CHAPTER 5

Jacobians: velocities and static forces

5.1 INTRODUCTION

In this chapter, we expand our consideration of robot manipulators beyond static-positioning problems. We examine the notions of linear and angular velocity of a rigid body and use these concepts to analyze the motion of a manipulator. We also will consider forces acting on a rigid body, and then use these ideas to study the application of static forces with manipulators.

It turns out that the study of both velocities and static forces leads to a matrix entity called the **Jacobian**[1] of the manipulator, which will be introduced in this chapter.

The field of kinematics of mechanisms is not treated in great depth here. For the most part, the presentation is restricted to only those ideas which are fundamental to the particular problem of robotics. The interested reader is urged to study further from any of several texts on mechanics [1–3].

5.2 NOTATION FOR TIME-VARYING POSITION AND ORIENTATION

Before investigating the description of the motion of a rigid body, we briefly discuss some basics: the differentiation of vectors, the representation of angular velocity, and notation.

[1] Mathematicians call it the "Jacobian matrix," but roboticists usually shorten it to simply "Jacobian."

Differentiation of position vectors

As a basis for our consideration of velocities (and, in Chapter 6, accelerations), we need the following notation for the derivative of a vector:

$$^{B}V_{Q} = \frac{d}{dt}\,^{B}Q = \lim_{\Delta t \to 0} \frac{^{B}Q(t+\Delta t) - {^{B}Q(t)}}{\Delta t}. \tag{5.1}$$

The velocity of a position vector can be thought of as the linear velocity of the point in space represented by the position vector. From (5.1), we see that we are calculating the derivative of Q relative to frame $\{B\}$. For example, if Q is not changing in time relative to $\{B\}$, then the velocity calculated is zero—even if there is some other frame in which Q is varying. Thus, it is important to indicate the frame in which the vector is differentiated.

As with any vector, a velocity vector can be described in terms of any frame, and this frame of reference is noted with a leading superscript. Hence, the velocity vector calculated by (5.1), when expressed in terms of frame $\{A\}$, would be written

$$^{A}(^{B}V_{Q}) = \frac{^{A}d}{dt}\,^{B}Q. \tag{5.2}$$

So we see that, in the general case, a velocity vector is associated with a point in space, but the numerical values describing the velocity of that point depend on two frames: one with respect to which the differentiation was done, and one in which the resulting velocity vector is expressed.

In (5.1), the calculated velocity is written in terms of the frame of differentiation, so the result could be indicated with a leading B superscript, but, for simplicity, when both superscripts are the same, we needn't indicate the outer one; that is, we write

$$^{B}(^{B}V_{Q}) = {^{B}V_{Q}}. \tag{5.3}$$

Finally, we can always remove the outer, leading superscript by explicitly including the rotation matrix that accomplishes the change in reference frame (see Section 2.10); that is, we write

$$^{A}(^{B}V_{Q}) = {^{A}_{B}R}\,^{B}V_{Q}. \tag{5.4}$$

We will usually write expressions in the form of the right-hand side of (5.4) so that the symbols representing velocities will always mean the velocity in the frame of differentiation and will not have outer, leading superscripts.

Rather than considering a general point's velocity relative to an arbitrary frame, we will very often consider the velocity of the *origin of a frame* relative to some understood universe reference frame. For this special case, we define a shorthand notation,

$$\upsilon_C = {^{U}V_{CORG}}, \tag{5.5}$$

where the point in question is the origin of frame $\{C\}$ and the reference frame is $\{U\}$. For example, we can use the notation υ_C to refer to the velocity of the origin of frame $\{C\}$; then $^{A}\upsilon_C$ is the velocity of the origin of frame $\{C\}$ expressed in terms of frame $\{A\}$ (though differentiation was done relative to $\{U\}$).

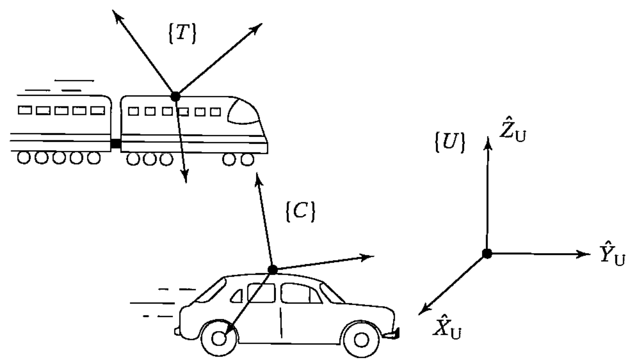

FIGURE 5.1: Example of some frames in linear motion.

EXAMPLE 5.1

Figure 5.1 shows a fixed universe frame, $\{U\}$, a frame attached to a train traveling at 100 mph, $\{T\}$, and a frame attached to a car traveling at 30 mph, $\{C\}$. Both vehicles are heading in the $\hat{X}$ direction of $\{U\}$. The rotation matrices, ${}^U_T R$ and ${}^U_C R$, are known and constant.

What is $\frac{{}^U d}{dt}\,{}^U P_{CORG}$?

$$\frac{{}^U d}{dt}\,{}^U P_{CORG} = {}^U V_{CORG} = v_C = 30\hat{X}.$$

What is ${}^C({}^U V_{TORG})$?

$${}^C({}^U V_{TORG}) = {}^C v_T = {}^C_U R v_T = {}^C_U R(100\hat{X}) = {}^U_C R^{-1}\,100\hat{X}.$$

What is ${}^C({}^T V_{CORG})$?

$${}^C({}^T V_{CORG}) = {}^C_T R\,{}^T V_{CORG} = -{}^U_C R^{-1}\,{}^U_T R\,70\hat{X}.$$

The angular velocity vector

We now introduce an **angular velocity vector**, using the symbol Ω. Whereas linear velocity describes an attribute of a point, angular velocity describes an attribute of a body. We always attach a frame to the bodies we consider, so we can also think of angular velocity as describing rotational motion of a frame.

In Fig. 5.2, ${}^A\Omega_B$ describes the rotation of frame $\{B\}$ relative to $\{A\}$. Physically, at any instant, the direction of ${}^A\Omega_B$ indicates the instantaneous axis of rotation of $\{B\}$ relative to $\{A\}$, and the magnitude of ${}^A\Omega_B$ indicates the speed of rotation. Again, like any vector, an angular velocity vector may be expressed in any coordinate system, and so another leading superscript may be added; for example, ${}^C({}^A\Omega_B)$ is the angular velocity of frame $\{B\}$ relative to $\{A\}$ expressed in terms of frame $\{C\}$.

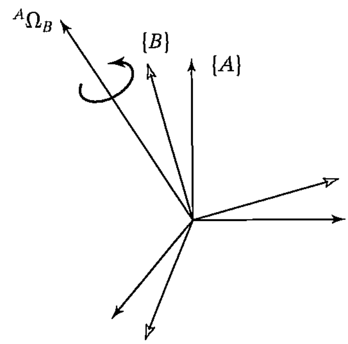

FIGURE 5.2: Frame $\{B\}$ is rotating with angular velocity ${}^A\Omega_B$ relative to frame $\{A\}$.

Again, we introduce a simplified notation for an important special case. This is simply the case in which there is an understood reference frame, so that it need not be mentioned in the notation:

$$\omega_C = {}^U\Omega_C. \tag{5.6}$$

Here, ω_C is the angular velocity of frame $\{C\}$ relative to some understood reference frame, $\{U\}$. For example, ${}^A\omega_C$ is the angular velocity of frame $\{C\}$ expressed in terms of $\{A\}$ (though the angular velocity is with respect to $\{U\}$).

5.3 LINEAR AND ROTATIONAL VELOCITY OF RIGID BODIES

In this section, we investigate the description of motion of a rigid body, at least as far as velocity is concerned. These ideas extend the notions of translations and orientations described in Chapter 2 to the time-varying case. In Chapter 6, we will further extend our study to considerations of acceleration.

As in Chapter 2, we attach a coordinate system to any body that we wish to describe. Then, motion of rigid bodies can be equivalently studied as the motion of frames relative to one another.

Linear velocity

Consider a frame $\{B\}$ attached to a rigid body. We wish to describe the motion of ${}^BQ$ relative to frame $\{A\}$, as in Fig. 5.3. We may consider $\{A\}$ to be fixed.

Frame $\{B\}$ is located relative to $\{A\}$, as described by a position vector, ${}^AP_{BORG}$, and a rotation matrix, ${}^A_BR$. For the moment, we will assume that the orientation, ${}^A_BR$, is not changing with time—that is, the motion of point Q relative to $\{A\}$ is due to ${}^AP_{BORG}$ or ${}^BQ$ changing in time.

Solving for the linear velocity of point Q in terms of $\{A\}$ is quite simple. Just express both components of the velocity in terms of $\{A\}$, and sum them:

$${}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q. \tag{5.7}$$

Equation (5.7) is for only that case in which relative orientation of $\{B\}$ and $\{A\}$ remains constant.

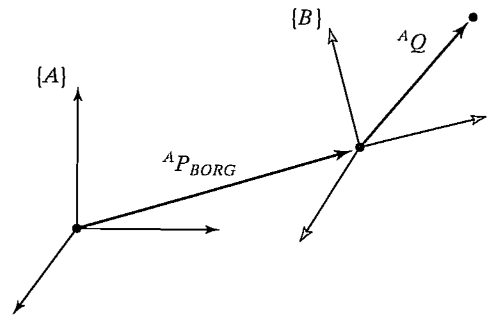

FIGURE 5.3: Frame $\{B\}$ is translating with velocity $^{A}V_{BORG}$ relative to frame $\{A\}$.

Rotational velocity

Now let us consider two frames with coincident origins and with zero linear relative velocity; their origins will remain coincident for all time. One or both could be attached to rigid bodies, but, for clarity, the rigid bodies are not shown in Fig. 5.4.

The orientation of frame $\{B\}$ with respect to frame $\{A\}$ is changing in time. As indicated in Fig. 5.4, rotational velocity of $\{B\}$ relative to $\{A\}$ is described by a vector called $^{A}\Omega_{B}$. We also have indicated a vector ^{B}Q that locates a point fixed in $\{B\}$. Now we consider the all-important question: How does a vector change with time as viewed from $\{A\}$ when it is fixed in $\{B\}$ and the systems are rotating?

Let us consider that the vector Q is constant as viewed from frame $\{B\}$; that is,

$$^{B}V_{Q} = 0. \tag{5.8}$$

Even though it is constant relative to $\{B\}$, it is clear that point Q will have a velocity as seen from $\{A\}$ that is due to the rotational velocity $^{A}\Omega_{B}$. To solve for the velocity of point Q, we will use an intuitive approach. Figure 5.5 shows two instants of time as vector Q rotates around $^{A}\Omega_{B}$. This is what an observer in $\{A\}$ would observe.

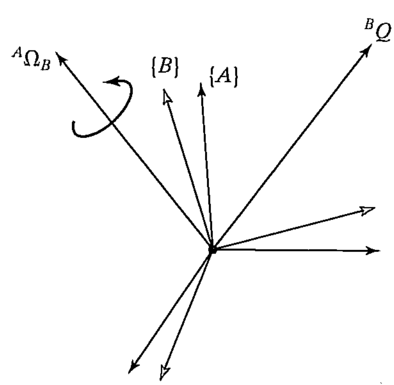

FIGURE 5.4: Vector ^{B}Q, fixed in frame $\{B\}$, is rotating with respect to frame $\{A\}$ with angular velocity $^{A}\Omega_{B}$.

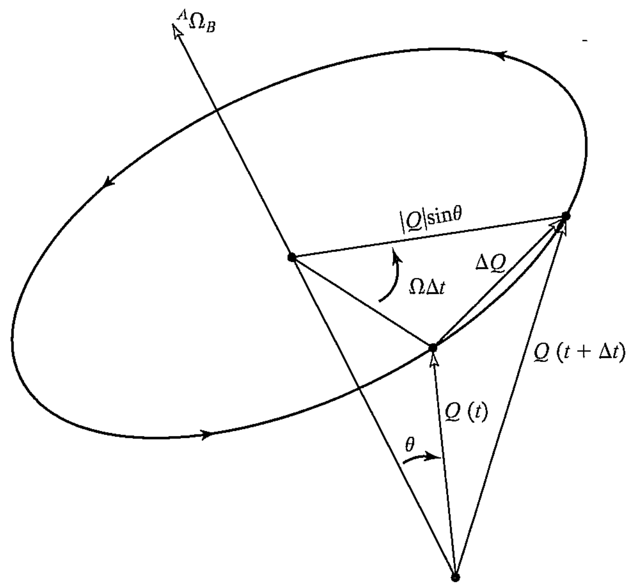

FIGURE 5.5: The velocity of a point due to an angular velocity.

By examining Fig. 5.5, we can figure out both the direction and the magnitude of the change in the vector as viewed from $\{A\}$. First, it is clear that the differential change in ${}^{A}Q$ must be perpendicular to both ${}^{A}\Omega_B$ and ${}^{A}Q$. Second, we see from Fig. 5.5 that the magnitude of the differential change is

$$|\Delta Q| = (|{}^{A}Q|\sin\theta)(|{}^{A}\Omega_B|\Delta t). \tag{5.9}$$

These conditions on magnitude and direction immediately suggest the vector cross-product. Indeed, our conclusions about direction and magnitude are satisfied by the computational form

$$ {}^{A}V_Q = {}^{A}\Omega_B \times {}^{A}Q. \tag{5.10}$$

In the general case, the vector Q could also be changing with respect to frame $\{B\}$, so, adding this component, we have

$$ {}^{A}V_Q = {}^{A}({}^{B}V_Q) + {}^{A}\Omega_B \times {}^{A}Q. \tag{5.11}$$

Using a rotation matrix to remove the dual-superscript, and noting that the description of ${}^{A}Q$ at any instant is ${}^{A}_{B}R\,{}^{B}Q$, we end with

$$ {}^{A}V_Q = {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q. \tag{5.12}$$

Simultaneous linear and rotational velocity

We can very simply expand (5.12) to the case where origins are not coincident by adding on the linear velocity of the origin to (5.12) to derive the general formula for velocity of a vector fixed in frame $\{B\}$ as seen from frame $\{A\}$:

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q \tag{5.13}$$

Equation (5.13) is the final result for the derivative of a vector in a moving frame as seen from a stationary frame.

5.4 MORE ON ANGULAR VELOCITY

In this section, we take a deeper look at angular velocity and, in particular, at the derivation of (5.10). Whereas the previous section took a geometric approach toward showing the validity of (5.10), here we take a mathematical approach. This section may be skipped by the first-time reader.

A property of the derivative of an orthonormal matrix

We can derive an interesting relationship between the derivative of an orthonormal matrix and a certain skew-symmetric matrix as follows. For any $n \times n$ orthonormal matrix, R, we have

$$RR^T = I_n \tag{5.14}$$

where I_n is the $n \times n$ identity matrix. Our interest, by the way, is in the case where $n = 3$ and R is a *proper* orthonormal matrix, or rotation matrix. Differentiating (5.14) yields

$$\dot{R}R^T + R\dot{R}^T = 0_n, \tag{5.15}$$

where 0_n is the $n \times n$ zero matrix. Eq. (5.15) may also be written

$$\dot{R}R^T + (\dot{R}R^T)^T = 0_n. \tag{5.16}$$

Defining

$$S = \dot{R}R^T, \tag{5.17}$$

we have, from (5.16), that

$$S + S^T = 0_n. \tag{5.18}$$

So, we see that S is a skew-symmetric matrix. Hence, a property relating the derivative of orthonormal matrices with skew-symmetric matrices exists and can be stated as

$$S = \dot{R}R^{-1}. \tag{5.19}$$

Velocity of a point due to rotating reference frame

Consider a fixed vector ${}^BP$ unchanging with respect to frame $\{B\}$. Its description in another frame $\{A\}$ is given as

$${}^AP = {}^A_BR\,{}^BP. \tag{5.20}$$

If frame $\{B\}$ is rotating (i.e., the derivative ${}^A_B\dot{R}$ is nonzero), then ${}^AP$ will be changing even though ${}^BP$ is constant; that is,

$${}^A\dot{P} = {}^A_B\dot{R}\,{}^BP, \tag{5.21}$$

or, using our notation for velocity,

$${}^AV_P = {}^A_B\dot{R}\,{}^BP. \tag{5.22}$$

Now, rewrite (5.22) by substituting for ${}^B P$, to obtain

$$ {}^A V_P = {}^A_B\dot{R}\; {}^A_B R^{-1}\; {}^A P. \tag{5.23} $$

Making use of our result (5.19) for orthonormal matrices, we have

$$ {}^A V_P = {}^A_B S\; {}^A P, \tag{5.24} $$

where we have adorned S with sub- and superscripts to indicate that it is the skew-symmetric matrix associated with the particular rotation matrix ${}^A_B R$. Because of its appearance in (5.24), and for other reasons to be seen shortly, the skew-symmetric matrix we have introduced is called the **angular-velocity matrix**.

Skew-symmetric matrices and the vector cross-product

If we assign the elements in a skew-symmetric matrix S as

$$ S = \begin{bmatrix} 0 & -\Omega_x & \Omega_y \\ \Omega_x & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}, \tag{5.25} $$

and define the 3×1 column vector

$$ \Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}, \tag{5.26} $$

then it is easily verified that

$$ SP = \Omega \times P, \tag{5.27} $$

where P is any vector, and $\times$ is the vector cross-product.

The 3×1 vector Ω, which corresponds to the 3×3 angular-velocity matrix, is called the **angular-velocity vector** and was already introduced in Section 5.2.

Hence, our relation (5.24) can be written

$$ {}^A V_P = {}^A\Omega_B \times {}^A P, \tag{5.28} $$

where we have shown the notation for Ω indicating that it is the angular-velocity vector specifying the motion of frame $\{B\}$ with respect to frame $\{A\}$.

Gaining physical insight concerning the angular-velocity vector

Having concluded that there exists some vector Ω such that (5.28) is true, we now wish to gain some insight as to its physical meaning. Derive Ω by direct differentiation of a rotation matrix; that is,

$$ \dot{R} = \lim_{\Delta t \to 0} \frac{R(t + \Delta t) - R(t)}{\Delta t}. \tag{5.29} $$

Now, write $R(t + \Delta t)$ as the composition of two matrices, namely,

$$ R(t + \Delta t) = R_K(\Delta\theta) R(t), \tag{5.30} $$

where, over the interval Δt, a small rotation of $\Delta\theta$ has occurred about axis $\hat{K}$. Using (5.30), write (5.29) as

$$\dot{R} = \lim_{\Delta t \to 0} \left(\frac{R_K(\Delta\theta) - I_3}{\Delta t} R(t) \right); \tag{5.31}$$

that is,

$$\dot{R} = \left(\lim_{\Delta t \to 0} \frac{R_K(\Delta\theta) - I_3}{\Delta t} \right) R(t). \tag{5.32}$$

Now, from small angle substitutions in (2.80), we have

$$R_K(\Delta\theta) = \begin{bmatrix} 1 & -k_z\Delta\theta & k_y\Delta\theta \\ k_z\Delta\theta & 1 & -k_x\Delta\theta \\ -k_y\Delta\theta & k_x\Delta\theta & 1 \end{bmatrix}. \tag{5.33}$$

So, (5.32) may be written

$$\dot{R} = \left(\lim_{\Delta t \to 0} \frac{\begin{bmatrix} 0 & -k_z\Delta\theta & k_y\Delta\theta \\ k_z\Delta\theta & 0 & -k_x\Delta\theta \\ -k_y\Delta\theta & k_x\Delta\theta & 0 \end{bmatrix}}{\Delta t} \right) R(t). \tag{5.34}$$

Finally, dividing the matrix through by Δt and then taking the limit, we have

$$\dot{R} = \begin{bmatrix} 0 & -k_z\dot{\theta} & k_y\dot{\theta} \\ k_z\dot{\theta} & 0 & -k_x\dot{\theta} \\ -k_y\dot{\theta} & k_x\dot{\theta} & 0 \end{bmatrix} R(t). \tag{5.35}$$

Hence, we see that

$$\dot{R}R^{-1} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}, \tag{5.36}$$

where

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} k_x\dot{\theta} \\ k_y\dot{\theta} \\ k_z\dot{\theta} \end{bmatrix} = \dot{\theta}\hat{K}. \tag{5.37}$$

The physical meaning of the angular-velocity vector Ω is that, at any instant, the change in orientation of a rotating frame can be viewed as a rotation about some axis $\hat{K}$. This **instantaneous axis of rotation**, taken as a unit vector and then scaled by the speed of rotation about that axis ($\dot{\theta}$), yields the angular-velocity vector.

Other representations of angular velocity

Other representations of angular velocity are possible; for example, imagine that the angular velocity of a rotating body is available as rates of the set of Z–Y–Z Euler angles:

$$\dot{\Theta}_{Z'Y'Z'} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}. \tag{5.38}$$

Given this style of description, or any other using one of the 24 **angle sets**, we would like to derive the equivalent angular-velocity vector.

We have seen that

$$\dot{R}R^T = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}. \tag{5.39}$$

From this matrix equation, one can extract three independent equations, namely,

$$\begin{aligned} \Omega_x &= \dot{r}_{31}r_{21} + \dot{r}_{32}r_{22} + \dot{r}_{33}r_{23}, \\ \Omega_y &= \dot{r}_{11}r_{31} + \dot{r}_{12}r_{32} + \dot{r}_{13}r_{33}, \\ \Omega_z &= \dot{r}_{21}r_{11} + \dot{r}_{22}r_{12} + \dot{r}_{23}r_{13}. \end{aligned} \tag{5.40}$$

From (5.40) and a symbolic description of R in terms of an angle set, one can derive the expressions that relate the angle-set velocities to the equivalent angular-velocity vector. The resulting expressions can be cast in matrix form—for example, for Z–Y–Z Euler angles,

$$\Omega = E_{Z'Y'Z'}(\Theta_{Z'Y'Z'})\dot{\Theta}_{Z'Y'Z'}. \tag{5.41}$$

That is, $E(\cdot)$ is a Jacobian relating an angle-set velocity vector to the angular-velocity vector and is a function of the instantaneous values of the angle set. The form of $E(\cdot)$ depends on the particular angle set it is developed for; hence, a subscript is added to indicate which.

EXAMPLE 5.2

Construct the E matrix that relates Z–Y–Z Euler angles to the angular-velocity vector; that is, find $E_{Z'Y'Z'}$ in (5.41).

Using (2.72) and (5.40) and doing the required symbolic differentiations yields

$$E_{Z'Y'Z'} = \begin{bmatrix} 0 & -s\alpha & c\alpha s\beta \\ 0 & c\alpha & s\alpha s\beta \\ 1 & 0 & c\beta \end{bmatrix}. \tag{5.42}$$

5.5 MOTION OF THE LINKS OF A ROBOT

In considering the motions of robot links, we will always use link frame {0} as our reference frame. Hence, v_i is the linear velocity of the origin of link frame $\{i\}$, and ω_i is the angular velocity of link frame $\{i\}$.

At any instant, each link of a robot in motion has some linear and angular velocity. Figure 5.6 indicates these vectors for link i. In this case, it is indicated that they are written in frame $\{i\}$.

5.6 VELOCITY "PROPAGATION" FROM LINK TO LINK

We now consider the problem of calculating the linear and angular velocities of the links of a robot. A manipulator is a chain of bodies, each one capable of motion

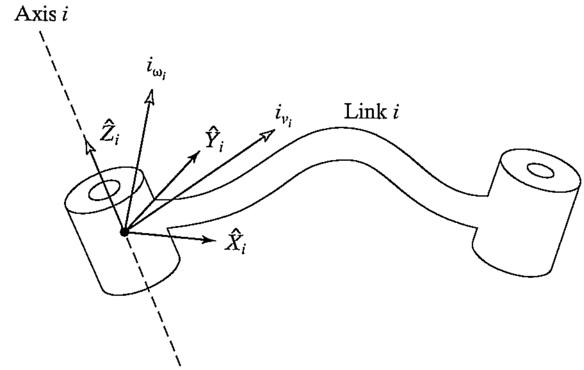

FIGURE 5.6: The velocity of link i is given by vectors v_i and ω_i, which may be written in any frame, even frame $\{i\}$.

relative to its neighbors. Because of this structure, we can compute the velocity of each link in order, starting from the base. The velocity of link $i+1$ will be that of link i, plus whatever new velocity components were added by joint $i+1$.[2]

As indicated in Fig. 5.6, let us now think of each link of the mechanism as a rigid body with linear and angular velocity vectors describing its motion. Further, we will express these velocities with respect to the link frame itself rather than with respect to the base coordinate system. Figure 5.7 shows links i and $i+1$, along with their velocity vectors defined in the link frames.

Rotational velocities can be added when both ω vectors are written with respect to the same frame. Therefore, the angular velocity of link $i+1$ is the same

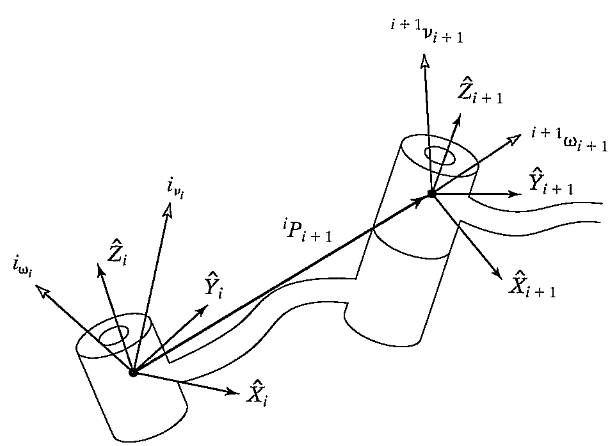

FIGURE 5.7: Velocity vectors of neighboring links.

[2]Remember that linear velocity is associated with a point, but angular velocity is associated with a body. Hence, the term "velocity of a link" here means the linear velocity of the origin of the link frame and the rotational velocity of the link.

as that of link i plus a new component caused by rotational velocity at joint $i+1$. This can be written in terms of frame $\{i\}$ as

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_{i} + {}^{i}_{i+1}R\,\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}. \tag{5.43}$$

Note that

$$\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}. \tag{5.44}$$

We have made use of the rotation matrix relating frames $\{i\}$ and $\{i+1\}$ in order to represent the added rotational component due to motion at the joint in frame $\{i\}$. The rotation matrix rotates the axis of rotation of joint $i+1$ into its description in frame $\{i\}$, so that the two components of angular velocity can be added.

By premultiplying both sides of (5.43) by ${}^{i+1}_{i}R$ we can find the description of the angular velocity of link $i+1$ with respect to frame $\{i+1\}$:

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}. \tag{5.45}$$

The linear velocity of the origin of frame $\{i+1\}$ is the same as that of the origin of frame $\{i\}$ plus a new component caused by rotational velocity of link i. This is exactly the situation described by (5.13), with one term vanishing because ${}^{i}P_{i+1}$ is constant in frame $\{i\}$. Therefore, we have

$$ {}^{i}v_{i+1} = {}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}. \tag{5.46}$$

Premultiplying both sides by ${}^{i+1}_{i}R$, we compute

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}). \tag{5.47}$$

Equations (5.45) and (5.47) are perhaps the most important results of this chapter. The corresponding relationships for the case that joint $i+1$ is prismatic are

$$\begin{aligned} {}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\,{}^{i}\omega_{i}, \\ {}^{i+1}v_{i+1} &= {}^{i+1}_{i}R({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}. \end{aligned} \tag{5.48}$$

Applying these equations successively from link to link, we can compute ${}^{N}\omega_{N}$ and ${}^{N}v_{N}$, the rotational and linear velocities of the last link. Note that the resulting velocities are expressed in terms of frame $\{N\}$. This turns out to be useful, as we will see later. If the velocities are desired in terms of the base coordinate system, they can be rotated into base coordinates by multiplication with ${}^{0}_{N}R$.

EXAMPLE 5.3

A two-link manipulator with rotational joints is shown in Fig. 5.8. Calculate the velocity of the tip of the arm as a function of joint rates. Give the answer in two forms—in terms of frame $\{3\}$ and also in terms of frame $\{0\}$.

Frame $\{3\}$ has been attached at the end of the manipulator, as shown in Fig. 5.9, and we wish to find the velocity of the origin of this frame expressed in frame $\{3\}$. As a second part of the problem, we will express these velocities in frame $\{0\}$ as

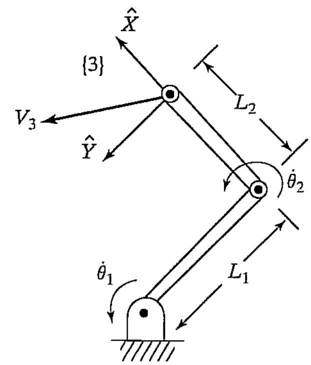

FIGURE 5.8: A two-link manipulator.

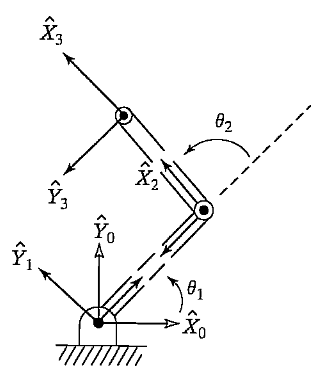

FIGURE 5.9: Frame assignments for the two-link manipulator.

well. We will start by attaching frames to the links as we have done before (shown in Fig. 5.9).

We will use (5.45) and (5.47) to compute the velocity of the origin of each frame, starting from the base frame {0}, which has zero velocity. Because (5.45) and (5.47) will make use of the link transformations, we compute them:

$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{5.49}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Note that these correspond to the manipulator of Example 3.3 with joint 3 permanently fixed at zero degrees. The final transformation between frames {2} and {3} need not be cast as a standard link transformation (though it might be helpful to do so). Then, using (5.45) and (5.47) sequentially from link to link, we calculate

$$ {}^{1}\omega_{1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \tag{5.50} $$

$$ {}^{1}v_{1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{5.51} $$

$$ {}^{2}\omega_{2} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}, \tag{5.52} $$

$$ {}^{2}v_{2} = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix}, \tag{5.53} $$

$$ {}^{3}\omega_{3} = {}^{2}\omega_{2}, \tag{5.54} $$

$$ {}^{3}v_{3} = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}. \tag{5.55} $$

Equation (5.55) is the answer. Also, the rotational velocity of frame {3} is found in (5.54).

To find these velocities with respect to the nonmoving base frame, we rotate them with the rotation matrix ${}^{0}_{3}R$, which is

$$ {}^{0}_{3}R = {}^{0}_{1}R \ {}^{1}_{2}R \ {}^{2}_{3}R = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.56} $$

This rotation yields

$$ {}^{0}v_{3} = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}. \tag{5.57} $$

It is important to point out the two distinct uses for (5.45) and (5.47). First, they can be used as a means of deriving analytical expressions, as in Example 5.3 above. Here, we manipulate the symbolic equations until we arrive at a form such as (5.55), which will be evaluated with a computer in some application. Second, they can be used directly to compute (5.45) and (5.47) as they are written. They can easily be written as a subroutine, which is then applied iteratively to compute link velocities. As such, they could be used for any manipulator, without the need of deriving the equations for a particular manipulator. However, the computation then yields a numeric result with the structure of the equations hidden. We are often interested in the structure of an analytic result such as (5.55). Also, if we bother to do the work (that is, (5.50) through (5.57)), we generally will find that there are fewer computations left for the computer to perform in the final application.

5.7 JACOBIANS

The Jacobian is a multidimensional form of the derivative. Suppose, for example, that we have six functions, each of which is a function of six independent variables:

$$\begin{aligned} y_1 &= f_1(x_1, x_2, x_3, x_4, x_5, x_6), \\ y_2 &= f_2(x_1, x_2, x_3, x_4, x_5, x_6), \\ &\vdots \\ y_6 &= f_6(x_1, x_2, x_3, x_4, x_5, x_6). \end{aligned} \tag{5.58}$$

We could also use vector notation to write these equations:

$$Y = F(X). \tag{5.59}$$

Now, if we wish to calculate the differentials of y_i as a function of differentials of x_j, we simply use the chain rule to calculate, and we get

$$\begin{aligned} \delta y_1 &= \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6, \\ \delta y_2 &= \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6}\delta x_6, \\ &\vdots \\ \delta y_6 &= \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6, \end{aligned} \tag{5.60}$$

which again might be written more simply in vector notation:

$$\delta Y = \frac{\partial F}{\partial X}\delta X. \tag{5.61}$$

The 6×6 matrix of partial derivatives in (5.61) is what we call the Jacobian, J. Note that, if the functions $f_1(X)$ through $f_6(X)$ are nonlinear, then the partial derivatives are a function of the x_i, so, we can use the notation

$$\delta Y = J(X)\delta X. \tag{5.62}$$

By dividing both sides by the differential time element, we can think of the Jacobian as mapping velocities in X to those in Y:

$$\dot{Y} = J(X)\dot{X}. \tag{5.63}$$

At any particular instant, X has a certain value, and $J(X)$ is a linear transformation. At each new time instant, X has changed, and therefore, so has the linear transformation. Jacobians are time-varying linear transformations.

In the field of robotics, we generally use Jacobians that relate joint velocities to Cartesian velocities of the tip of the arm—for example,

$$^{0}\nu = {}^{0}J(\Theta)\dot{\Theta}, \tag{5.64}$$

where Θ is the vector of joint angles of the manipulator and ν is a vector of Cartesian velocities. In (5.64), we have added a leading superscript to our Jacobian notation to indicate in which frame the resulting Cartesian velocity is expressed. Sometimes this superscript is omitted when the frame is obvious or when it is unimportant to the development. Note that, for any given configuration of the manipulator, joint rates are related to velocity of the tip in a linear fashion, yet this is only an instantaneous relationship—in the next instant, the Jacobian has changed slightly. For the general case of a six-jointed robot, the Jacobian is 6×6, $\dot{\Theta}$ is 6×1, and $^{0}\nu$ is 6×1. This 6×1 Cartesian velocity vector is the 3×1 linear velocity vector and the 3×1 rotational velocity vector stacked together:

$$^{0}\nu = \begin{bmatrix} ^{0}v \\ ^{0}\omega \end{bmatrix}. \tag{5.65}$$

Jacobians of any dimension (including nonsquare) can be defined. The number of rows equals the number of degrees of freedom in the Cartesian space being considered. The number of columns in a Jacobian is equal to the number of joints of the manipulator. In dealing with a planar arm, for example, there is no reason for the Jacobian to have more than three rows, although, for redundant planar manipulators, there could be arbitrarily many columns (one for each joint).

In the case of a two-link arm, we can write a 2×2 Jacobian that relates joint rates to end-effector velocity. From the result of Example 5.3, we can easily determine the Jacobian of our two-link arm. The Jacobian written in frame {3} is seen (from (5.55)) to be

$$^{3}J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}, \tag{5.66}$$

and the Jacobian written in frame {0} is (from (5.57))

$$^{0}J(\Theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}. \tag{5.67}$$

Note that, in both cases, we have chosen to write a square matrix that relates joint rates to end-effector velocity. We could also consider a 3×2 Jacobian that would include the angular velocity of the end-effector.

Considering (5.58) through (5.62), which define the Jacobian, we see that the Jacobian might also be found by directly differentiating the kinematic equations of the mechanism. This is straightforward for linear velocity, but there is no 3×1 orientation vector whose derivative is ω. Hence, we have introduced a method to derive the Jacobian by using successive application of (5.45) and (5.47). There are several other methods that can be used (see, for example, [4]), one of which will be introduced shortly in Section 5.8. One reason for deriving Jacobians via the method presented is that it helps prepare us for material in Chapter 6, in which we will find that similar techniques apply to calculating the dynamic equations of motion of a manipulator.

Changing a Jacobian's frame of reference

Given a Jacobian written in frame $\{B\}$, that is,

$$\begin{bmatrix} {}^B v \\ {}^B \omega \end{bmatrix} = {}^B \nu = {}^B J(\Theta)\dot{\Theta}, \tag{5.68}$$

we might be interested in giving an expression for the Jacobian in another frame, $\{A\}$. First, note that a 6×1 Cartesian velocity vector given in $\{B\}$ is described relative to $\{A\}$ by the transformation

$$\begin{bmatrix} {}^A v \\ {}^A \omega \end{bmatrix} = \left[\begin{array}{c|c} {}^A_B R & 0 \\ \hline 0 & {}^A_B R \end{array} \right] \begin{bmatrix} {}^B v \\ {}^B \omega \end{bmatrix}. \tag{5.69}$$

Hence, we can write

$$\begin{bmatrix} {}^A v \\ {}^A \omega \end{bmatrix} = \left[\begin{array}{c|c} {}^A_B R & 0 \\ \hline 0 & {}^A_B R \end{array} \right] {}^B J(\Theta)\dot{\Theta}. \tag{5.70}$$

Now it is clear that changing the frame of reference of a Jacobian is accomplished by means of the following relationship:

$${}^A J(\Theta) = \left[\begin{array}{c|c} {}^A_B R & 0 \\ \hline 0 & {}^A_B R \end{array} \right] {}^B J(\Theta). \tag{5.71}$$

5.8 SINGULARITIES

Given that we have a linear transformation relating joint velocity to Cartesian velocity, a reasonable question to ask is: Is this matrix invertible? That is, is it nonsingular? If the matrix is nonsingular, then we can invert it to calculate joint rates from given Cartesian velocities:

$$\dot{\Theta} = J^{-1}(\Theta)\nu. \tag{5.72}$$

This is an important relationship. For example, say that we wish the hand of the robot to move with a certain velocity vector in Cartesian space. Using (5.72), we could calculate the necessary joint rates at each instant along the path. The real question of invertibility is: Is the Jacobian invertible for all values of Θ? If not, where is it not invertible?

Most manipulators have values of Θ where the Jacobian becomes singular. Such locations are called **singularities of the mechanism** or **singularities** for short. All manipulators have singularities at the boundary of their workspace, and most have loci of singularities inside their workspace. An in-depth study of the classification of singularities is beyond the scope of this book—for more information, see [5]. For our purposes, and without giving rigorous definitions, we will class singularities into two categories:

1. **Workspace-boundary singularities** occur when the manipulator is fully stretched out or folded back on itself in such a way that the end-effector is at or very near the boundary of the workspace.

2. **Workspace-interior singularities** occur away from the workspace boundary; they generally are caused by a lining up of two or more joint axes.

When a manipulator is in a singular configuration, it has lost one or more degrees of freedom (as viewed from Cartesian space). This means that there is some direction (or subspace) in Cartesian space along which it is impossible to move the hand of the robot, no matter what joint rates are selected. It is obvious that this happens at the workspace boundary of robots.

EXAMPLE 5.4

Where are the singularities of the simple two-link arm of Example 5.3? What is the physical explanation of the singularities? Are they workspace-boundary singularities or workspace-interior singularities?

To find the singular points of a mechanism, we must examine the determinant of its Jacobian. Where the determinant is equal to zero, the Jacobian has lost full rank and is singular:

$$DET[J(\Theta)] = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} = l_1 l_2 s_2 = 0. \tag{5.73}$$

Clearly, a singularity of the mechanism exists when θ_2 is 0 or 180 degrees. Physically, when $\theta_2 = 0$, the arm is stretched straight out. In this configuration, motion of the end-effector is possible along only one Cartesian direction (the one perpendicular to the arm). Therefore, the mechanism has lost one degree of freedom. Likewise, when $\theta_2 = 180$, the arm is folded completely back on itself, and motion of the hand again is possible only in one Cartesian direction instead of two. We will class these singularities as workspace-boundary singularities, because they exist at the edge of the manipulator's workspace. Note that the Jacobian written with respect to frame {0}, or any other frame, would have yielded the same result.

The danger in applying (5.72) in a robot control system is that, at a singular point, the inverse Jacobian blows up! This results in joint rates approaching infinity as the singularity is approached.

EXAMPLE 5.5

Consider the two-link robot from Example 5.3 as it is moving its end-effector along the $\hat{X}$ axis at 1.0 m/s, as in Fig. 5.10. Show that joint rates are reasonable when far from a singularity, but that, as a singularity is approached at $\theta_2 = 0$, joint rates tend to infinity.

We start by calculating the inverse of the Jacobian written in {0}:

$$^{0}J^{-1}(\Theta) = \frac{1}{l_1 l_2 s_2} \begin{bmatrix} l_2 c_{12} & l_2 s_{12} \\ -l_1 c_1 - l_2 c_{12} & -l_1 s_1 - l_2 s_{12} \end{bmatrix}. \tag{5.74}$$

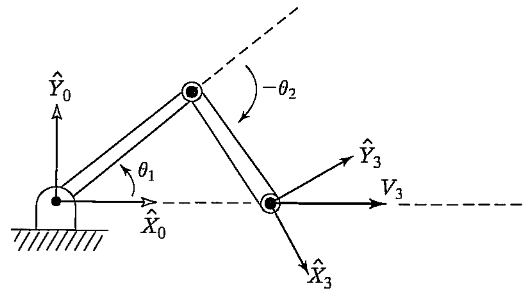

FIGURE 5.10: A two-link manipulator moving its tip at a constant linear velocity.

Then, using Eq. (5.74) for a velocity of 1 m/s in the $\hat{X}$ direction, we can calculate joint rates as a function of manipulator configuration:

$$\dot{\theta}_1 = \frac{c_{12}}{l_1 s_2}, \quad (5.75)$$

$$\dot{\theta}_2 = -\frac{c_1}{l_2 s_2} - \frac{c_{12}}{l_1 s_2}.$$

Clearly, as the arm stretches out toward $\theta_2 = 0$, both joint rates go to infinity.

EXAMPLE 5.6

For the PUMA 560 manipulator, give two examples of singularities that can occur.

There is singularity when θ_3 is near -90.0 degrees. Calculation of the exact value of θ_3 is left as an exercise. (See Exercise 5.14.) In this situation, links 2 and 3 are "stretched out," just like the singular location of the two-link manipulator in Example 5.3. This is classed as a workspace-boundary singularity.

Whenever $\theta_5 = 0.0$ degrees, the manipulator is in a singular configuration. In this configuration, joint axes 4 and 6 line up—both of their actions would result in the same end-effector motion, so it is as if a degree of freedom has been lost. Because this can occur interior to the workspace envelope, we will class it as a workspace-interior singularity.

5.9 STATIC FORCES IN MANIPULATORS

The chainlike nature of a manipulator leads us quite naturally to consider how forces and moments "propagate" from one link to the next. Typically, the robot is pushing on something in the environment with the chain's free end (the end-effector) or is perhaps supporting a load at the hand. We wish to solve for the joint torques that must be acting to keep the system in static equilibrium.

In considering static forces in a manipulator, we first lock all the joints so that the manipulator becomes a structure. We then consider each link in this structure and write a force-moment balance relationship in terms of the link frames. Finally, we compute what static torque must be acting about the joint axis in order for the manipulator to be in static equilibrium. In this way, we solve for the set of joint torques needed to support a static load acting at the end-effector.

In this section, we will not be considering the force on the links due to gravity (that will be left until chapter 6). The static forces and torques we are considering at the joints are those caused by a static force or torque (or both) acting on the last link—for example, as when the manipulator has its end-effector in contact with the environment.

We define special symbols for the force and torque exerted by a neighbor link:

f_i = force exerted on link i by link $i-1$,

n_i = torque exerted on link i by link $i-1$.

We will use our usual convention for assigning frames to links. Figure 5.11 shows the static forces and moments (excluding the gravity force) acting on link i. Summing the forces and setting them equal to zero, we have

$$ {}^{i}f_i - {}^{i}f_{i+1} = 0. \tag{5.76}$$

Summing torques about the origin of frame $\{i\}$, we have

$$ {}^{i}n_i - {}^{i}n_{i+1} - {}^{i}P_{i+1} \times {}^{i}f_{i+1} = 0. \tag{5.77}$$

If we start with a description of the force and moment applied by the hand, we can calculate the force and moment applied by each link, working from the last link down to the base (link 0). To do this, we formulate the force-moment expressions (5.76) and (5.77) such that they specify iterations from higher numbered links to lower numbered links. The result can be written as

$$ {}^{i}f_i = {}^{i}f_{i+1}, \tag{5.78}$$

$$ {}^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1}. \tag{5.79}$$

In order to write these equations in terms of only forces and moments defined within their own link frames, we transform with the rotation matrix describing frame

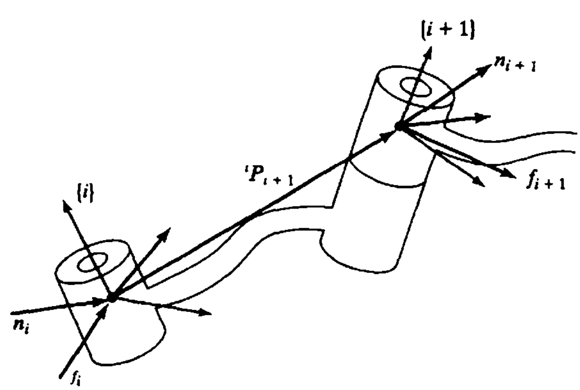

FIGURE 5.11: Static force-moment balance for a single link.

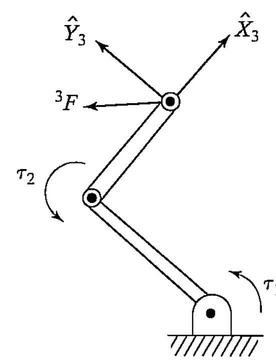

FIGURE 5.12: A two-link manipulator applying a force at its tip.

$\{i+1\}$ relative to frame $\{i\}$. This leads to our most important result for static force "propagation" from link to link:

$$ {}^{i}f_i = {}^{i}_{i+1}R \, {}^{i+1}f_{i+1}, \tag{5.80}$$

$$ {}^{i}n_i = {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i. \tag{5.81}$$

Finally, this important question arises: What torques are needed at the joints in order to balance the reaction forces and moments acting on the links? All components of the force and moment vectors are resisted by the structure of the mechanism itself, except for the torque about the joint axis. Therefore, to find the joint torque required to maintain the static equilibrium, the dot product of the joint-axis vector with the moment vector acting on the link is computed:

$$ \tau_i = {}^{i}n_i^T \, {}^{i}\hat{Z}_i. \tag{5.82}$$

In the case that joint i is prismatic, we compute the joint actuator force as

$$ \tau_i = {}^{i}f_i^T \, {}^{i}\hat{Z}_i. \tag{5.83}$$

Note that we are using the symbol τ even for a linear joint force.

As a matter of convention, we generally define the positive direction of joint torque as the direction which would tend to move the joint in the direction of increasing joint angle.

Equations (5.80) through (5.83) give us a means to compute the joint torques needed to apply any force or moment with the end-effector of a manipulator in the static case.

EXAMPLE 5.7

The two-link manipulator of Example 5.3 is applying a force vector ${}^3F$ with its end-effector. (Consider this force to be acting at the origin of $\{3\}$.) Find the required joint torques as a function of configuration and of the applied force. (See Fig. 5.12.)

We apply Eqs. (5.80) through (5.82), starting from the last link and going toward the base of the robot:

$$ {}^{2}f_{2} = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}, \tag{5.84} $$

$$ {}^{2}n_{2} = l_2 \hat{X}_2 \times \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix}, \tag{5.85} $$

$$ {}^{1}f_{1} = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix}, \tag{5.86} $$

$$ {}^{1}n_{1} = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} + l_1 \hat{X}_1 \times {}^{1}f_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix}. \tag{5.87} $$

Therefore, we have

$$ \tau_1 = l_1 s_2 f_x + (l_2 + l_1 c_2) f_y, \tag{5.88} $$

$$ \tau_2 = l_2 f_y. \tag{5.89} $$

This relationship can be written as a matrix operator:

$$ \tau = \begin{bmatrix} l_1 s_2 & l_2 + l_1 c_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}. \tag{5.90} $$

It is not a coincidence that this matrix is the transpose of the Jacobian that we found in (5.66)!

5.10 JACOBIANS IN THE FORCE DOMAIN

We have found joint torques that will exactly balance forces at the hand in the static situation. When forces act on a mechanism, work (in the technical sense) is done if the mechanism moves through a displacement. Work is defined as a force acting through a distance and is a scalar with units of energy. The principle of **virtual work** allows us to make certain statements about the static case by allowing the amount of this displacement to go to an infinitesimal. Work has the units of energy, so it must be the same measured in any set of generalized coordinates. Specifically, we can equate the work done in Cartesian terms with the work done in joint-space terms. In the multidimensional case, work is the dot product of a vector force or torque and a vector displacement. Thus, we have

$$ \mathcal{F} \cdot \delta\chi = \tau \cdot \delta\Theta, \tag{5.91} $$

where $\mathcal{F}$ is a 6×1 Cartesian force-moment vector acting at the end-effector, $\delta\chi$ is a 6×1 infinitesimal Cartesian displacement of the end-effector, τ is a 6×1 vector of torques at the joints, and $\delta\Theta$ is a 6×1 vector of infinitesimal joint displacements. Expression (5.91) can also be written as

$$\mathcal{F}^T \delta\chi = \tau^T \delta\Theta. \tag{5.92}$$

The definition of the Jacobian is

$$\delta\chi = J\delta\Theta, \tag{5.93}$$

so we may write

$$\mathcal{F}^T J\delta\theta = \tau^T \delta\Theta, \tag{5.94}$$

which must hold for all $\delta\Theta$; hence, we have

$$\mathcal{F}^T J = \tau^T. \tag{5.95}$$

Transposing both sides yields this result:

$$\tau = J^T \mathcal{F}. \tag{5.96}$$

Equation (5.96) verifies in general what we saw in the particular case of the two-link manipulator in Example 5.6: The Jacobian transpose maps Cartesian forces acting at the hand into equivalent joint torques. When the Jacobian is written with respect to frame {0}, then force vectors written in {0} can be transformed, as is made clear by the following notation:

$$\tau = {}^0J^T\, {}^0\mathcal{F}. \tag{5.97}$$

When the Jacobian loses full rank, there are certain directions in which the end-effector cannot exert static forces even if desired. That is, in (5.97), if the Jacobian is singular, $\mathcal{F}$ could be increased or decreased in certain directions (those defining the null-space of the Jacobian [6]) without effect on the value calculated for τ. This also means that, near singular configurations, mechanical advantage tends toward infinity, such that, with small joint torques, large forces could be generated at the end-effector.[3] Thus, singularities manifest themselves in the force domain as well as in the position domain.

Note that (5.97) is a very interesting relationship, in that it allows us to convert a Cartesian quantity into a joint-space quantity without calculating any inverse kinematic functions. We will make use of this when we consider the problem of control in later chapters.

5.11 CARTESIAN TRANSFORMATION OF VELOCITIES AND STATIC FORCES

We might wish to think in terms of 6×1 representations of general velocity of a body:

$$\nu = \begin{bmatrix} v \\ \omega \end{bmatrix}. \tag{5.98}$$

Likewise, we could consider 6×1 representations of general force vectors, such as

$$\mathcal{F} = \begin{bmatrix} F \\ N \end{bmatrix}, \tag{5.99}$$

[3] Consider a two-link planar manipulator nearly outstretched with the end-effector in contact with a reaction surface. In this configuration, arbitrarily large forces could be exerted by "small" joint torques.

where F is a 3×1 force vector and N is a 3×1 moment vector. It is then natural to think of 6×6 transformations that map these quantities from one frame to another. This is exactly what we have already done in considering the propagation of velocities and forces from link to link. Here, we write (5.45) and (5.47) in matrix-operator form to transform general velocity vectors in frame $\{A\}$ to their description in frame $\{B\}$.

The two frames involved here are rigidly connected, so $\dot{\theta}_{i+1}$, appearing in (5.45), is set to zero in deriving the relationship

$$\begin{bmatrix} {}^{B}v_{B} \\ {}^{B}\omega_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & -{}^{B}_{A}R\, {}^{A}P_{BORG}\times \\ 0 & {}^{B}_{A}R \end{bmatrix} \begin{bmatrix} {}^{A}v_{A} \\ {}^{A}\omega_{A} \end{bmatrix}, \tag{5.100}$$

where the cross product is understood to be the matrix operator

$$P\times = \begin{bmatrix} 0 & -p_x & p_y \\ p_x & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}. \tag{5.101}$$

Now, (5.100) relates velocities in one frame to those in another, so the 6×6 operator will be called a **velocity transformation**; we will use the symbol T_v. In this case, it is a velocity transformation that maps velocities in $\{A\}$ into velocities in $\{B\}$, so we use the following notation to express (5.100) compactly:

$${}^{B}\nu_{B} = {}^{B}_{A}T_{v}\, {}^{A}\nu_{A}. \tag{5.102}$$

We can invert (5.100) in order to compute the description of velocity in terms of $\{A\}$, given the quantities in $\{B\}$:

$$\begin{bmatrix} {}^{A}v_{A} \\ {}^{A}\omega_{A} \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{BORG}\times {}^{A}_{B}R \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}v_{B} \\ {}^{B}\omega_{B} \end{bmatrix}, \tag{5.103}$$

or

$${}^{A}\nu_{A} = {}^{A}_{B}T_{v}\, {}^{B}\nu_{B}. \tag{5.104}$$

Note that these mappings of velocities from frame to frame depend on ${}^{A}_{B}T$ (or its inverse) and so must be interpreted as instantaneous results, unless the relationship between the two frames is static. Similarly, from (5.80) and (5.81), we write the 6×6 matrix that transforms general force vectors written in terms of $\{B\}$ into their description in frame $\{A\}$, namely,

$$\begin{bmatrix} {}^{A}F_{A} \\ {}^{A}N_{A} \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ {}^{A}P_{BORG}\times {}^{A}_{B}R & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}F_{B} \\ {}^{B}N_{B} \end{bmatrix}, \tag{5.105}$$

which may be written compactly as

$${}^{A}\mathcal{F}_{A} = {}^{A}_{B}T_{f}\, {}^{B}\mathcal{F}_{B}, \tag{5.106}$$

where T_f is used to denote a **force-moment transformation**.

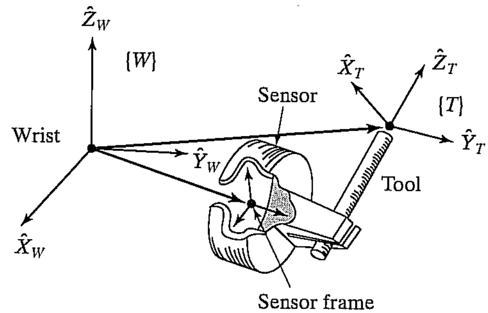

FIGURE 5.13: Frames of interest with a force sensor.

Velocity and force transformations are similar to Jacobians in that they relate velocities and forces in different coordinate systems. As with Jacobians, we have

$$ {}^A_BT_f = {}^A_BT_v^T, \tag{5.107}$$

as can be verified by examining (5.105) and (5.103).

EXAMPLE 5.8

Figure 5.13 shows an end-effector holding a tool. Located at the point where the end-effector attaches to the manipulator is a force-sensing wrist. This is a device that can measure the forces and torques applied to it.

Consider the output of this sensor to be a 6 × 1 vector, ${}^S\mathcal{F}$, composed of three forces and three torques expressed in the sensor frame, $\{S\}$. Our real interest is in knowing the forces and torques applied at the tip of the tool, ${}^T\mathcal{F}$. Find the 6 × 6 transformation that transforms the force-moment vector from $\{S\}$ to the tool frame, $\{T\}$. The transform relating $\{T\}$ to $\{S\}$, ${}^S_TT$, is known. (Note that $\{S\}$ here is the sensor frame, not the station frame.)

This is simply an application of (5.106). First, from ${}^S_TT$, we calculate the inverse, ${}^T_ST$, which is composed of ${}^T_SR$ and ${}^TP_{SORG}$. Then we apply (5.106) to obtain

$$ {}^T\mathcal{F}_T = {}^T_ST_f \, {}^S\mathcal{F}_S, \tag{5.108}$$

where

$$ {}^T_ST_f = \begin{bmatrix} {}^T_SR & 0 \\ {}^TP_{SORG} \times {}^T_SR & {}^T_SR \end{bmatrix}. \tag{5.109}$$

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EXERCISES

5.1 [10] Repeat Example 5.3, but using the Jacobian written in frame {0}. Are the results the same as those of Example 5.3?

5.2 [25] Find the Jacobian of the manipulator with three degrees of freedom from Exercise 3 of Chapter 3. Write it in terms of a frame {4} located at the tip of the hand and having the same orientation as frame {3}.

5.3 [35] Find the Jacobian of the manipulator with three degrees of freedom from Exercise 3 of Chapter 3. Write it in terms of a frame {4} located at the tip of the hand and having the same orientation as frame {3}. Derive the Jacobian in three different ways: velocity propagation from base to tip, static force propagation from tip to base, and by direct differentiation of the kinematic equations.

5.4 [8] Prove that singularities in the force domain exist at the same configurations as singularities in the position domain.

5.5 [39] Calculate the Jacobian of the PUMA 560 in frame {6}.

5.6 [47] Is it true that any mechanism with three revolute joints and nonzero link lengths must have a locus of singular points interior to its workspace?

5.7 [7] Sketch a figure of a mechanism with three degrees of freedom whose linear velocity Jacobian is the 3×3 identity matrix over all configurations of the manipulator. Describe the kinematics in a sentence or two.

5.8 [18] General mechanisms sometimes have certain configurations, called "isotropic points," where the columns of the Jacobian become orthogonal and of equal magnitude [7]. For the two-link manipulator of Example 5.3, find out if any isotropic points exist. Hint: Is there a requirement on l_1 and l_2?

5.9 [50] Find the conditions necessary for isotropic points to exist in a general manipulator with six degrees of freedom. (See Exercise 5.8.)

5.10 [7] For the two-link manipulator of Example 5.2, give the transformation that would map joint torques into a 2×1 force vector, ${}^3F$, at the hand.

5.11 [14] Given

$$
{}^A_BT = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 0.0 \\ 0.000 & 0.000 & 1.000 & 5.0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

if the velocity vector at the origin of $\{A\}$ is

$$ {}^{A}\nu = \begin{bmatrix} 0.0 \\ 2.0 \\ -3.0 \\ 1.414 \\ 1.414 \\ 0.0 \end{bmatrix}, $$

find the 6×1 velocity vector with reference point the origin of $\{B\}$.

5.12 [15] For the three-link manipulator of Exercise 3.3, give a set of joint angles for which the manipulator is at a workspace-boundary singularity and another set of angles for which the manipulator is at a workspace-interior singularity.

5.13 [9] A certain two-link manipulator has the following Jacobian:

$$ {}^{0}J(\Theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}. $$

Ignoring gravity, what are the joint torques required in order that the manipulator will apply a static force vector ${}^{0}F = 10\hat{X}_0$?

5.14 [18] If the link parameter a_3 of the PUMA 560 were zero, a workspace-boundary singularity would occur when $\theta_3 = -90.0°$. Give an expression for the value of θ_3 where the singularity occurs, and show that, if a_3 were zero, the result would be $\theta_3 = -90.0°$. *Hint*: In this configuration, a straight line passes through joint axes 2 and 3 and the point where axes 4, 5, and 6 intersect.

5.15 [24] Give the 3×3 Jacobian that calculates linear velocity of the tool tip from the three joint rates for the manipulator of Example 3.4 in Chapter 3. Give the Jacobian in frame $\{0\}$.

5.16 [20] A $3R$ manipulator has kinematics that correspond exactly to the set of Z–Y–Z Euler angles (i.e., the forward kinematics are given by (2.72) with $\alpha = \theta_1$, $\beta = \theta_2$, and $\gamma = \theta_3$). Give the Jacobian relating joint velocities to the angular velocity of the final link.

5.17 [31] Imagine that, for a general 6-DOF robot, we have available ${}^{0}\hat{Z}_i$ and ${}^{0}P_{i\,org}$ for all i—that is, we know the values for the unit Z vectors of each link frame in terms of the base frame and we know the locations of the origins of all link frames in terms of the base frame. Let us also say that we are interested in the velocity of the tool point (fixed relative to link n) and that we know ${}^{0}P_{tool}$ also. Now, for a revolute joint, the velocity of the tool tip due to the velocity of joint i is given by

$$ {}^{0}v_i = \dot{\theta}_i \, {}^{0}\hat{Z}_i \times ({}^{0}P_{tool} - {}^{0}P_{i\,org}) \tag{5.110} $$

and the angular velocity of link n due to the velocity of this joint is given by

$$ {}^{0}\omega_i = \dot{\theta}_i \, {}^{0}\hat{Z}_i. \tag{5.111} $$

The total linear and angular velocity of the tool is given by the sum of the ${}^{0}v_i$ and ${}^{0}\omega_i$ respectively. Give equations analogous to (5.110) and (5.111) for the case of joint i prismatic, and write the 6×6 Jacobian matrix of an arbitrary 6-DOF manipulator in terms of the $\hat{Z}_i$, $P_{i\,org}$, and P_{tool}.

5.18 [18] The kinematics of a $3R$ robot are given by

$$ {}^{0}_{3}T = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & l_1 c_1 + l_2 c_1 c_2 \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & l_1 s_1 + l_2 s_1 c_2 \\ s_{23} & c_{23} & 0 & l_2 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

Find ${}^{0}J(\Theta)$, which, when multiplied by the joint velocity vector, gives the linear velocity of the origin of frame {3} relative to frame {0}.

5.19 [15] The position of the origin of link 2 for an *RP* manipulator is given by

$$ {}^{0}P_{2ORG} = \begin{bmatrix} a_1c_1 - d_2s_1 \\ a_1s_1 + d_2c_1 \\ 0 \end{bmatrix}. $$

Give the 2×2 Jacobian that relates the two joint rates to the linear velocity of the origin of frame {2}. Give a value of Θ where the device is at a singularity.

5.20 [20] Explain what might be meant by the statement: "An n-DOF manipulator at a singularity can be treated as a redundant manipulator in a space of dimensionality $n-1$."

PROGRAMMING EXERCISE (PART 5)

1. Two frames, {A} and {B}, are not moving relative to one another—that is, ${}^{A}_{B}T$ is constant. In the planar case, we define the velocity of frame {A} as

$$ {}^{A}\nu_A = \begin{bmatrix} {}^{A}\dot{x}_A \\ {}^{A}\dot{y}_A \\ {}^{A}\dot{\theta}_A \end{bmatrix}. $$

Write a routine that, given ${}^{A}_{B}T$ and ${}^{A}\nu_A$, computes ${}^{B}\nu_B$. *Hint:* This is the planar analog of (5.100). Use a procedure heading something like (or equivalent C):

```
Procedure Veltrans (VAR brela: frame; VAR vrela, vrelb: vec3);
```

where "vrela" is the velocity relative to frame {A}, or ${}^{A}\nu_A$, and "vrelb" is the output of the routine (the velocity relative to frame {B}), or ${}^{B}\nu_B$.

2. Determine the 3×3 Jacobian of the three-link planar manipulator (from Example 3.3). In order to derive the Jacobian, you should use velocity-propagation analysis (as in Example 5.2) or static-force analysis (as in Example 5.6). Hand in your work showing how you derived the Jacobian.

Write a routine to compute the Jacobian in frame {3}—that is, ${}^{3}J(\Theta)$—as a function of the joint angles. Note that frame {3} is the standard link frame with origin on the axis of joint 3. Use a procedure heading something like (or equivalent C):

```
Procedure Jacobian (VAR theta: vec3; Var Jac: mat33);
```

The manipulator data are $l_2 = l_2 = 0.5$ meters.

3. A tool frame and a station frame are defined as follows by the user for a certain task (units are meters and degrees):

$$ {}^{W}_{T}T = [x\ y\ \theta] = [0.1\ 0.2\ 30.0], $$

$$ {}^{B}_{S}T = [x\ y\ \theta] = [0.0\ 0.0\ 0.0]. $$

At a certain instant, the tool tip is at the position

$$ {}^{S}_{T}T = [x\ y\ \theta] = [0.6\ \ -0.3\ 45.0]. $$

At the same instant, the joint rates (in deg/sec) are measured to be

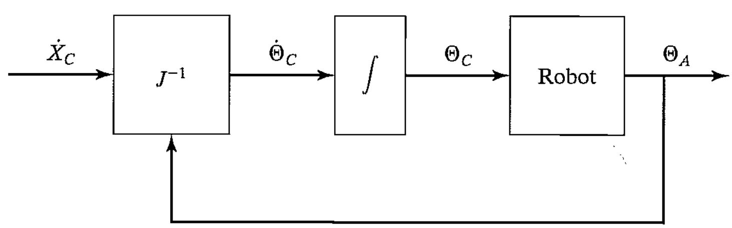
$$\dot{\Theta} = [\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3] = [20.0\ -10.0\ 12.0].$$

Calculate the linear and angular velocity of the tool tip relative to its own frame, that is, ${}^{T}\nu_{T}$. If there is more than one possible answer, calculate all possible answers.

MATLAB EXERCISE 5

This exercise focuses on the Jacobian matrix and determinant, simulated resolved-rate control, and inverse statics for the planar 3-DOF, 3R robot. (See Figures 3.6 and 3.7; the DH parameters are given in Figure 3.8.)

The resolved-rate control method [9] is based on the manipulator velocity equation ${}^{k}\dot{X} = {}^{k}J\dot{\Theta}$, where ${}^{k}J$ is the Jacobian matrix, $\dot{\Theta}$ is the vector of relative joint rates, ${}^{k}\dot{X}$ is the vector of commanded Cartesian velocities (both translational and rotational), and k is the frame of expression for the Jacobian matrix and Cartesian velocities. This figure shows a block diagram for simulating the resolved-rate control algorithm:



Resolved-Rate-Algorithm Block Diagram

As is seen in the figure, the resolved-rate algorithm calculates the required commanded joint rates $\dot{\Theta}_C$ to provide the commanded Cartesian velocities $\dot{X}_C$; this diagram must be calculated at every simulated time step. The Jacobian matrix changes with configuration Θ_A. For simulation purposes, assume that the commanded joint angles Θ_C are always identical to the actual joint angles achieved, Θ_A (a result rarely true in the real world). For the planar 3-DOF, 3R robot assigned, the velocity equations ${}^{k}\dot{X} = {}^{k}J\dot{\Theta}$ for $k = 0$ are

$${}^{0}\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \omega_z \end{Bmatrix} = {}^{0}\begin{bmatrix} -L_1s_1 - L_2s_{12} - L_3s_{123} & -L_2s_{12} - L_3s_{123} & -L_3s_{123} \\ L_1c_1 + L_2c_{12} + L_3c_{123} & L_2c_{12} + L_3c_{123} & L_3c_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix},$$

where $s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$, $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$, and so on. Note that ${}^{0}\dot{X}$ gives the Cartesian velocities of the origin of the hand frame (at the center of the grippers in Figure 3.6) with respect to the origin of the base frame {0}, expressed in {0} coordinates.

Now, most industrial robots cannot command $\dot{\Theta}_C$ directly, so we must first integrate these commanded relative joint rates to commanded joint angles Θ_C, which can be commanded to the robot at every time step. In practice, the simplest possible integration scheme works well, assuming a small control time step Δt: $\Theta_{new} = \Theta_{old} + \dot{\Theta}\Delta t$. In your MATLAB resolved-rate simulation, assume that the commanded Θ_{new} can be achieved perfectly by the virtual robot. (Chapters 6 and 9 present dynamics and control material for which we do not have to make this simplifying assumption.) Be sure to update the

Jacobian matrix with the new configuration Θ_{new} before completing the resolved-rate calculations for the next time step.

Develop a MATLAB program to calculate the Jacobian matrix and to simulate resolved-rate control for the planar 3R robot. Given the robot lengths $L_1 = 4$, $L_2 = 3$, and $L_3 = 2$ (m); the initial joint angles $\Theta = \{\theta_1\ \theta_2\ \theta_3\}^T = \{10^\circ\ 20^\circ\ 30^\circ\}^T$, and the constant commanded Cartesian rates ${}^0\{\dot{X}\} = \{\dot{x}\ \dot{y}\ \omega_z\}^T = \{0.2\ -0.3\ -0.2\}^T$ (m/s, m/s, rad/s), simulate for exactly 5 sec, using time steps of exactly $dt = 0.1$ sec. In the same program loop, calculate the inverse-statics problem—that is, calculate the joint torques $\mathrm{T} = \{\tau_1\ \tau_2\ \tau_3\}^T$ (Nm), given the constant commanded Cartesian wrench ${}^0\{W\} = \{f_x\ f_y\ m_z\}^T = \{1\ 2\ 3\}^T$ (N, N, Nm). Also, in the same loop, animate the robot to the screen during each time step, so that you can watch the simulated motion to verify that it is correct.

a) For the specific numbers assigned, present five plots (each set on a separate graph, please):

1. the three active joint rates $\dot{\Theta} = \{\dot{\theta}_1\ \dot{\theta}_2\ \dot{\theta}_3\}^T$ vs. time;
2. the three active joint angles $\Theta = \{\theta_1\ \theta_2\ \theta_3\}^T$ vs. time;
3. the three Cartesian components of ${}^0_H T$, $X = \{x\ y\ \phi\}^T$ (rad is fine for ϕ so that it will fit) vs. time;
4. the Jacobian matrix determinant $|J|$ vs. time—comment on nearness to singularities during the simulated resolved-rate motion;
5. the three active joint torques $\mathrm{T} = \{\tau_1\ \tau_2\ \tau_3\}^T$ vs. time.

Carefully label (by hand is fine!) each component on each plot; also, label the axes with names and units.

b) Check your Jacobian matrix results for the initial and final joint-angle sets by means of the Corke MATLAB Robotics Toolbox. Try function *jacob0()*. **Caution**: The toolbox Jacobian functions are for motion of {3} with respect to {0}, not for $\{H\}$ with respect to {0} as in the problem assignment. The preceding function gives the Jacobian result in {0} coordinates; *jacobn()* would give results in {3} coordinates.