Matrix Structural Analysis of Frame

Chapter 1 Introduction

Structural type to be studied : Frames using line elements (beam, column, truss etc)

Other types of elements, plates, shells, and solids, will be studied in the Finite Element Analysis.

Linear analysis versus Nonlinear analysis

Structural Analysis can be broken down into five parts.

1) Basic Mechanics :
   Fundamental relationship of stress and strain (constitutive relationship)
   Equilibrium
   Compatibility
2) Finite element mechanics ; exact or approximate solution of the differential equations of the element
3) Equation formulation : establishment of the governing algebraic equations of the system (Force versus displacement)
4) Solution : computational method and algorithms
5) Solution Interpretation : presentation of results in a form useful in design

1.1 a brief history of structural analysis

1850 – 1875 : Maxwell, Castigliano, Mohr
1875 – 1920 : Timoshenko (strength of material)
1920's ; displacement method (unknowns : displacement)
1932 : moment distribution method (Hardy cross)
Early 1950's : development of computer
1960's : development of matrix analysis
    Argyris and Kelsey, Turner, Clough, etc
1960 : geometric stiffness matrix for uniform axial force member

Turner, Dill, martin and melosh

1945 – 1965 ; material nonlinearity

1.2 Computer Programs

Computerization can relieve the burdens of the rote operations in design and analysis.

But it cannot relieve the engineer of responsibility for the design results.

1.2.1 computational flow and general purpose programs

1) Input ; geometry, material, loading, boundary conditions.
2) Library of elements : generation of mathematical models
3) Solution : construction and solution of mathematical models
4) Output : display of predicted displacements and forces.

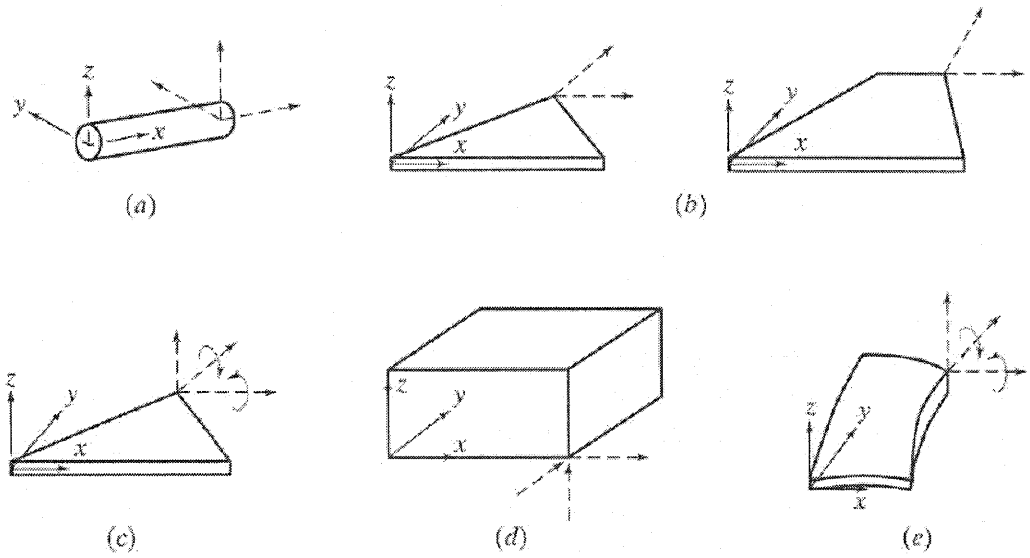
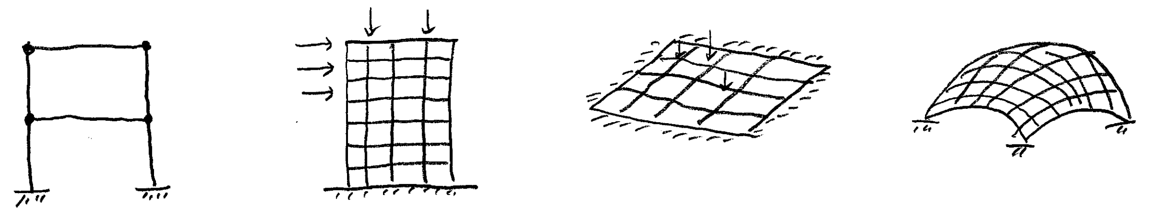
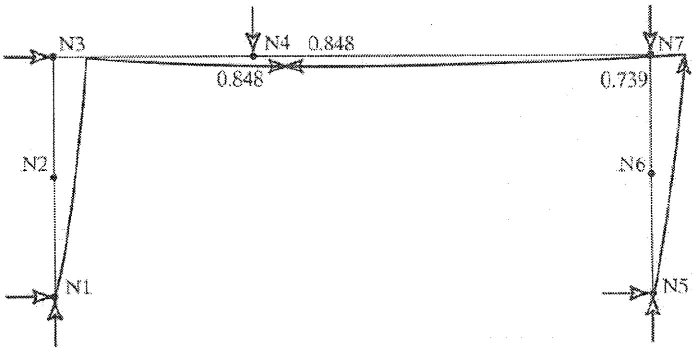


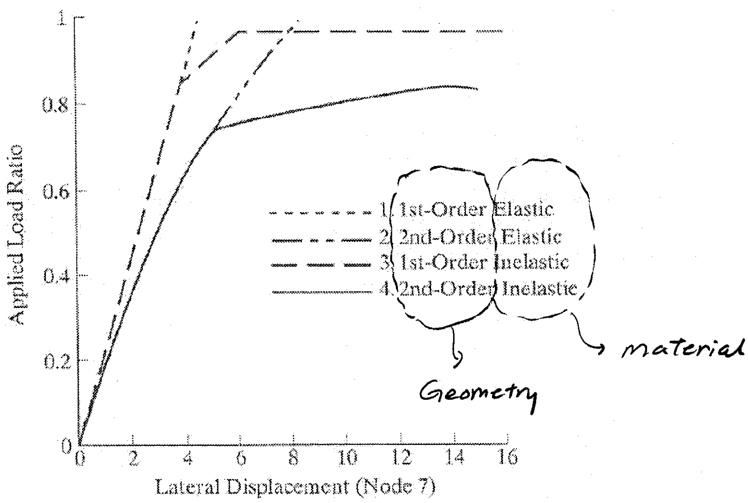
Figure 1.2 Sample finite elements. (*a*) Framework member. (*b*) Plane stress. (*c*) Flat plate bending. (*d*) Solid element. (*e*) Curved thin shell.



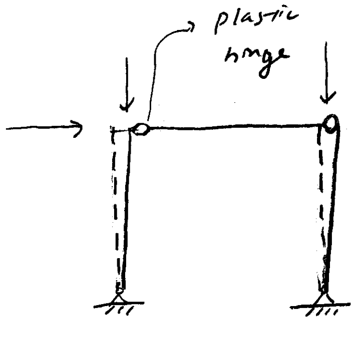
1.2.2 computer program



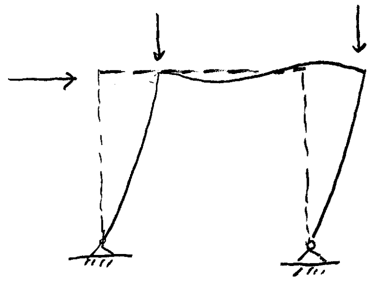
(a) Deformed shape and plastic hinge locations



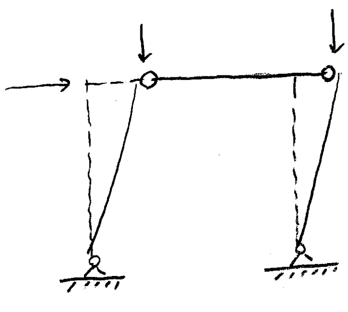
(b) Response curves



1st order inelastic analysis



2nd order elastic analysis



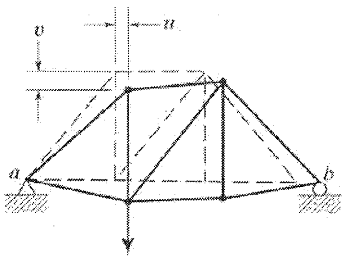
2nd order inelastic analysis

Chapter 2 Definitions and Concepts

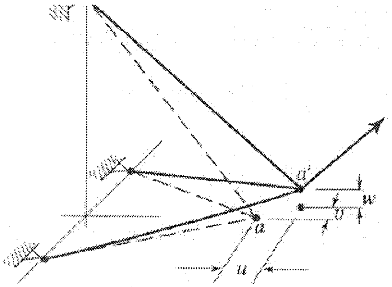
Displacement method – direct stiffness method $\underline{P} = \underline{K}\,\underline{U}$

Flexibility method – force method. $\underline{u}_f = \underline{f} \cdot \underline{P}_f$

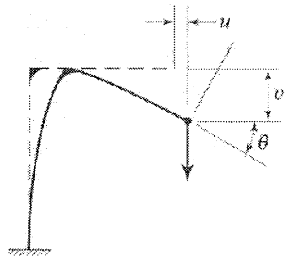
2.1 degrees of freedom

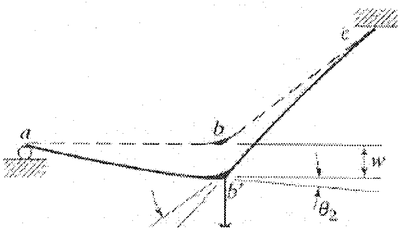


(a)



(b)





1) pin-jointed plane truss : u, v – translational displacements
2) pin-jointed space truss ; u, v, w
3) plane frame : u, v, θ (rotational displacement)
4) grid structure : w, θ_1, θ_2

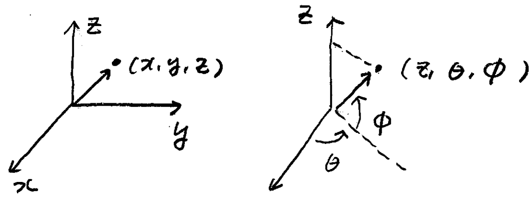
degree of freedom : displacement components needed to define its position in space at any time under any loading

Indeed, infinite number of dof exists in any structure. But these can be reduced to the finite number of dof for member ends which represents the behavior of the members.

2.2 coordinate systems and conditions of analysis

Cartesian coordinate system

Rotational coordinate system



Assumption for linear elastic analysis

Displacements and sufficiently small even after deformation

Therefore, geometry, material properties, and load conditions are unchanged.

All equations can be formulated with respect to the geometry of the original undeformed system.

Principle of superposition can be applied. convenient when a number of load combinations should be considered.

Nodes (joints)

Member (Element)

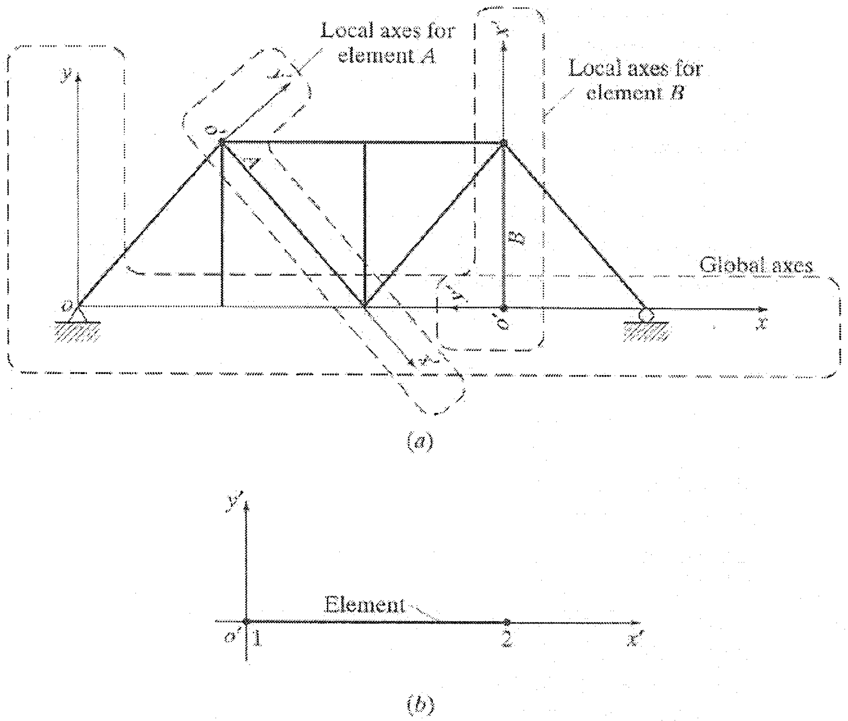


Figure 2.4 Coordinate axes and joint numbering. (*a*) Types of coordinate axes. (*b*) Joint numbering scheme in local coordinates.

Force vectors $\underline{F} = \begin{bmatrix} F_1 \\ F_2 \\ \\ \\ F_{16} \end{bmatrix} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ \\ \\ F_{y8} \end{bmatrix}$

Displacement vectors $\underline{U} = \begin{bmatrix} u_1 \\ u_2 \\ \\ \\ u_{16} \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ \\ \\ v_8 \end{bmatrix}$

Release : free from constraint

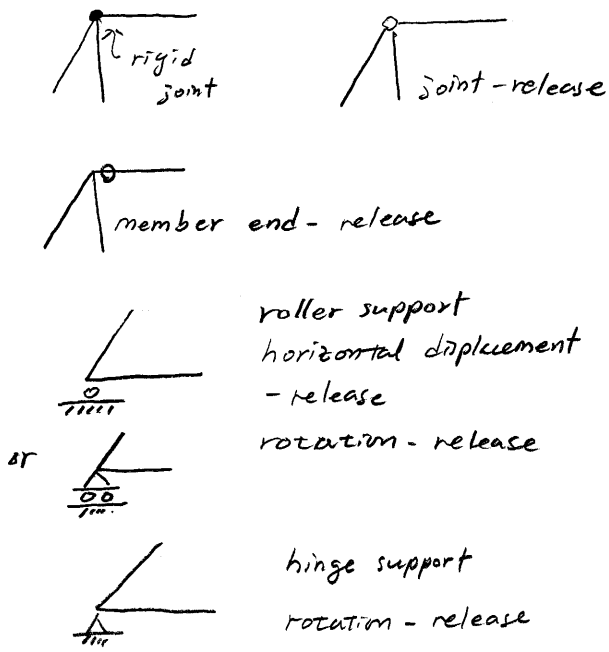
joint – release

member end-release

support - release

Global coordinates

Local coordinates



2.3 structure idealization

Structure configuration

Boundary condition

Loading condition

Degree of freedom

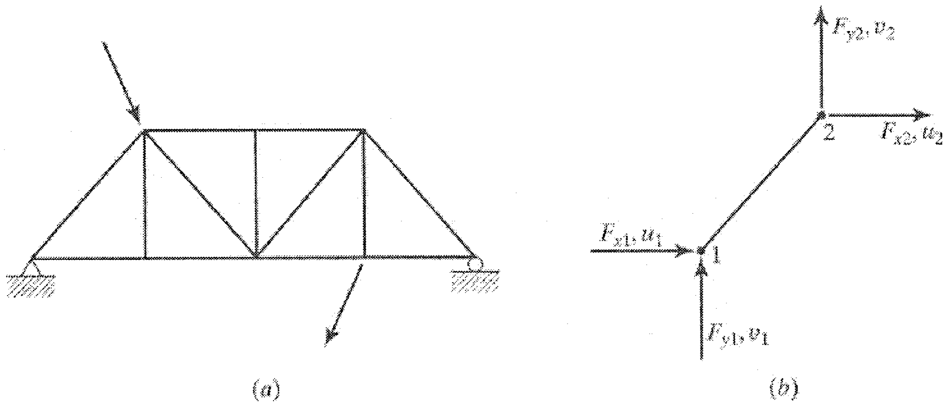


Figure 2.5 Idealized truss. (*a*) Truss. (*b*) Typical truss member.

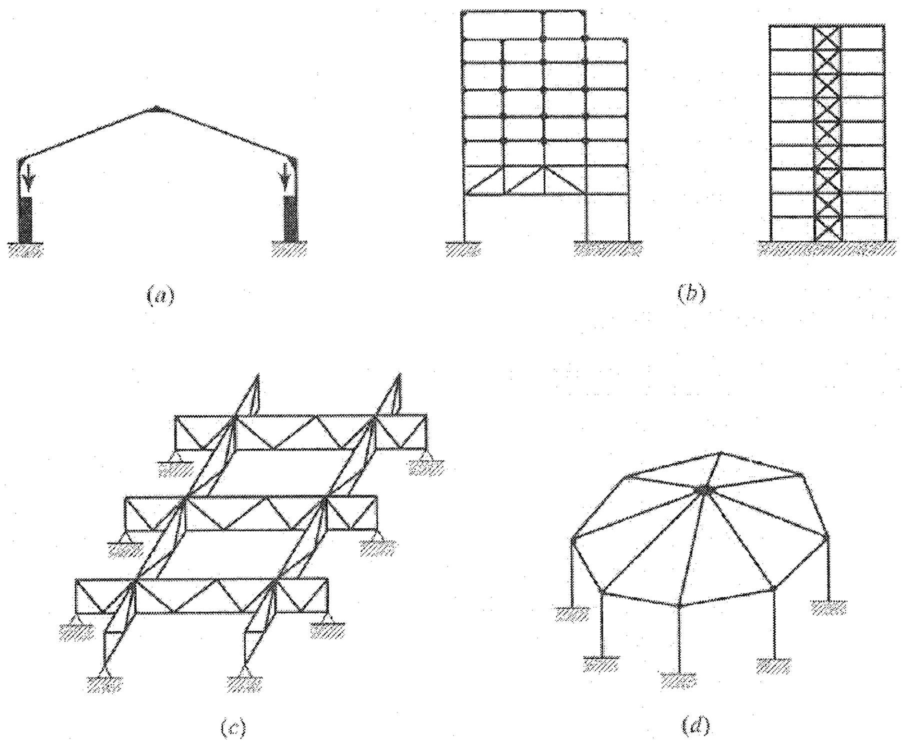
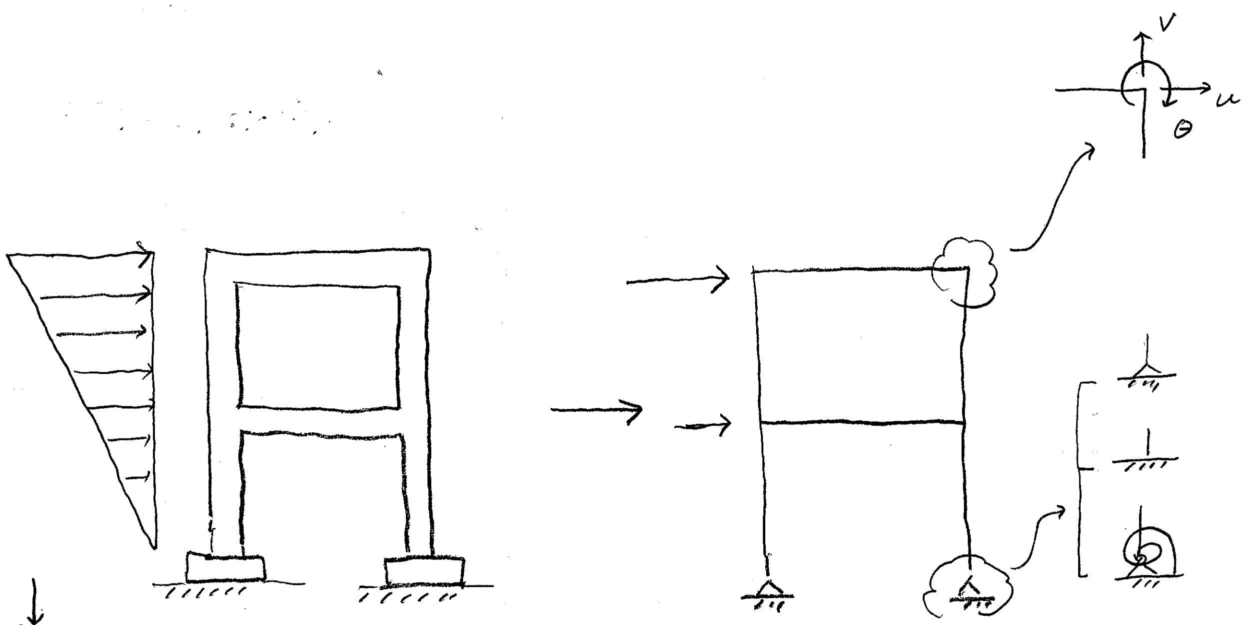
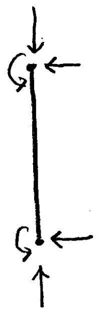


Figure 2.6 Typical framed structures. (*a*) Rigidly jointed plane frame. (*b*) Multistory frames—rigidly jointed and trussed. (*c*) Trussed space frame. (*d*) Rigidly jointed space frame.

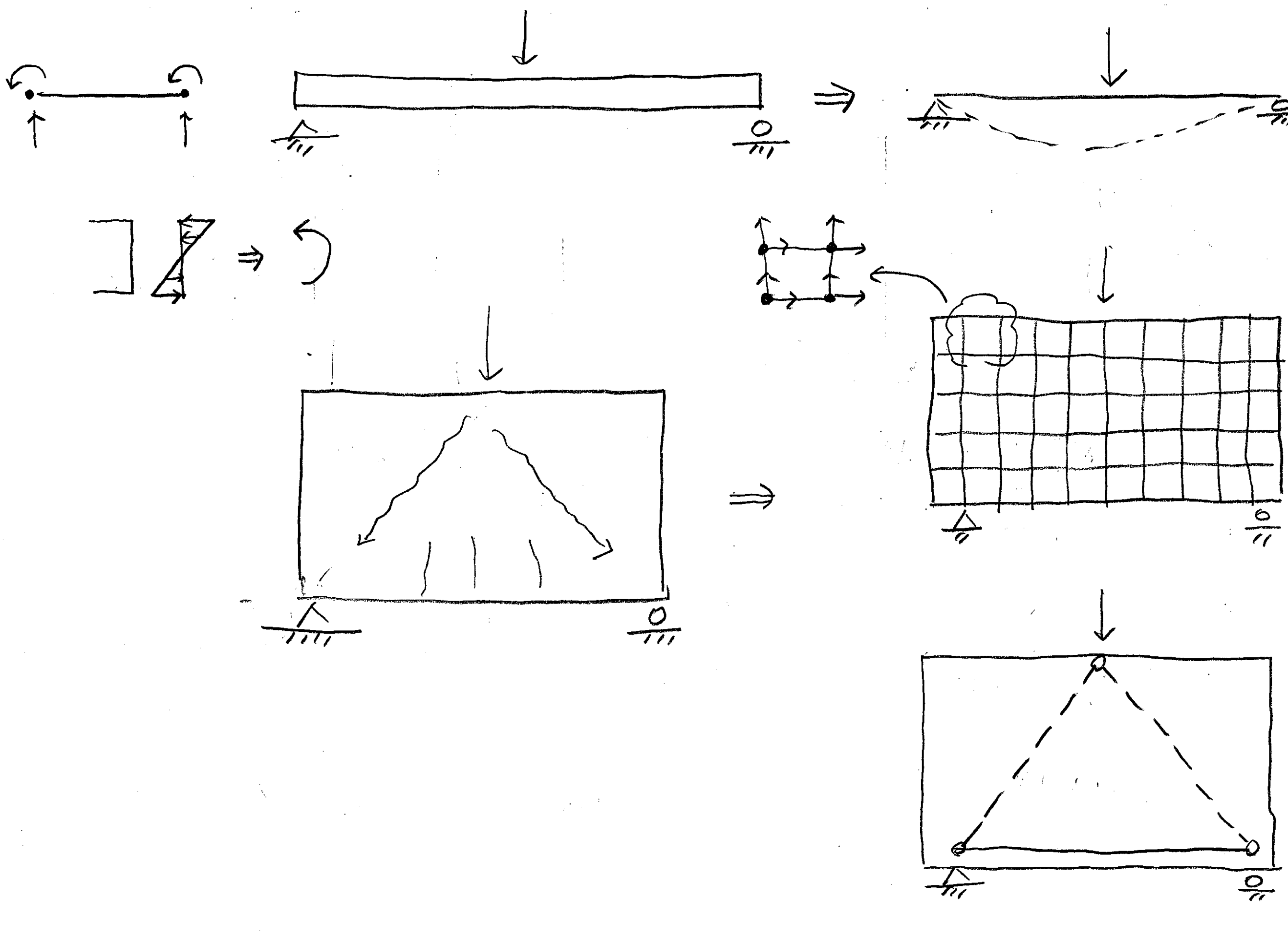




3D Element → 1-D line element or 2-D plate

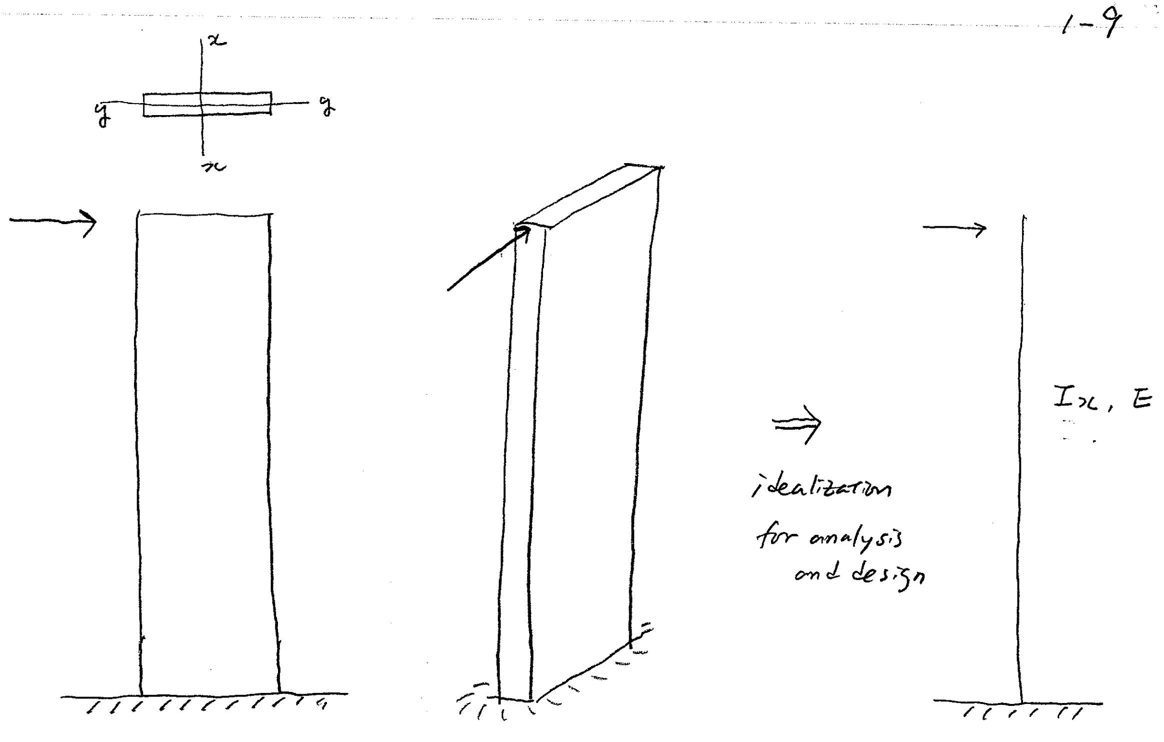
foundation → Hinge (idealized B/c)

그림으로 나타내기 쉽도록 역학적 거동을 고려하여 단순하게 표현



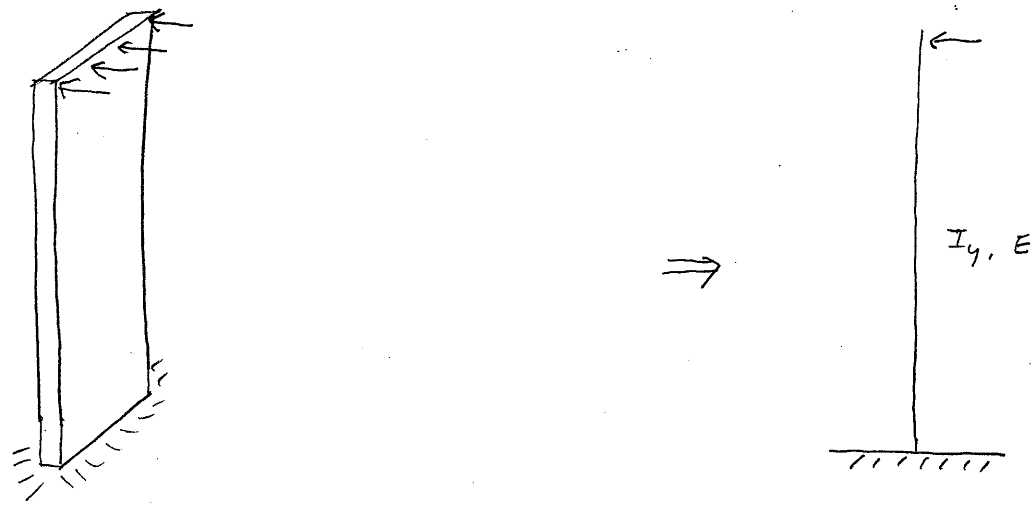
Finite element analysis

Strut-and-tie model



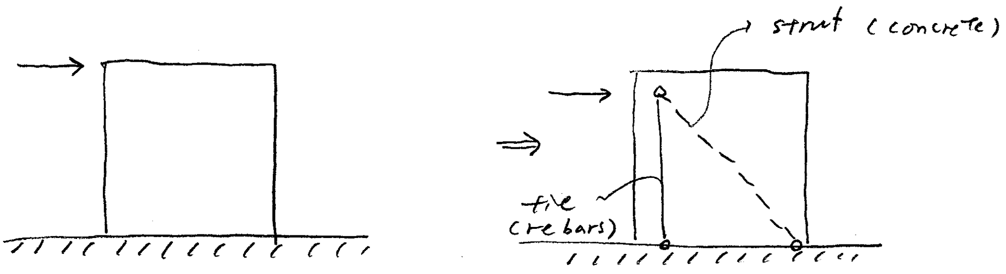
slender wall subjected to inplane loading

cantilever beam



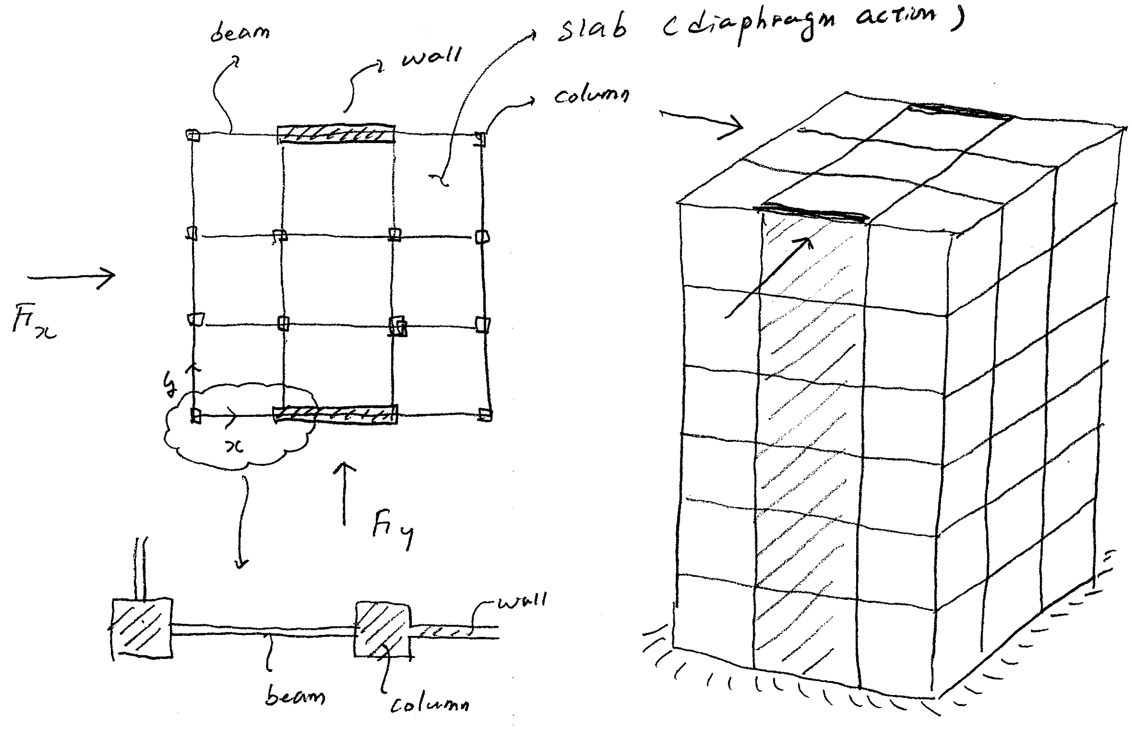
slender wall subjected to out of plane loading

cantilever beam bent w.r.t weak axis

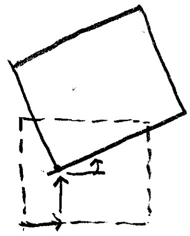


squat wall subjected to inplane loading (reinforced concrete)

Strut-and-tie model (RC squat wall)

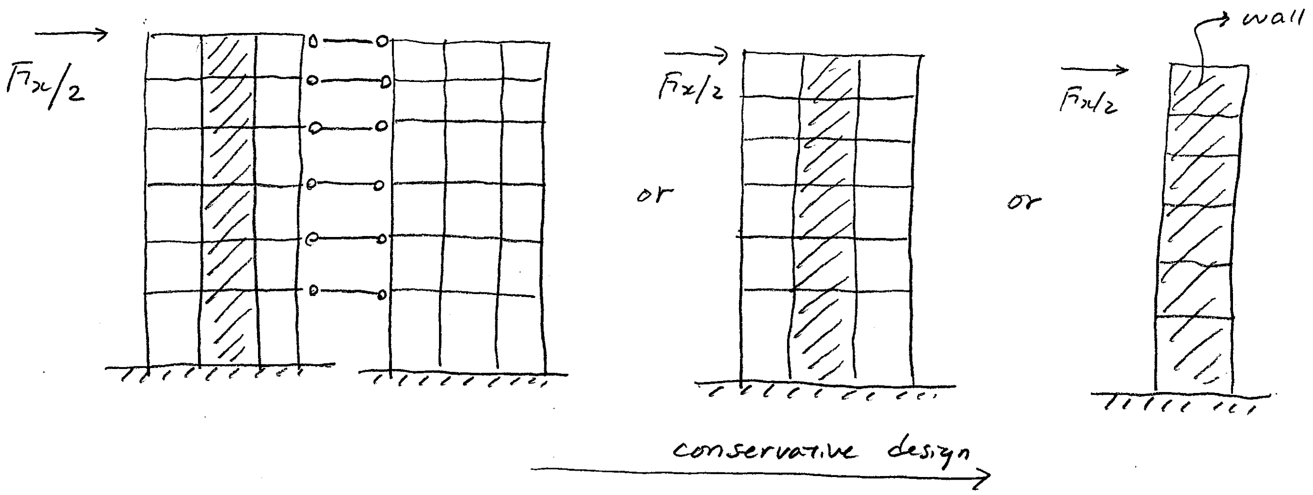


3-dimensional Model

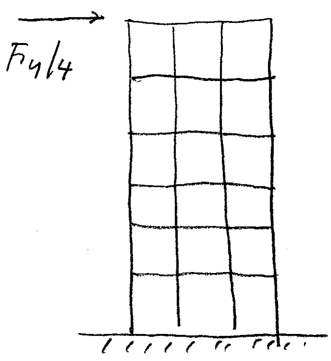


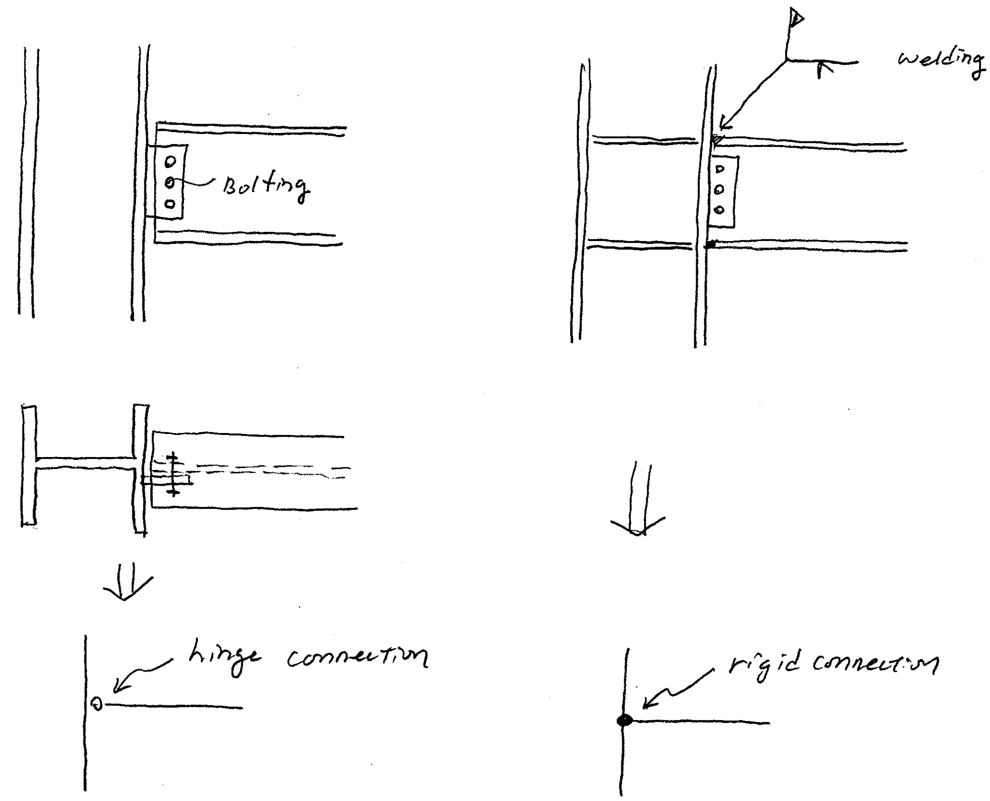
rigid diaphragm action

2-dimensional model for F_x



2-dimensional model for F_y

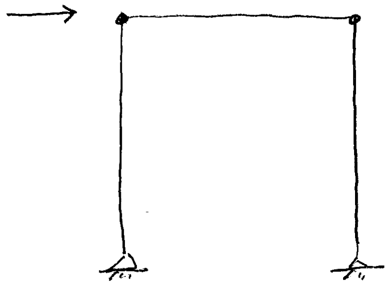


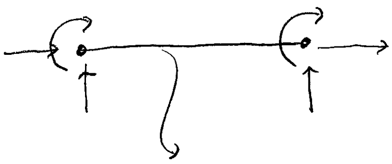

Bolting
welding
hinge connection
rigid connection

frame analysis vs finite element analysis

- computer matrix method

line elements (truss, frame)

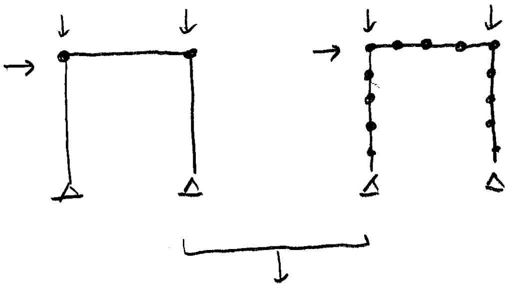




displacement function
v

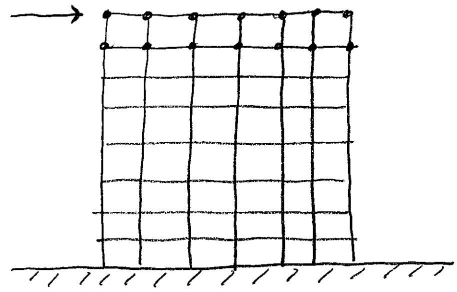
⇒ exact

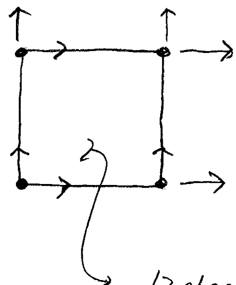
⇒ satisfy equilibrium function



produce the same result

2-D, 3-D elements

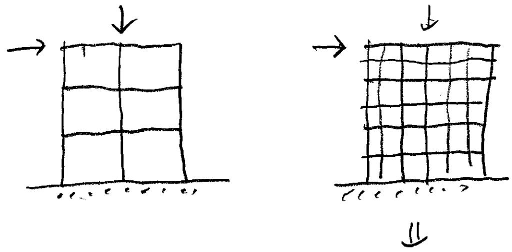




displacement function
u, v

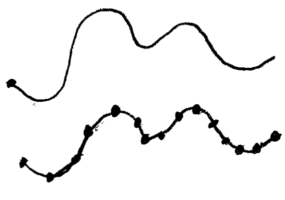
⇒ not exact

⇒ not satisfy equilibrium function, Generally



⇓

more accurate solution



⇒ complex displ.

⇒ simplified displ.

2.4 Axial Force Element (truss element)

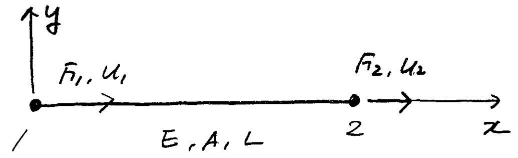
2.4.1 element stiffness equation

$\underline{F} = \underline{k}\ \underline{U}$

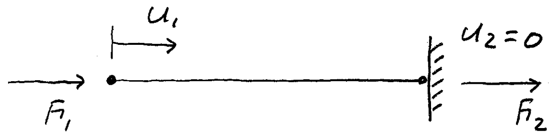
$\underline{F} =$ element force vector

$\underline{k} =$ element stiffness matrix

$\underline{U} =$ element displacement vector



$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



$$\sigma = \frac{F_1}{A}, \quad \varepsilon = \frac{F_1}{EA}, \quad u_1 = \varepsilon L = \frac{F_1 L}{EA}$$

$$F_1 = \frac{EA}{L} u_1 = k_{11} u_1, \quad F_2 = -F_1 = -\frac{EA}{L} u_1 = k_{21} u_1$$

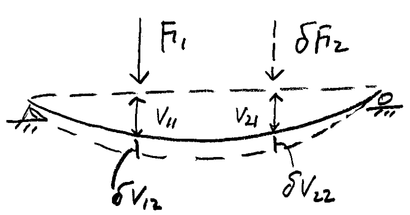
In the same way,

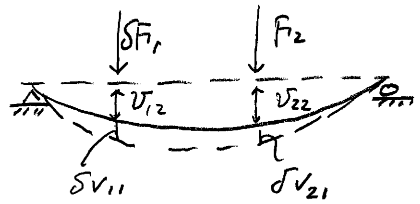
$$F_2 = \frac{EA}{L} u_2 = k_{22} u_2, \quad F_1 = -\frac{EA}{L} u_2 = k_{12} u_2$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Characteristics of stiffness matrix

1) stiffness matrix is symmetric due to reciprocity of displacements





Incremental energy due to δF_2 | Inc. energy due to δF_1

$F_1\delta v_{12} + \delta F_2 \delta v_{22} \frac{1}{2}$ | $\frac{1}{2}\delta F_1 \delta v_{11} + F_2 \delta v_{21}$

$= \int \delta\sigma_2 \varepsilon_1 dV$ | $= \int \delta\sigma_1 \varepsilon_2 dV$

$= \frac{1}{E}\int \delta\varepsilon_2 \varepsilon_1 dV$ (if linear elastic system) | $= \frac{1}{E}\int \delta\varepsilon_1 \varepsilon_2 dV$

Since $\frac{1}{E}\int \delta\varepsilon_2\varepsilon_1 dV = \frac{1}{E}\int \delta\varepsilon_1\varepsilon_2 dV$ and neglecting very small terms,

$F_1\delta v_{12} = F_2\delta v_{21}$. ($F_1 f_{12} \delta F_2$) ($F_2 f_{21} \delta F_1$)

From $F_1\delta v_{12} = \delta(F_1 f_{12} F_2)$ and $F_2 \delta v_{21} = \delta(F_2 f_{21} F_1)$, $f_{12} = f_{21}$.

Therefore, $k_{12} = k_{21}$.

$\because \underset{\sim}{f} = \underset{\sim}{k}^{-1}$

2) stiffness matrix is singular. Infinite number of solutions exist.

$|\underline{k}| = 0$, $\frac{EA}{L}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$

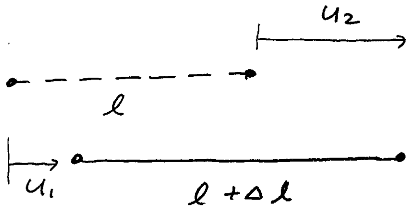
2nd row depends on 1st row

$\frac{EA}{L}\underbrace{(u_1 - u_2)}_{\Delta = \text{deformation}} = F_1$

$F_2 = -F_1$, i.e. $\sum F_i = 0$

$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ depends only on the difference between the displacements, i.e. deformation $(u_1 - u_2)$

3) Displacement = rigid body translation + deformation



Deformation = $u_2 - u_1$

Rigid body translation = u_1

Only deformation develops member forces.

$\Delta\ell = u_2 - u_1$

2.4.2 Element flexibility equations

mathmatically, $\underline{k}^{-1}$ does not exist.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

X

$F_1 \rightarrow \dfrac{(u_1 - u_2)}{\Delta}$

$\underline{U}_f = \underline{d}\ \underline{F}_f$

$\underline{U}_f$ = displacement vector for the dofs that are free to displace

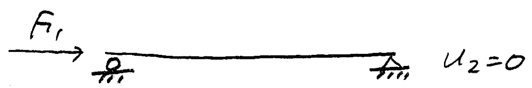
$\underline{F}_f$ = force vector for the dofs that are free to displace

$\underline{d}$ = flexibility matrix ⇒ Basically relationship between member force and deformati[on]

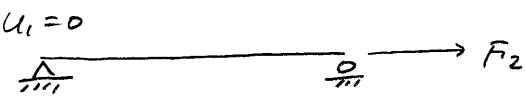
d_{ij} = flexibility coefficient; the value of the displacement u_i caused by a unit value of the force F_j

Flexibility equations are defined only for the dofs that are free to displace. (i.e. deformation)
Flexibility can be written only for the elements supported in a stable manner.

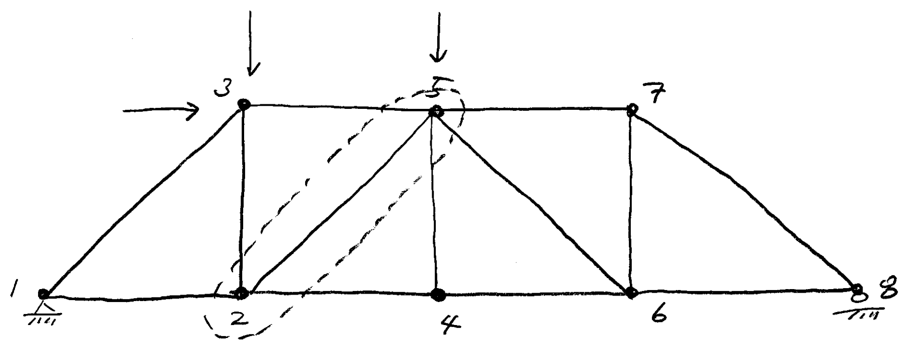
$u_1 = \dfrac{L}{EA} F_1$

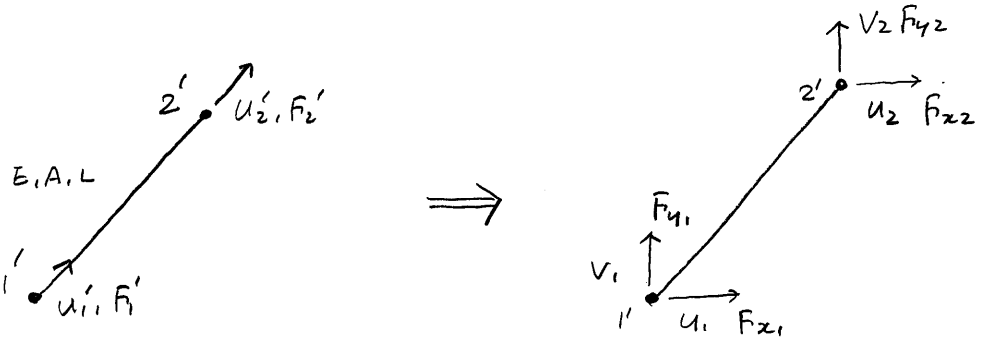


$u_2 = \dfrac{L}{EA} F_2$

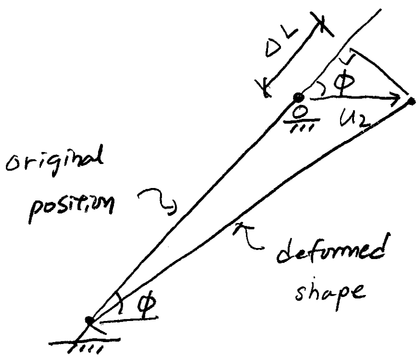


2.5 Axial force element - Global stiffness equations





$$\begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$



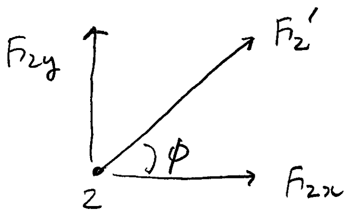
$u_1 = v_1 = v_2 = 0$

elongation $\Delta L = u_2 \cos\phi$

$$F_2' = \frac{EA}{L}\Delta L$$

$$= \frac{EA}{L}\cos\phi\, u_2$$

$$F_1' = -F_2' = -\frac{EA}{L}\cos\phi\, u_2$$



$$F_{2x} = F_2' \cos\phi = \frac{EA}{L}\cos^2\phi \, u_2$$

$$F_{2y} = F_2' \sin\phi = \frac{EA}{L}\cos\phi\sin\phi \, u_2$$

$$F_{1x} = F_1' \cos\phi = -\frac{EA}{L}\cos^2\phi \, u_2$$

$$F_{1y} = F_1' \sin\phi = -\frac{EA}{L}\cos\phi\sin\phi \, u_2$$

$$\Rightarrow \quad k_{31} = -\frac{EA}{L}\cos^2\phi$$

$$k_{32} = -\frac{EA}{L}\cos\phi\sin\phi$$

$$k_{33} = \frac{EA}{L}\cos^2\phi$$

$$k_{34} = \frac{EA}{L}\cos\phi\sin\phi$$

$$\underset{\sim}{k} = \frac{EA}{L}\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi & -\cos^2\phi & -\sin\phi\cos\phi \\ & \sin^2\phi & -\sin\phi\cos\phi & -\sin^2\phi \\ & \text{sym.} & \cos^2\phi & \sin\phi\cos\phi \\ & & & \sin^2\phi \end{bmatrix} \quad \text{Eq. (2.5)}$$

If $\phi = 0$, Eq. (2.5) is reduced to Eq. (2.3)

$$\underset{\sim}{k} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

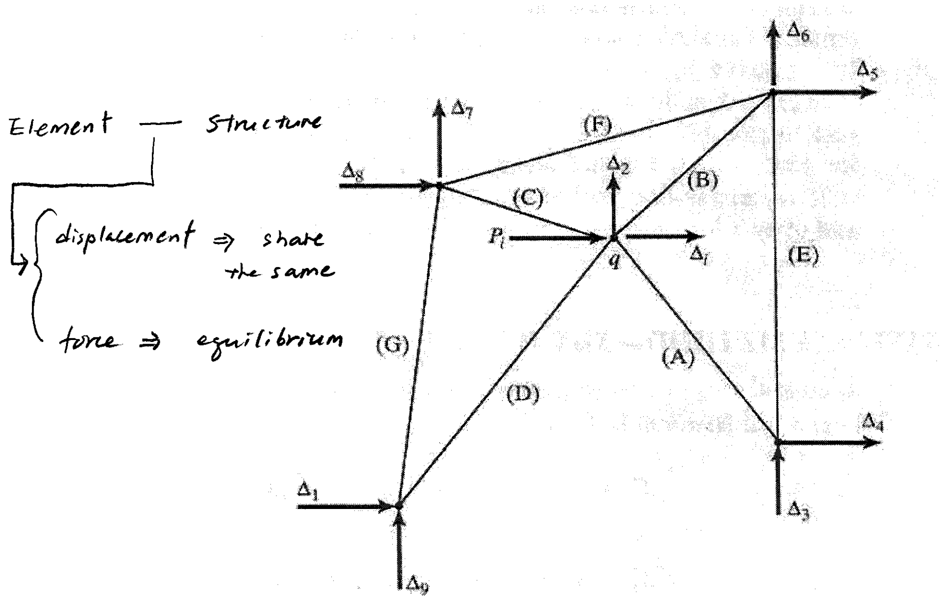
Chapter 3 Formation of the global analysis equations

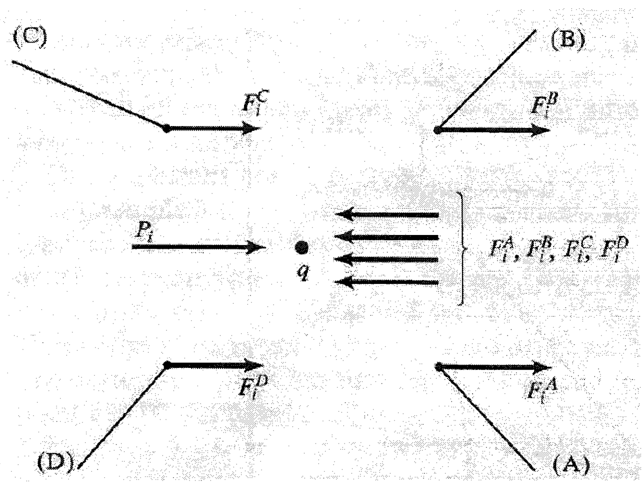
3.1 direct stiffness method

The internal force of each element can be expressed as a function of the nodal displacements of the element : $\underline{f} = \underline{k}\ \underline{u}$

Since the sum of the internal force of the elements are the total internal force, and the nodal displacements are the same at all the elements (displacement-compatibility), by assembling the internal forces of the elements, the total internal force can be expressed by a function of the nodal displacements: $\underline{F} = \underline{K}\underline{U}$.

Using the force equilibrium between the external force and the internal force $\underline{F} = \underline{P}$, the formulation of the stiffness method can be established: $\underline{P} = \underline{K}\underline{U}$

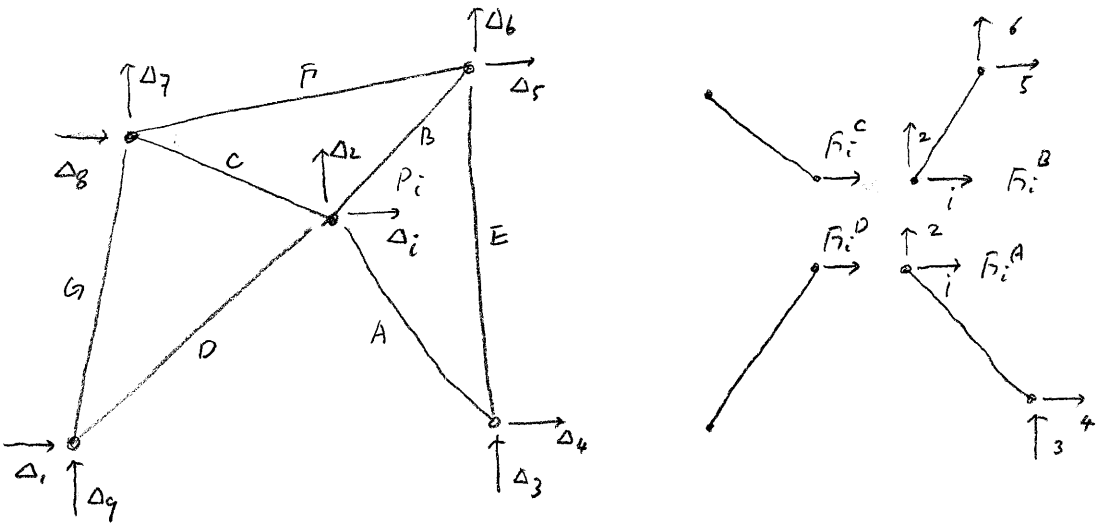




Chapter 3.

Formation of the Global Analysis Equations

3.1 Direct Stiffness Method



By equilibrium. $P_i = F_i^A + F_i^B + F_i^C + F_i^D$

In Element A.

$$\begin{bmatrix} F_i^A \\ F_2^A \\ F_4^A \\ F_3^A \end{bmatrix} = \begin{bmatrix} k_{ii}^A & k_{i2}^A & k_{i4}^A & k_{i3}^A \\ k_{2i}^A & k_{22}^A & k_{24}^A & k_{23}^A \\ k_{4i}^A & k_{42}^A & k_{44}^A & k_{43}^A \\ k_{3i}^A & k_{32}^A & k_{34}^A & k_{33}^A \end{bmatrix} \begin{bmatrix} \Delta_i \\ \Delta_2 \\ \Delta_4 \\ \Delta_3 \end{bmatrix}$$

$$P_i = (k_{ii}^A \Delta_i + k_{i2}^A \Delta_2 + k_{i4}^A \Delta_4 + k_{i3}^A \Delta_3) \rightarrow F_i^A$$

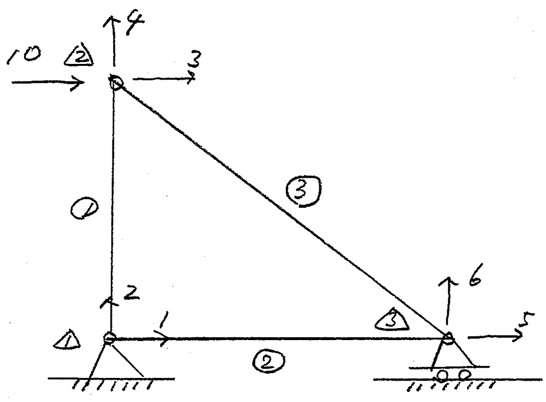
$$+ (k_{ii}^B \Delta_i + k_{i2}^B \Delta_2 + k_{i5}^B \Delta_5 + k_{i6}^B \Delta_6) \rightarrow F_i^B$$

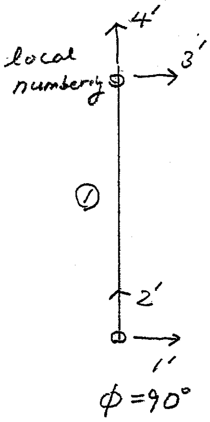
$$+ \cdots$$

$$= (k_{ii}^A + k_{ii}^B + k_{ii}^C + k_{ii}^D)\Delta_i + \cdots$$

$$= K_{ii}\Delta_i + K_{i1}\Delta_1 + \cdots + K_{i9}\Delta_9$$

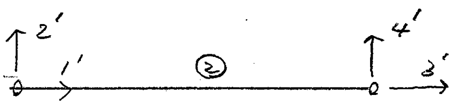
K_{ii} : Global Stiffness Coefficient





$$\begin{matrix} & 1 & 2 & 3 & 4 & \rightarrow \text{Global} \\ & 1' & 2' & 3' & 4' & \rightarrow \text{local} \end{matrix}$$

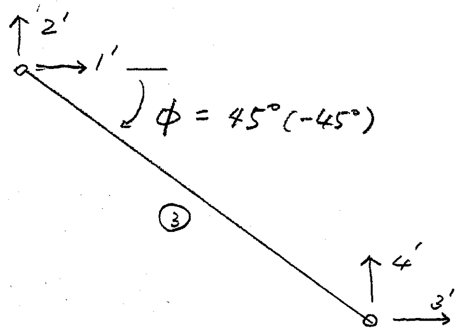
$$\begin{vmatrix} F_1^{①} \\ F_2^{①} \\ F_3^{①} \\ F_4^{①} \end{vmatrix} = \underbrace{\begin{vmatrix} k_{11}^{①} & k_{12}^{①} & k_{13}^{①} & k_{14}^{①} \\ & \ddots & & \\ & & & \\ & & & k_{44}^{①} \end{vmatrix}}_{\tilde{k}^{①}} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{vmatrix}$$



$\phi = 0$

$$\begin{matrix} 1 & 2 & 5 & 6 \\ 1' & 2' & 3' & 4' \end{matrix}$$

$$\tilde{k}^{②} = \begin{vmatrix} k_{11}^{②} & k_{12}^{②} & k_{13}^{②} & k_{14}^{②} \\ & \ddots & & \\ & & & \\ & & & k_{44}^{②} \end{vmatrix}$$



$$\begin{matrix} 3 & 4 & 5 & 6 \\ 1' & 2' & 3' & 4' \end{matrix}$$

$$\tilde{k}^{③} = \begin{vmatrix} k_{11}^{③} & k_{12}^{③} & k_{13}^{③} & k_{14}^{③} \\ & \ddots & & \\ & & & \\ & & & k_{44}^{③} \end{vmatrix}$$

By force - equilibrium

At node △1

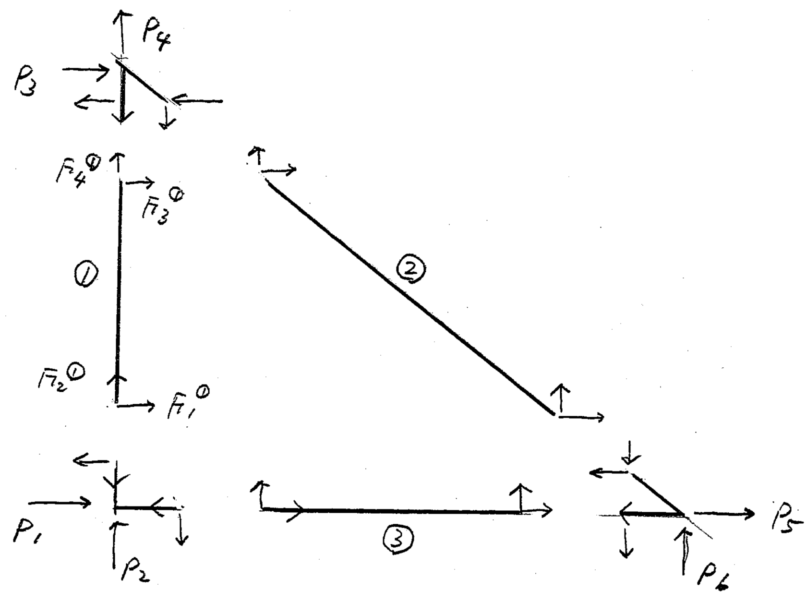
$$\begin{vmatrix} P_1 \\ P_2 \end{vmatrix} = \begin{vmatrix} F_1^{①} + F_1^{②} \\ F_2^{①} + F_2^{②} \end{vmatrix} = \begin{vmatrix} (k_{11}^{①} + k_{11}^{②})u_1 + (k_{12}^{①} + k_{12}^{②})u_2 + k_{13}^{①}u_3 + k_{14}^{①}u_4 + k_{13}^{②}u_5 + k_{14}^{②}u_6 \\ (k_{21}^{①} + k_{21}^{②})u_1 + (k_{22}^{①} + k_{22}^{②})u_2 + k_{23}^{①}u_3 + k_{24}^{①}u_4 + k_{23}^{②}u_5 + k_{24}^{②}u_6 \end{vmatrix}$$

At node △2

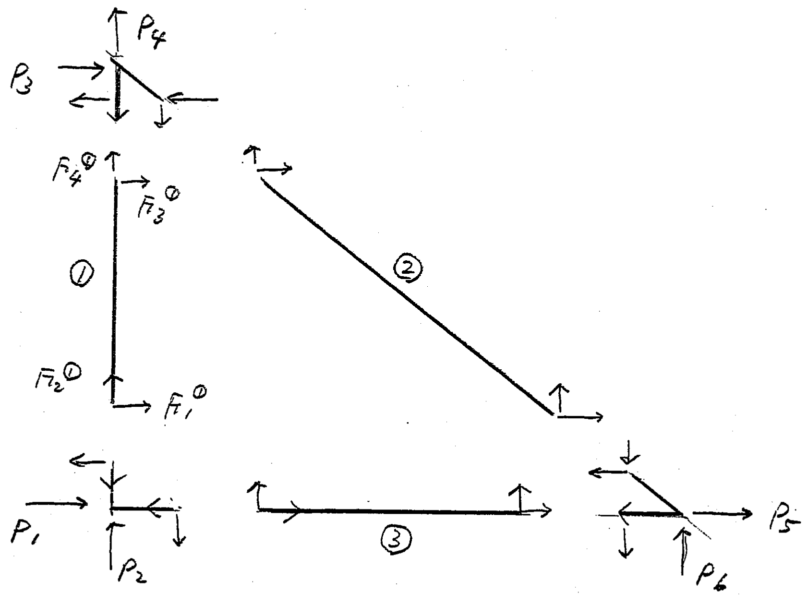
$$\begin{vmatrix} P_3 \\ P_4 \end{vmatrix} = \begin{vmatrix} F_3^{①} + F_1^{③} \\ F_4^{①} + F_4^{③} \end{vmatrix}$$

At node △3

$$\begin{vmatrix} P_5 \\ P_6 \end{vmatrix} = \begin{vmatrix} F_3^{②} + F_3^{③} \\ F_4^{②} + F_4^{③} \end{vmatrix}$$



		1	2	3	4	5	6		
P_1		$k_{11}^{(1)} + k_{11}^{(2)}$	$k_{12}^{(1)} + k_{12}^{(2)}$	$k_{13}^{(1)}$	$k_{14}^{(1)}$	$k_{13}^{(2)}$	$k_{14}^{(2)}$		u_1
P_2		$k_{21}^{(1)} + k_{21}^{(2)}$	$k_{22}^{(1)} + k_{22}^{(2)}$	$k_{23}^{(1)}$	$k_{24}^{(1)}$	$k_{23}^{(2)}$	$k_{24}^{(2)}$		u_2
P_3		$k_{31}^{(1)}$	$k_{32}^{(1)}$	$k_{33}^{(1)} + k_{11}^{(3)}$	$k_{34}^{(1)} + k_{12}^{(3)}$	$k_{13}^{(3)}$	$k_{14}^{(3)}$		u_3
P_4		$k_{41}^{(1)}$	$k_{42}^{(1)}$	$k_{43}^{(1)} + k_{21}^{(3)}$	$k_{44}^{(1)} + k_{22}^{(3)}$	$k_{23}^{(3)}$	$k_{24}^{(3)}$		u_4
P_5		$k_{31}^{(2)}$	$k_{32}^{(2)}$	$k_{31}^{(3)}$	$k_{32}^{(3)}$	$k_{33}^{(2)} + k_{33}^{(3)}$	$k_{34}^{(2)} + k_{34}^{(3)}$		u_5
P_6		$k_{41}^{(2)}$	$k_{42}^{(2)}$	$k_{41}^{(3)}$	$k_{42}^{(3)}$	$k_{43}^{(2)} + k_{43}^{(3)}$	$k_{44}^{(2)} + k_{44}^{(3)}$		u_6



Load	1	2	3	4	5	6	Displacement	
P_1							0	EQ for reactions
P_2							0	
10	$k_{31}^{(1)}$	$k_{32}^{(1)}$	$k_{33}^{(1)} + k_{11}^{(3)}$	$k_{34}^{(1)} + k_{12}^{(3)}$	$k_{13}^{(3)}$	$k_{14}^{(3)}$	u_3	Equations for free displacements
0	$k_{41}^{(1)}$	$k_{42}^{(1)}$	$k_{43}^{(1)} + k_{21}^{(3)}$	$k_{44}^{(1)} + k_{22}^{(3)}$	$k_{23}^{(3)}$	$k_{24}^{(3)}$	u_4	
0	$k_{31}^{(2)}$	$k_{32}^{(2)}$	$k_{31}^{(3)}$	$k_{32}^{(3)}$	$k_{33}^{(2)} + k_{33}^{(3)}$	$k_{34}^{(2)} + k_{34}^{(3)}$	u_5	
P_6							0	EQ for reactions

3.2 direct stiffness method – the general procedure

1) construct element stiffness
2) assemble the element stiffness coefficients into the structural stiffness
3) modify the stiffness matrix considering boundary conditions
4) solve displacements
5) element forces and reactions

$\underline{P} = \underline{K}\underline{U}$

$$\begin{bmatrix} \underline{P}_f \\ \underline{P}_s \end{bmatrix} = \begin{bmatrix} \underline{k}_{ff} & \underline{k}_{fs} \\ \underline{k}_{sf} & \underline{k}_{ss} \end{bmatrix} \begin{bmatrix} \underline{u}_f \\ \underline{u}_s \end{bmatrix}$$

$\underline{u}_f, \underline{P}_s$ = unknowns   subscript f = free in displacement

$\underline{u}_s, \underline{P}_f$ = knowns s = supports

In the 1st equation, $\underline{P}_f = \underline{k}_{ff}\underline{u}_f + \underline{k}_{fs}\underline{u}_s$,

$\underline{k}_{ff}\underline{u}_f = \underline{P}_f - \underline{k}_{fs}\underline{u}_s$, solve $\underline{u}_f$

In the 2nd equation, $\underline{P}_s = \underline{k}_{sf}\underline{u}_f + \underline{k}_{ss}\underline{u}_s$ solve $\underline{P}_s$

Solve element forces $\underline{f} = \underline{k}\ \underline{u}$.

3.3 some features of the stiffness equations

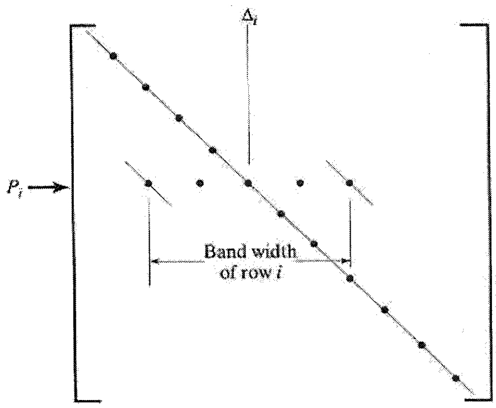
1) stiffness matrix is symmetric

2) nonzero stiffness coefficients

Zero stiffness coefficients

3) bandwidth

4) skyline



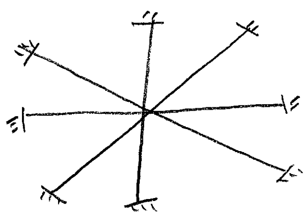
3.4 Indeterminancy

Static indeterminancy – force – flexibility method

Kinematic indeterminancy- displacement – stiffness method

The displacement method appears more automatic.

Number of redundant forces is more than the number of nodal displacements. Therefore the displacement method has advantage for the case that a large number of equations are involved.



Grid structure

n = number of element

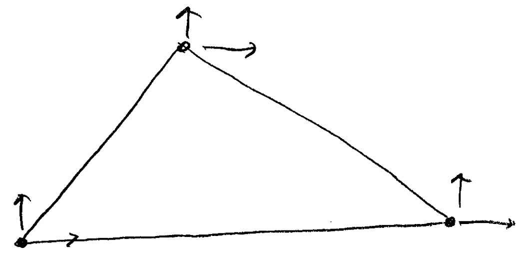
for force method,

number of dof = n × 3

for displacement method

number of dof = 3

Example 3.1 Rigid body motion



$\underset{\sim}{P} = \underset{\sim}{K}\,\underset{\sim}{u}$ $\quad$ $\underset{\sim}{u} \neq 0$, $\underset{\sim}{P} = 0 \Rightarrow$ Rigid Body motion

$\underset{\sim}{K}\,\underset{\sim}{u} = \underset{\sim}{0}$ $\quad$ homogeneous equation

only when $|\underset{\sim}{K}| = 0$, this equation has solutions

$$\left.\begin{aligned} 2x + 4y &= 0 \\ 4x + 8y &= 0 \end{aligned}\right\} \Rightarrow \text{has solution} \quad \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 0$$

$$\left.\begin{aligned} 2x + 4y &= 0 \\ 4x + 7y &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \text{only solution} \quad \begin{vmatrix} 2 & 4 \\ 4 & 7 \end{vmatrix} \neq 0$$

when $|K| = 0$, the rows and colums of $\underset{\sim}{K}$ are dependent each other.

No-zero displacements exist for making $\underset{\sim}{P} = \underset{\sim}{0}$.