(\frac{\partial v_2}{\partial x}dt - \frac{\partial v_1}{\partial y}dt\right) = \frac{1}{2}(d\theta_2 - d\theta_2) = 0$$

⇨ $\omega_3 = 0$

## ☑ Physical Meaning of Curl (회전)

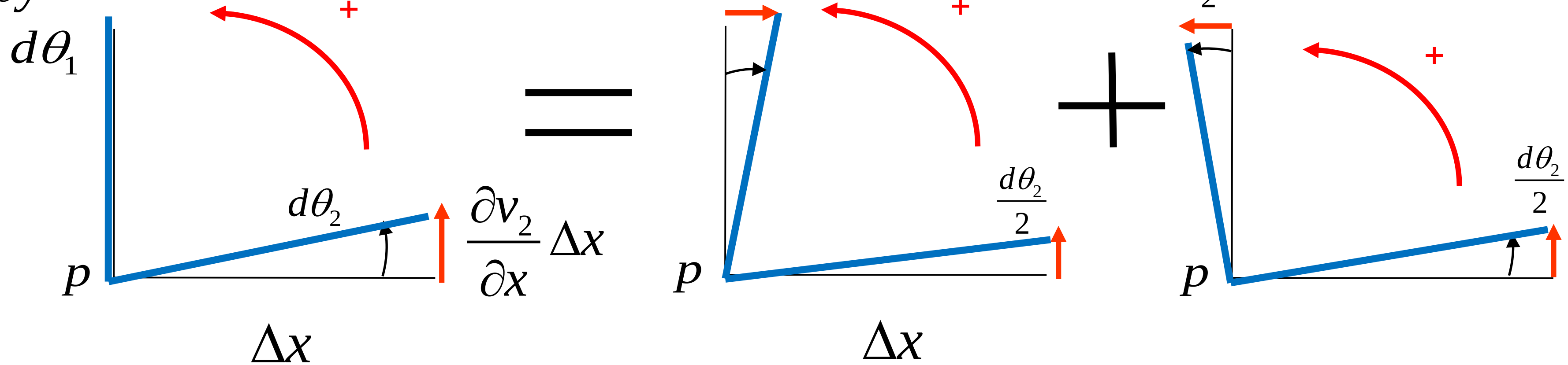
- Focusing on xy-plane

$\mathbf{v}=[v_1, v_2, v_3]$ $\qquad \operatorname{curl}\mathbf{v}=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\mathbf{k}$

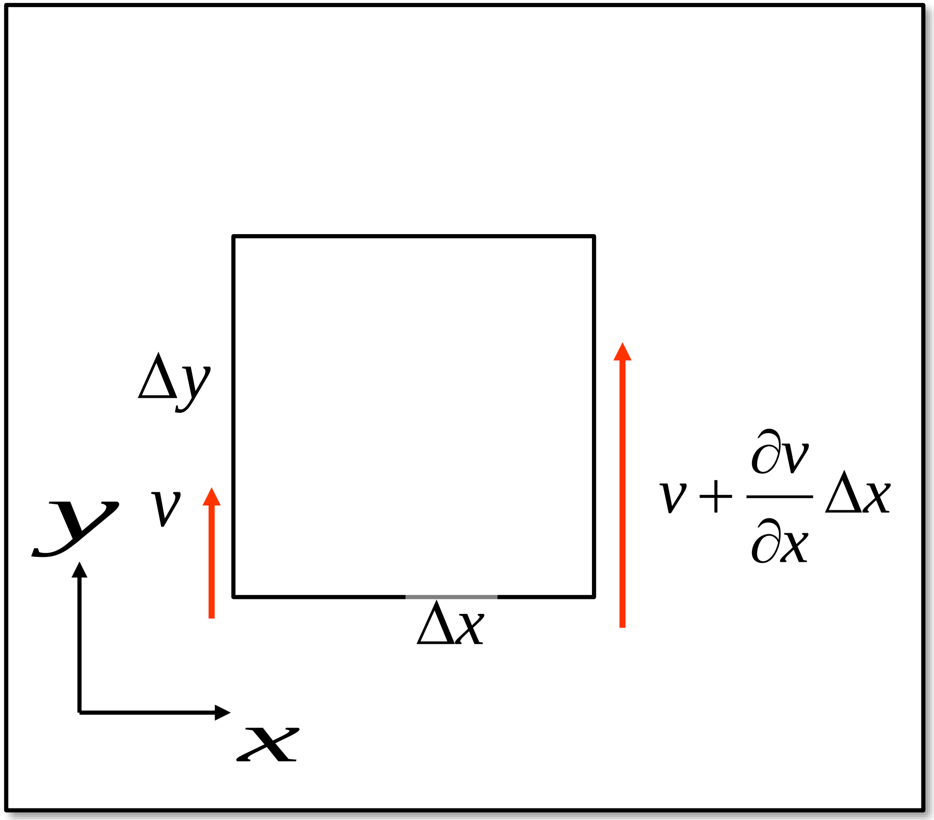
$$d\theta=\frac{1}{2}\left(\frac{\partial v_2}{\partial x}\Delta t-\cancel{\frac{\partial v_1}{\partial y}}\Delta t\right)=\frac{1}{2}\frac{\partial v_2}{\partial x}dt=\frac{1}{2}d\theta_2 \Rightarrow \omega_3=\frac{d\theta}{dt}=\frac{1}{2}\frac{\partial v_2}{\partial x}$$

**복합** (shear + rotation ) **순수 전단 변형** **순수 회전** (rotation)

$$-\frac{\partial v_1}{\partial y}\Delta y=0$$



# ❖ Curl and Rotation in Fluids




✓ 유체 입자의 왼쪽 아래의 $y$방향 속도를 $v$라고 할 때, $x$축으로 $\triangle x$ 만큼 떨어진 지점에서의 $y$방향 속도

$$v = v(x, y, t)$$

$y,t$ 가 고정이라면,

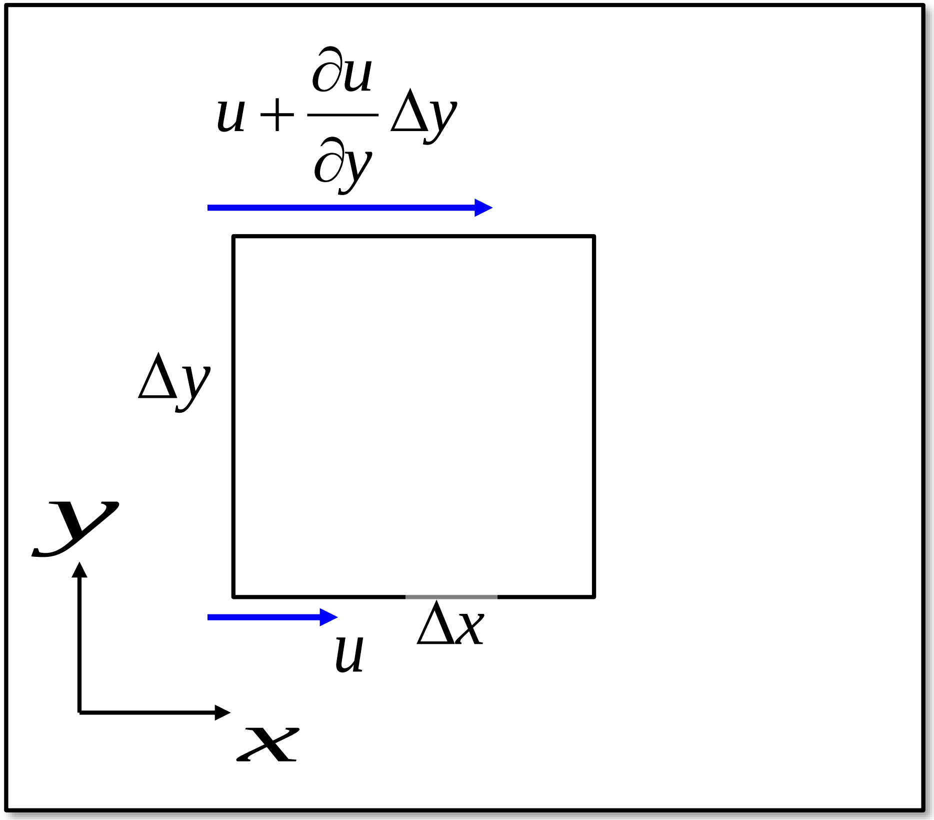
$$v = v(x)$$

$\triangle x$가 작다고 가정하면,

$$v(x+\Delta x) = v(x) + \frac{\partial v}{\partial x}\Delta x$$

(반시계 방향)

# ❖ Curl and Rotation in Fluids




✓ 유체 입자의 왼쪽 아래의 $x$방향 속도를 $u$라고 할 때, $y$축으로 $\triangle y$ 만큼 떨어진 지점에서의 $x$방향 속도

$$u = u(x, y, t)$$

$x,t$ 가 고정이라면,

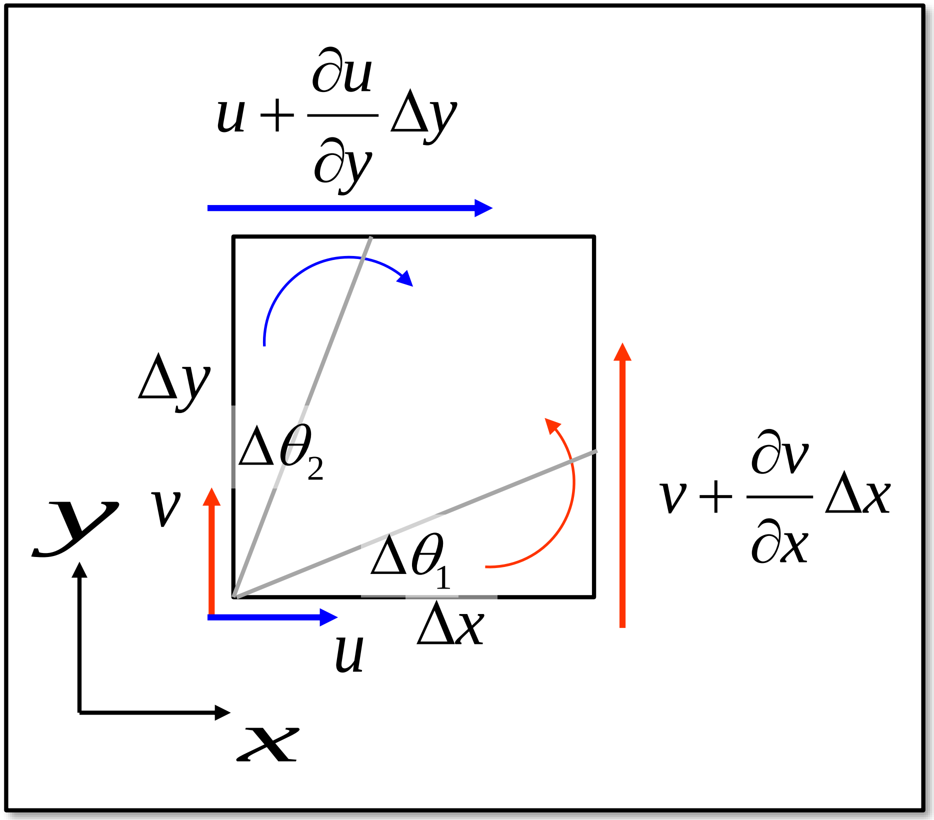
$$u = u(y)$$

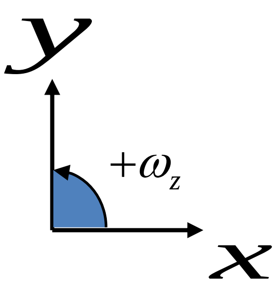
$\triangle y$가 작다고 가정하면,

$$u(y+\Delta y) = u(y) + \frac{\partial u}{\partial y}\Delta y$$

(시계 방향)

# ❖ Curl and Rotation in Fluids




✓ 각속도 (속도 차만큼 변위 발생)

$$\omega_{z1}=\frac{\Delta\theta_1}{\Delta t}\approx\frac{\tan\Delta\theta_1}{\Delta t}=\frac{\Delta v\Delta t}{\Delta x}\cdot\frac{1}{\Delta t}$$

$$=\frac{v+\frac{\partial v}{\partial x}\Delta x-v}{\Delta x}=\frac{\partial v}{\partial x}$$ (반시계 방향)

$$\omega_{z2}=-\frac{\Delta\theta_2}{\Delta t}\approx-\frac{\tan\Delta\theta_2}{\Delta t}=-\frac{\Delta u\Delta t}{\Delta y}\cdot\frac{1}{\Delta t}$$

$$=-\frac{u+\frac{\partial u}{\partial y}\Delta y-u}{\Delta y}=-\frac{\partial u}{\partial y}$$ (시계 방향)

✓ z축에 대한 각속도는 두 각속도의 평균으로 정의하면

$$\omega_z=\frac{1}{2}\left(\omega_{z1}+\omega_{z2}\right)=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

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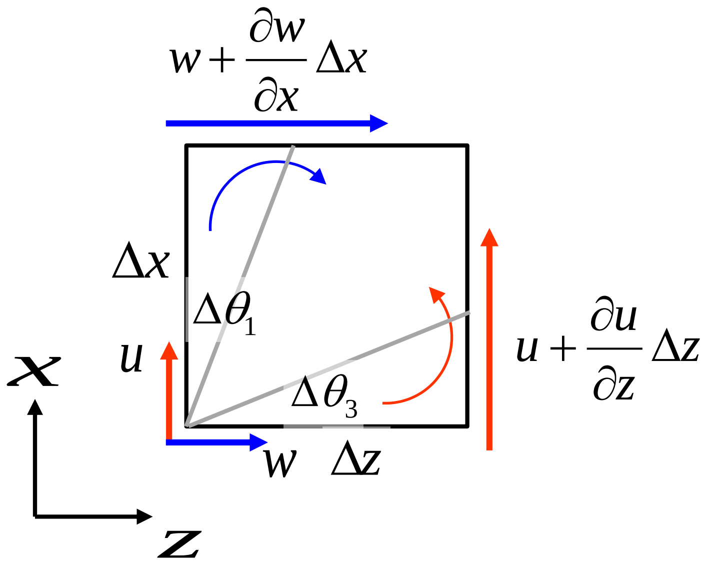
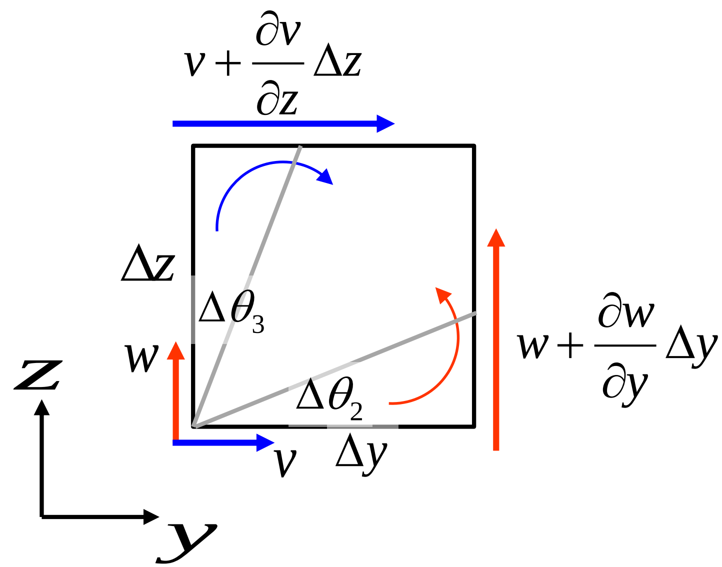


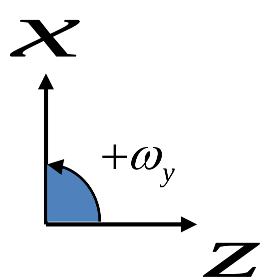




✓ y축에 대한 각속도

(반시계 방향) $\omega_{y1}=\frac{\partial u}{\partial z}$

(시계 방향) $\omega_{y2}=-\frac{\partial w}{\partial x}$

$$\therefore \omega_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

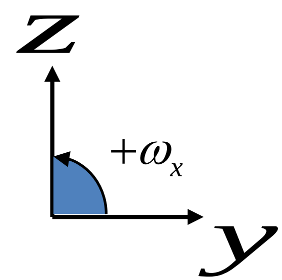



✓ x축에 대한 각속도

(반시계 방향) $\omega_{x1}=\frac{\partial w}{\partial y}$

(시계 방향) $\omega_{x2}=-\frac{\partial v}{\partial z}$

$$\therefore \omega_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$



# Curl and Rotation in Fluids

✓ curl의 정의

$$\text{curl}\,\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$

$$= 2\left[\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}\right]$$

$$= 2\omega_x\mathbf{i} + 2\omega_y\mathbf{j} + 2\omega_z\mathbf{k} = 2\boldsymbol{\omega}$$

✓ x축에 대한 각속도

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

✓ y축에 대한 각속도

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

✓ z축에 대한 각속도

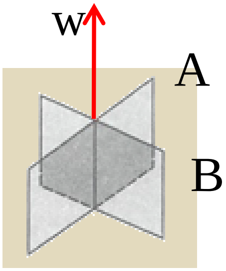
$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

=> curl v **크기는 유체 입자의 회전 각속도x2에 해당**

# Curl of a Vector Fields

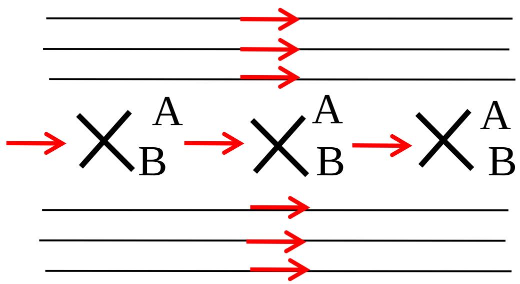
## ❖ Paddle device

If a paddle device is inserted in a flowing fluid, then the curl of the velocity field $\mathbf{v}$ is a measure of the tendency of the fluid to turn the device about its vertical axis $w$.



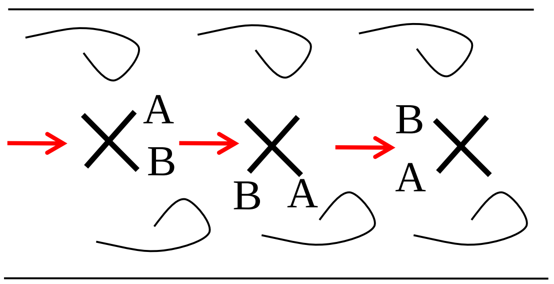

**Paddle device**

**Physical Interpretations**




**Irrotational flow**

$(\text{curl }\mathbf{F} = 0)$




**Rotational flow**

$(\text{curl }\mathbf{F} \neq 0)$

# ❖ Gradient, Divergence, Curl in Fluid Mechanics

## ☑ Velocity Potential

$$\mathbf{V} = \text{grad}\,\phi \qquad \mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$ “Gradient is used here”

+

$$\nabla \bullet \mathbf{V} = 0 \quad \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right)$$ “divergence is used here”

||

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Continuity Equation
(Laplace Equation for Incompressible Fluid)

**For any scalar function** $\phi = \phi(x, y, z)$, $\text{curl}(\text{grad}\,\phi) = \mathbf{0}$ **is always true.**

$$\text{curl}(\text{grad}\,\phi) = \text{curl}\left( \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} \right)$$

$$= \left( \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right)\mathbf{i} - \left( \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right)\mathbf{j} + \left( \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right)\mathbf{k} = 0 \Rightarrow$$ Irrotational

“Curl is used here”

“Velocity potential을 사용하기 위해서는 irrotational 가정이 반드시 필요하다“
⇒Turbulence 유체 운동은 풀 수 없다!

# Key Summary

## ☑ Proof

$$r = ((x-x_0)^2 + (y-y_0)^2 + (y-y_0)^2)^{1/2}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-2(x-x_0)}{2\left[(x-x_0)^2 + (y-y_0)^2 + (y-y_0)^2\right]^{3/2}} = -\frac{x-x_0}{r^3} \Rightarrow \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(x-x_0)^2}{r^5}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y-y_0}{r^3} \Rightarrow \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(y-y_0)^2}{r^5}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z-z_0}{r^3} \Rightarrow \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(z-z_0)^2}{r^5}$$

$$3(x-x_0)^2 + 3(y-y_0)^2 + 3(y-y_0)^2 = 3r^2$$

$$\boxed{\therefore \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{3}{r^3} + \frac{3r^2}{r^5} = 0}$$

# ❖ Gradient, Divergence, Curl in Fluid Mechanics

## ☑ Velocity Potential

$$\mathbf{V} = \operatorname{grad}\phi \qquad \mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

"Gradient is used here"

$+$

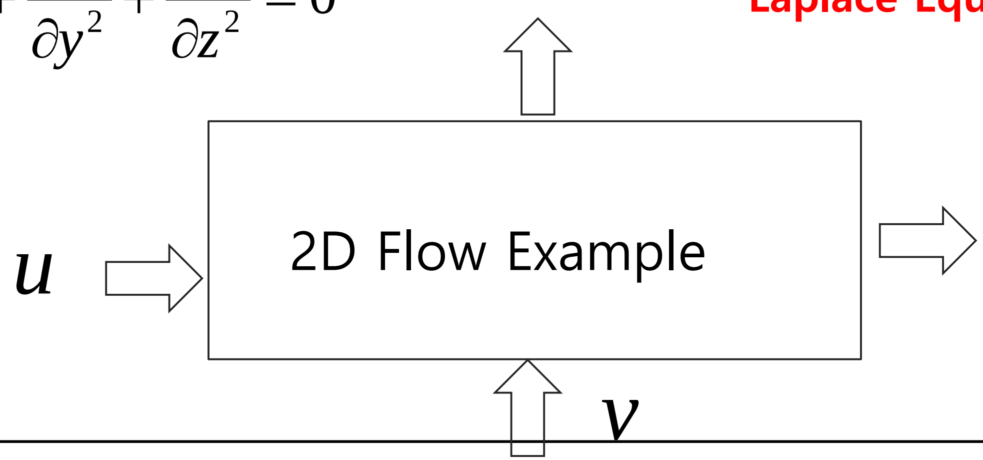
$$\nabla \bullet \mathbf{V} = 0 \quad \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right)$$

**"divergence is used here"**

$||$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

**"Laplace Equation"**



# Curl of a Vector Field

## ☑ Curl (회전):

- Let $\mathbf{v}(x, y, z) = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a differentiable vector function of Cartesian coordinates x, y, z.
- Then the curl of the vector function **v** or of the vector field given by **v** is defined by the "symbolic" determinant.

$$\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

# 9.9 Curl of a Vector Field

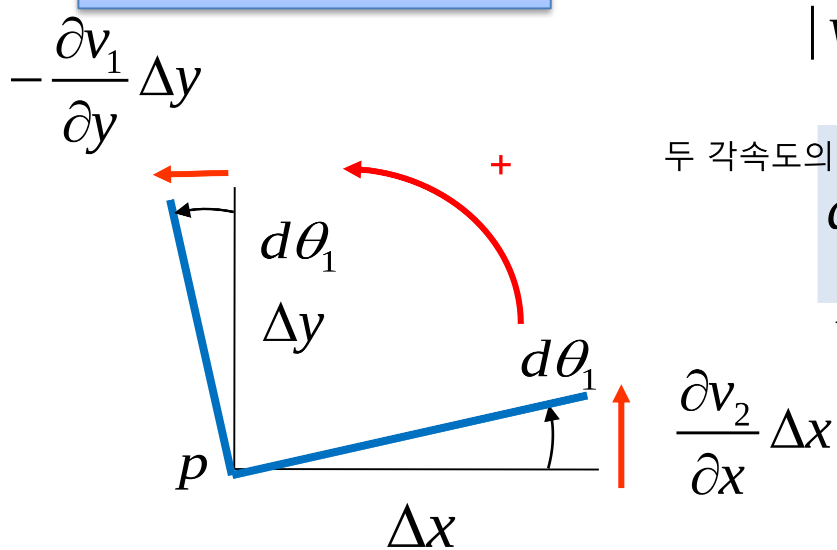
## ☑ Physical Meaning of Curl (회전)

- Focusing on xy-plane

$\mathbf{v}=[v_1, v_2, v_3]$

$$\operatorname{curl}\mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

**순수 회전** (rotation)




$|v_2| = |v_1|$ $\tan(d\theta_1)$

두 각속도의 평균

$$d\theta = \frac{1}{2}\left(\boxed{\frac{\partial v_2}{\partial x}} dt - \frac{\partial v_1}{\partial y} dt\right) = \frac{1}{2} 2 \frac{\partial v_2}{\partial x} dt = d\theta_1$$

속도 가속도 시간

$$\omega_3 = \frac{d\theta}{dt} = \frac{\partial v_2}{\partial x}$$

# 9.9 Curl of a Vector Field

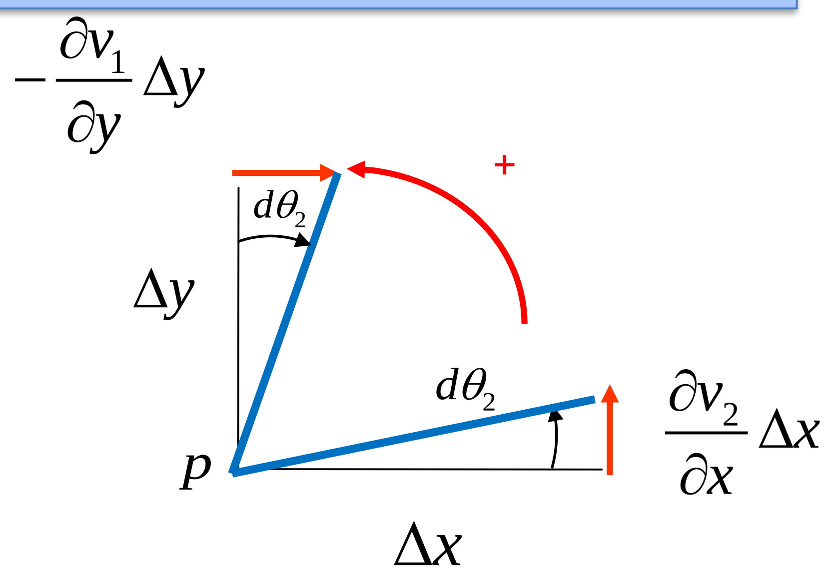
## ☑ Physical Meaning of Curl (회전)

- Focusing on xy-plane

  $\mathbf{v}=[v_1, v_2, v_3]$

$$\operatorname{curl} \mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

**순수 전단 변형** (shear deformation )




$$|v_2| = |v_1|$$

$$d\theta = \frac{1}{2}\left(\frac{\partial v_2}{\partial x}dt - \frac{\partial v_1}{\partial y}dt\right) = \frac{1}{2}(d\theta_2 - d\theta_2) = 0$$

⇨ $\omega_3 = 0$

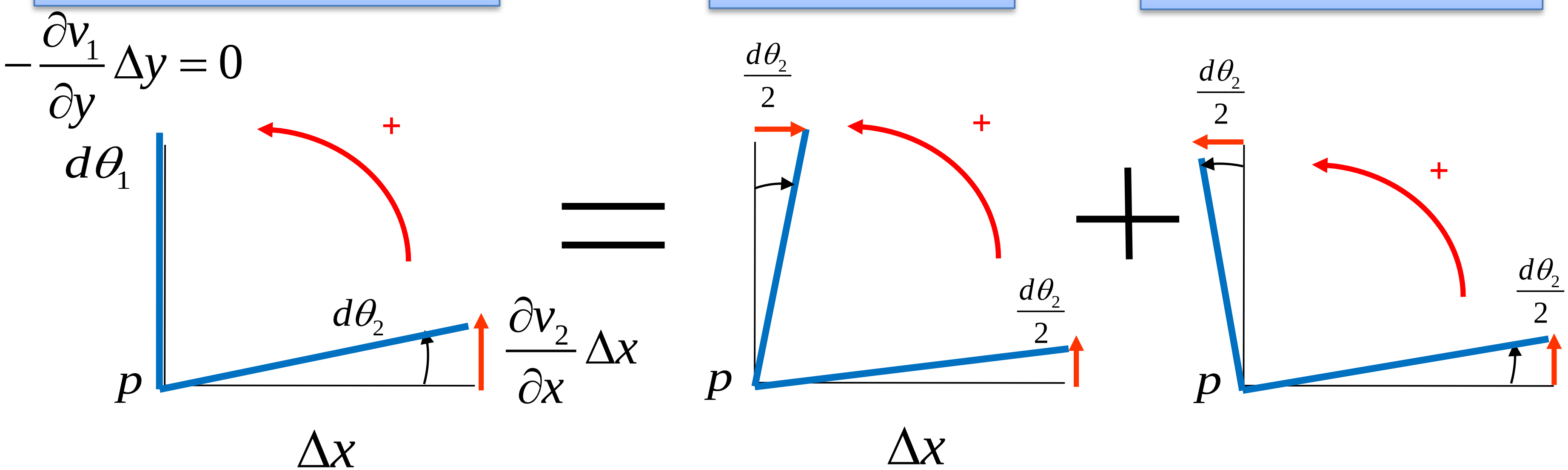
## ☑ Physical Meaning of Curl (회전)

- Focusing on xy-plane

$\mathbf{v}=[v_1, v_2, v_3] \qquad \operatorname{curl}\mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$

$$d\theta = \frac{1}{2}\left(\frac{\partial v_2}{\partial x}\Delta t - \frac{\partial v_1}{\partial y}\Delta t\right) = \frac{1}{2}\frac{\partial v_2}{\partial x}dt = \frac{1}{2}d\theta_2 \quad \Rightarrow \quad \omega_3 = \frac{d\theta}{dt} = \frac{1}{2}\frac{\partial v_2}{\partial x}$$

복합 (shear + rotation ) = 순수 전단 변형 + 순수 회전 (rotation)



# ❖ Gradient, Divergence, Curl in Fluid Mechanics

## ☑ Velocity Potential

$$\mathbf{V} = \operatorname{grad}\phi \qquad \mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

"Gradient is used here"

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Laplace Equation for Incompressible Fluid

For any scalar function $\phi = \phi(x, y, z)$, $\operatorname{curl}(\operatorname{grad}\phi) = \mathbf{0}$ is always true.

$$\operatorname{curl}(\operatorname{grad}\phi) = \operatorname{curl}\left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right)$$

"Curl is used here"

$$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)\mathbf{k} = 0 \Rightarrow$$

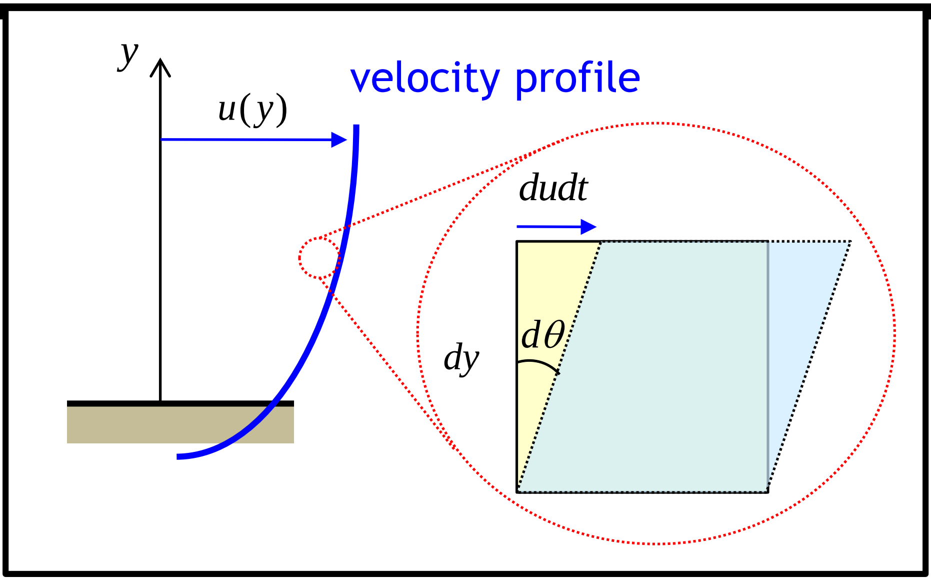
Irrotational

"Velocity potential을 사용하기 위해서는 irrotational 가정이 반드시 필요하다"

⇒Turbulence 유체 운동은 풀 수 없다!

# Spare Slides

# 뉴턴 유체1) (Newtonian Fluid)




✓ 미소 구간에서의 전단변형율

$$d\theta \approx \tan d\theta = \frac{dudt}{dy} = \frac{du}{dy}dt$$

✓ 전단변형율의 시간변화율은 속도 구배와 같음

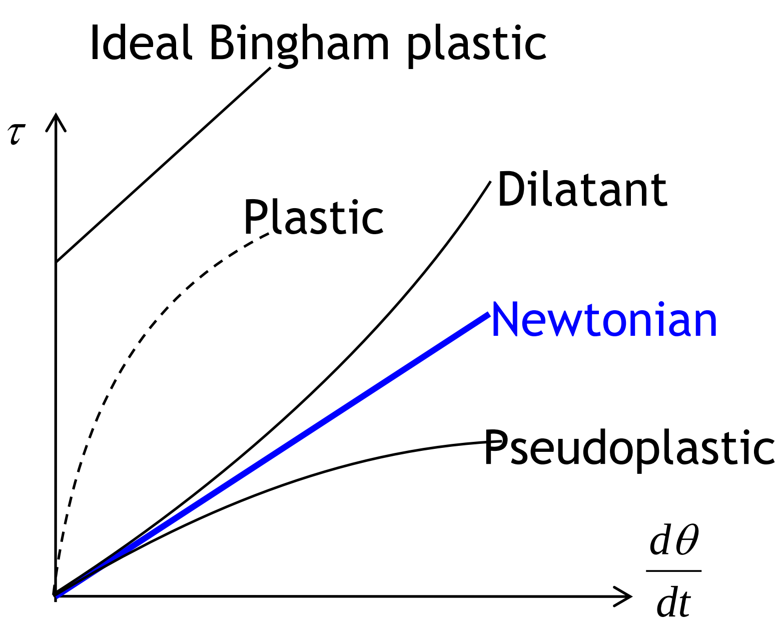
$$\frac{d\theta}{dt} = \frac{du}{dy} \quad \text{---①}$$

✓ 뉴턴 유체(Newtonian fluid)
: 전단응력이 전단변형율의 시간변화율에 비례

$$\tau \propto \frac{d\theta}{dt} \quad \text{---②}$$

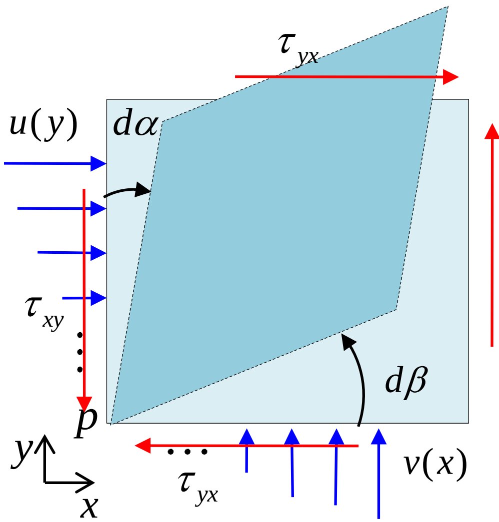
✓ 뉴턴 유체(Newtonian fluid)의 특징
: ①,②에 의해, 전단응력은 속도구배에 비례함
(비례 상수 $\mu$ : 점성 계수)

$$\tau \propto \frac{du}{dy} \Rightarrow \tau = \mu \frac{du}{dy}$$

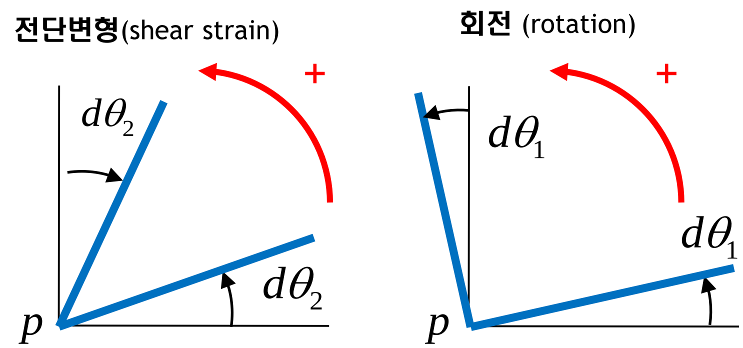


1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

# Curl and Rotation in Fluids - 다른 유도 방법




회전과 전단변형으로 나타낼 수 있음




$$-d\alpha = -d\theta_2 + d\theta_1 \quad \cdots ①$$

$$d\beta = d\theta_2 + d\theta_1 \quad \cdots ②$$

Given : $d\alpha, d\beta$

Find : 회전각속도 $\omega_z$

(1) ②와 ①을 더하면, $d\beta - d\alpha = 2d\theta_1$

$$d\theta_1 = \frac{1}{2}(d\beta - d\alpha)$$

➡ z축에 대한 회전 각도

(2) $dt$로 나누면, $\frac{d\theta_1}{dt} = \frac{1}{2}\left(\frac{d\beta}{dt} - \frac{d\alpha}{dt}\right)$

➡ z축에 대한 회전 각속도

(3) 전단변형율의 시간 변화율은 속도 구배와 같음

$$\omega_z = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)$$

(5) 동일한 방법에 의해 다음도 성립함

➡ y축에 대한 회전각속도 $\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

➡ x축에 대한 회전각속도 $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$