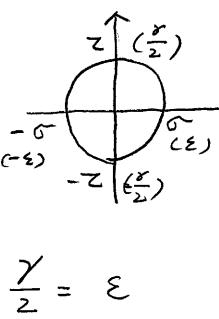
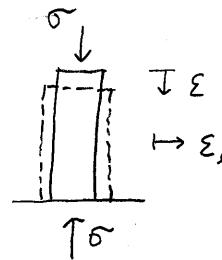


Chapter 4. Stiffness Analysis of Frames - I

4.1 Stress - strain relationships

$$\epsilon = \frac{\sigma}{E}$$

in pure shear $\epsilon_e = -\nu \epsilon = -\nu \frac{\sigma}{E}$



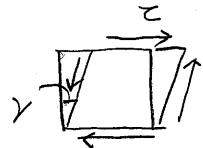
$$\epsilon = \frac{\sigma}{E} - \nu \frac{(-\sigma)}{E} = \frac{\sigma}{E} (1 + \nu) = \frac{\gamma}{2}$$

$$\tau = \sigma = \frac{E}{2(1+\nu)} \gamma = G \gamma$$

expansive \rightarrow same

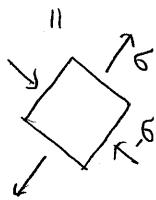
$$\nu = 0.3 \quad E = 200,000 \text{ MPa}$$

$$\nu = 0.15 \quad E = 4700 \sqrt{f_{ck}} \text{ MPa}$$



$$\gamma = \frac{\tau}{G}$$

$$G = \frac{E}{2(1+\nu)}$$

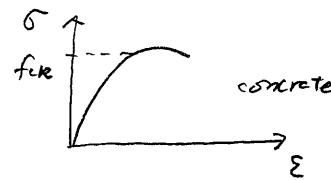
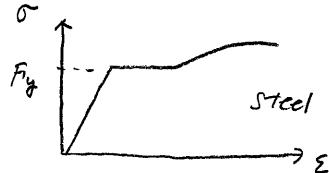


Steel

$$f_y = 250 \sim 700 \text{ MPa}$$

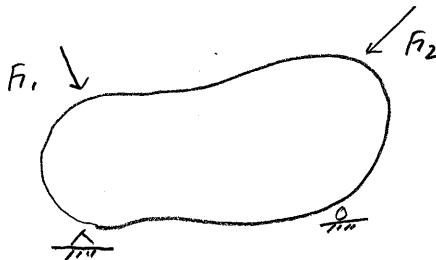
Concrete

$$f_{ck} = 21 \sim 80 \text{ MPa}$$

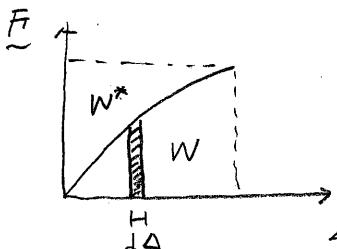


Very small tensile strength of concrete

4.2 Work and Energy



External Work



$$W = \text{work} = \int F d\Delta$$

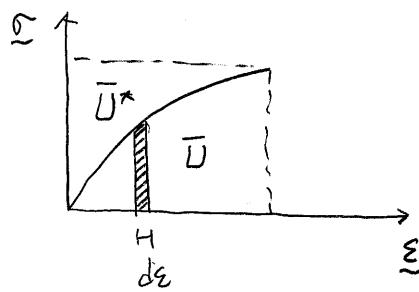
$$W^* = \text{complementary work} = \int \Delta dF$$

Δ For linear elastic material

$$W = \frac{1}{2} \underline{F} \cdot \underline{\Delta} = \frac{1}{2} \underline{\Delta}^T K \underline{\Delta}$$

$$W^* = \frac{1}{2} \underline{F} \cdot \underline{\Delta} = \frac{1}{2} \underline{F}_f^T \underline{\Delta} \underline{F}_f$$

Internal Work



\bar{U} = strain energy density

$$= \int \underline{\sigma} d\underline{\epsilon}$$

U = strain energy (internal energy)

$$= \int \bar{U} dV$$

\bar{U}^* = complimentary strain energy density

$$= \int \underline{\epsilon} d\underline{\sigma}$$

$$U^* = \int \bar{U}^* dV$$

For linear elastic material,

$$\bar{U} = \frac{1}{2} \underline{\sigma} \cdot \underline{\epsilon} = \bar{U}^*$$

Energy Conservation

External Work = Internal Work

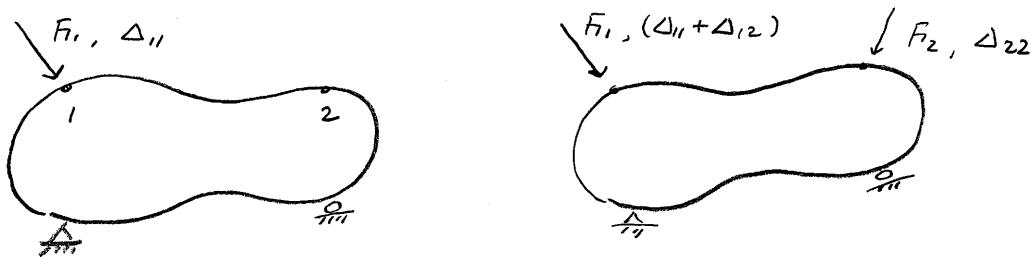
$$W = U \quad \text{and} \quad W^* = U^*$$

Example 4.1

4.3 Reciprocity

$\underbrace{\text{L}}_{\text{L}} : k_{ij} = k_{ji}$ $\underbrace{\text{d}}_{\text{d}} : d_{ij} = d_{ji}$ $\Rightarrow \text{symmetry} \Rightarrow \text{computational efficiency}$

Maxwell's Reciprocal Theorem



Apply F_1 and then $F_2 \Rightarrow$ work W_I

$$W_{I1} (\text{By } F_1) = \frac{1}{2} \Delta_{11} F_1 = \frac{1}{2} d_{11} F_1 F_1$$

$W_{I2} (\text{By } F_2) = \frac{1}{2} \Delta_{22} F_2 + \Delta_{12} F_1$
 $= \frac{1}{2} d_{22} F_2 F_2 + d_{12} F_2 F_1$

$$W_I = W_{I1} + W_{I2} = \frac{1}{2} d_{11} F_1^2 + d_{12} F_1 F_2 + \frac{1}{2} d_{22} F_2^2$$

Apply F_2 and then $F_1 \Rightarrow$ work W_{II}

$$W_{II} = W_{II1} + W_{II2} = \frac{1}{2} d_{11} F_1^2 + d_{21} F_1 F_2 + \frac{1}{2} d_{22} F_2^2$$

For linear elastic system,

$$W_I \equiv W_{II}$$

$$\Rightarrow d_{12} \equiv d_{21} \Rightarrow k_{12} \equiv k_{21}$$

4.4 Flexibility - Stiffness Transformations

4.4.1 Stiffness - to - flexibility transformation

$$\begin{bmatrix} \tilde{F}_f \\ \tilde{F}_s \end{bmatrix} = \begin{bmatrix} \underline{k}_{ff} & \underline{k}_{fs} \\ \underline{k}_{sf} & \underline{k}_{ss} \end{bmatrix} \begin{bmatrix} \Delta_f \\ \Delta_s \end{bmatrix}$$

if $\Delta_s = 0$

$$\begin{bmatrix} \tilde{F}_f \\ \tilde{F}_s \end{bmatrix} = \begin{bmatrix} \underline{k}_{ff} \\ \underline{k}_{sf} \end{bmatrix} \Delta_f$$

$$\tilde{F}_f = \underline{k}_{ff} \Delta_f \Rightarrow \Delta_f = \underline{d} \tilde{F}_f$$

$$\underline{d} = \underline{k}_{ff}^{-1}$$

4.4.2 flexibility - to - stiffness transformation

$$\tilde{F}_f = \underline{d}^{-1} \Delta_f$$

$$= \underline{K}_{ff} \Delta_f$$

$$\underline{K}_{ff} = \underline{d}^{-1}$$

Other terms? $\underline{k}_{fs}, \underline{k}_{sf}, \underline{k}_{ss}?$

$$\underline{F}_s = \underline{\Phi} \underline{F}_f \quad \text{by force-equilibrium}$$

$\underline{\Phi}$ = equilibrium matrix

$$= \underline{\Phi} \underline{d}^{-1} \underline{\Delta}_f = \underline{k}_{sf} \underline{\Delta}_f$$

$$\underline{k}_{sf} = \underline{\Phi} \underline{d}^{-1}$$

From reciprocal theorem,

$$\underline{k}_{fs} = \underline{k}_{sf}^T = (\underline{d}^{-1})^T \underline{\Phi}^T = \underline{d}^{-1} \underline{\Phi}^T$$

\underline{d}^{-1} = symmetric

$$\underline{F}_s = \underline{\Phi} \underline{F}_f = \underline{\Phi} [\underline{k}_{ff} \underline{\Delta}_f + \underline{k}_{fs} \underline{\Delta}_s]$$

$$= \underline{\Phi} [\underline{d}^{-1} \underline{\Delta}_f + \underline{d}^{-1} \underline{\Phi}^T \underline{\Delta}_s]$$

$$\underline{F}_s = \underline{k}_{sf} \underline{\Delta}_f + \underline{k}_{ss} \underline{\Delta}_s$$

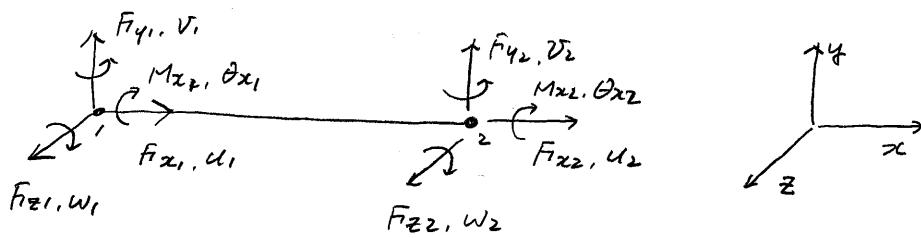
$$\underline{k}_{ss} = \underline{\Phi} \underline{d}^{-1} \underline{\Phi}^T$$

$$\underline{k} = \left[\begin{array}{c|c} \underline{k}_{ff} & \underline{k}_{fs} \\ \hline \underline{k}_{sf} & \underline{k}_{ss} \end{array} \right]$$

$$= \left[\begin{array}{c|c} \underline{d}^{-1} & \underline{d}^{-1} \underline{\Phi}^T \\ \hline \underline{\Phi} \underline{d}^{-1} & \underline{\Phi} \underline{d}^{-1} \underline{\Phi}^T \end{array} \right]$$

Example 4.2

4.5 Framework Element Stiffness Matrix



6 Degree of freedom per each node

For 3 dimensional frame element, 12 dof

$$\underline{u} = \langle u_1, v_1, w_1, \theta_{x1}, \theta_{y1}, \theta_{z1}, u_2, v_2, w_2, \theta_{x2}, \theta_{y2}, \theta_{z2} \rangle$$

$$\underline{F} = \langle F_{x1}, F_{y1}, F_{z1}, M_{x1}, M_{y1}, M_{z1}, F_{x2}, F_{y2}, F_{z2}, M_{x2}, M_{y2}, M_{z2} \rangle$$

Assuming $\left\{ \begin{array}{l} 1) \text{ small deformation} \\ 2) \text{ cross-section is bisymmetric} \\ 3) \text{ no warping (distortion of cross-section)} \end{array} \right.$

The behavior of a 3-D frame element can be uncoupled into four actions, which is independent

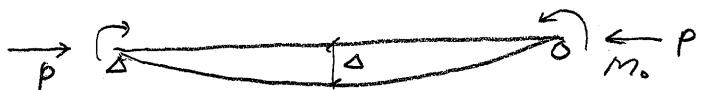
1) axial force $\underline{u} = \langle u_1, u_2 \rangle$

2) pure torsion $\underline{u} = \langle \theta_{x1}, \theta_{x2} \rangle$

3) major axis bending $\underline{u} = \langle v_1, \theta_{z1}, v_2, \theta_{z2} \rangle$

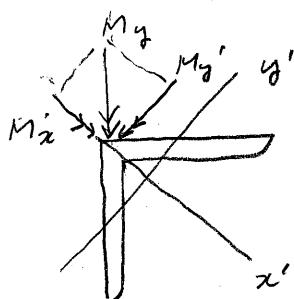
4) minor axis bending $\underline{u} = \langle w_1, \theta_{y1}, w_2, \theta_{y2} \rangle$

large deformation - coupled actions of flexure and axial compression



$$M \text{ at center} = M_0 + P \cdot \Delta$$

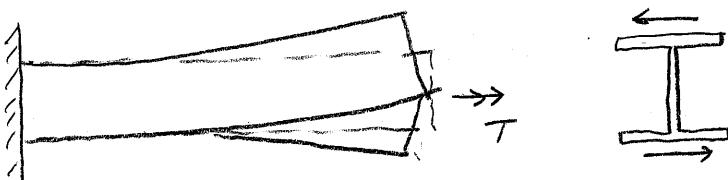
Asymmetric cross-section



M_y induces $M_{x'}$ on $M_{y'}$
in local axis.

coupled action of major axis
bending and minor axis bending

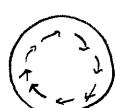
warping (warping torsion versus pure torsion)



coupled action of
torsion and minor
axis bending.

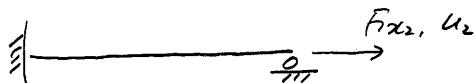
when a torsional moment is applied to a asymmetric section or open section, the cross-section is distorted.

As a result, normal stresses are developed by the torsion



\Rightarrow pure torsion causing shear stress only
no warping

4.5.1 Axial Force Member



$$u_2 = \int_0^L \varepsilon dx = \int_0^L \frac{\sigma}{E} dx = \int_0^L \frac{F_{xz2}}{EA} dx = \frac{F_{xz2} L}{EA}$$

$$\underline{d} = \frac{L}{EA} \quad k_{ff} = \underline{d}^{-1} = \frac{EA}{L}$$

$$F_{x1} = \underline{\phi} F_{xz2} \quad \underline{\phi} = -1$$

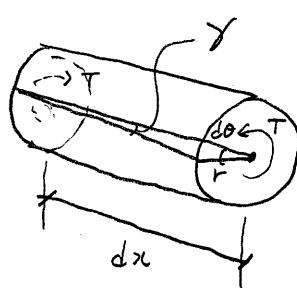
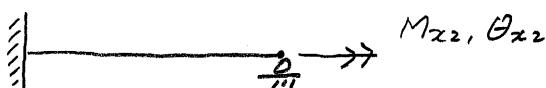
$$\underline{k}_{sf} = \underline{\phi} \underline{d}^{-1} = -\frac{EA}{L}$$

$$\underline{k}_{fs} = \underline{d}^{-1} \underline{\phi}^T = -\frac{EA}{L}$$

$$\underline{k}_{ss} = \underline{\phi} \underline{d}^{-1} \underline{\phi}^T = \frac{EA}{L}$$

$$\underline{k} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

4.5.2 Pure Torsional Member



$$\gamma \cdot dx = r d\theta$$

$$\gamma = r \frac{d\theta}{dx} = r \beta$$

$$\tau = G \gamma = G r \beta \quad dA = r d\phi dr$$

$$T = \int \tau \cdot r dA = \iint Gr^3 \beta d\phi dr$$

$$= G J \beta \quad J = \text{torsional constant} \\ = \iint r^3 dr d\phi$$

$$\text{rate of twist } \beta \left(= \frac{d\theta_x}{dx} \right) = \frac{M_{xz}}{GJ}$$

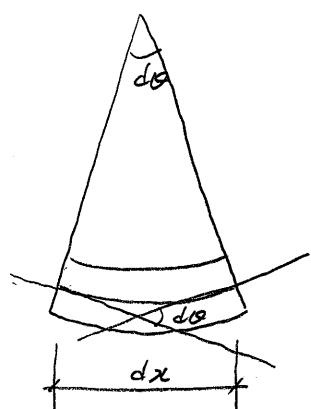
$$\theta_{xz} = \int_0^L \beta dx = \frac{L}{GJ} M_{xz}$$

$$\underline{d} = \frac{L}{GJ} \quad \underline{k}_{\text{eff}} = \underline{d}^{-1} = \frac{GJ}{L}$$

$$M_{x_1} = -M_{xz} \Rightarrow \underline{\phi} = -1$$

$$\underline{\underline{k}} = \frac{GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

4.5.3 Beam bent about its z axis



$$\begin{aligned} \frac{d\theta}{dx} &= \phi & \varepsilon &= -y \frac{d\phi}{dx} \\ & & &= -y \frac{d^2v}{dx^2} \end{aligned}$$

$$\sigma_x = -y \frac{d^2v}{dx^2}$$

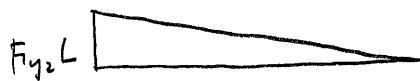
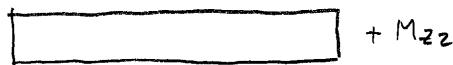
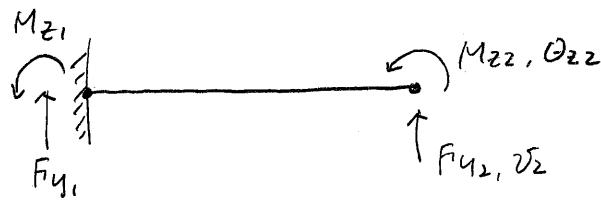
$$\sigma_x = -E y \frac{d^2v}{dx^2}$$

$$M_x = - \int \sigma_x y dA = EI_z \frac{d^2v}{dx^2} = EI_z \phi$$

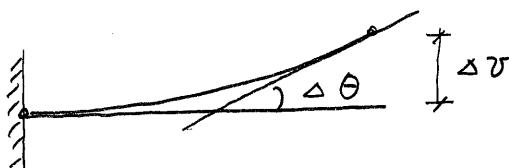
M_x = Generalized stress

$\frac{d^2v}{dx^2}$ = Generalized strain

EI_z = Generalized stiffness



Moment - area method (curvature area method)



$$\Delta\theta = \int \phi dx = \int \frac{M}{EI} dx (= \int d\theta)$$

$$\Delta v = \int d\theta \cdot x = \int \frac{M}{EI} \cdot x dx$$

$$\Theta_{zz} = \frac{M_{zz}}{EI} \cdot L + \frac{F_{y2}L}{EI} \cdot \frac{L}{2}$$

$$v_2 = \frac{M_{zz}}{EI} L \cdot \frac{L}{2} + \frac{F_{y2}L}{EI} \cdot \frac{L}{2} - \frac{2}{3} L$$

$$\begin{bmatrix} v_2 \\ \Theta_{zz} \end{bmatrix} = \underbrace{\frac{L}{EI_2} \begin{bmatrix} \frac{L^2}{3} & \frac{L}{2} \\ \frac{L}{2} & 1 \end{bmatrix}}_{d} \begin{bmatrix} F_{y2} \\ M_{zz} \end{bmatrix}$$

$$\underline{\Delta f} = \underline{d} \underline{F_f}$$

By force-equilibrium

$$F_{y_1} = -F_{y_2}$$

$$M_{z_1} = -F_{y_2} \cdot L - M_{z_2}$$

$$\begin{bmatrix} F_{y_1} \\ M_{z_1} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ -L & -1 \end{bmatrix}}_{\underline{k}} \begin{bmatrix} F_{y_2} \\ M_{z_2} \end{bmatrix}$$

$$\underline{F_s} = \underline{\Phi} \quad \underline{F_f}$$

$$\underline{k} = \left[\begin{array}{c|c} \underline{d}^{-1} & \underline{d}^T \underline{\Phi}^T \\ \hline \underline{\Phi} \underline{d}^{-1} & \underline{\Phi} \underline{d}^{-1} \underline{\Phi}^T \end{array} \right]$$

$$\text{Hence } \underline{k} = \frac{\underline{k}}{E I_z} \frac{L}{L} \left[\begin{array}{c|c|c|c} v_1 & \theta_1 & v_2 & \theta_2 \\ \hline 12/L^2 & 6/L & -12/L^2 & 6/L \\ 6/L & 4 & -6/L & 2 \\ \hline -12/L^2 & -6/L & 12/L^2 & -6/L \\ 6/L & 2 & -6/L & 4 \end{array} \right]$$

Eg. (4.32)

see examples 4.3 and 4.4

4.5.4 Beam bent about its y axis

\underline{k} for this action can be independently derived
in the same manner

see eg. (4.33)

4.5.5 complete Element Stiffness Matrix

$$\underline{\underline{K}} = \underline{\underline{k}}_{\text{axial}} + \underline{\underline{k}}_{\text{torsion}} + \underline{\underline{k}}_{z\text{-bending}} + \underline{\underline{k}}_{y\text{-bending}}$$

All actions can be superposed because they were assumed to be independent.

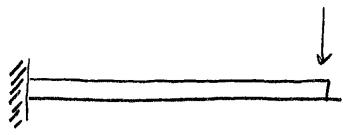
⇒ see Equation (4.34)

4.6 A commentary on deformations and displacement variables

4.6.1 Neglected deformations

□ Transverse shear

$$\Delta = \Delta_{\text{flexure}} + \Delta_{\text{shear}}$$

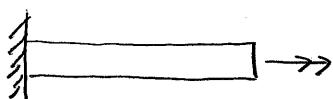


$$\Delta \approx \Delta_{\text{flexure}}$$



$$\Delta = \Delta_{\text{flexure}} + \Delta_{\text{shear}}$$

□ Warping torsion



PURE TORSION



PURE TORSION
+ WARPING TORSION

$$T \rightarrow Z$$

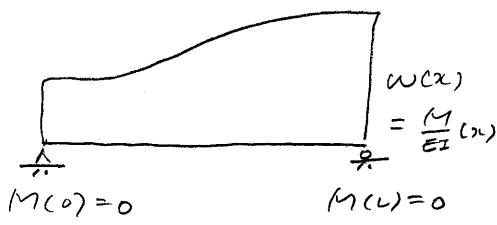
$$T \rightarrow Z \neq 0$$

TORSION + minor axis bending
(← warping torsion)

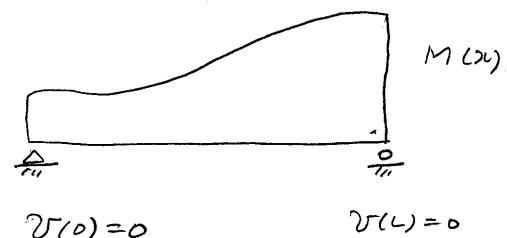
Example 4.5

conjugate beam method

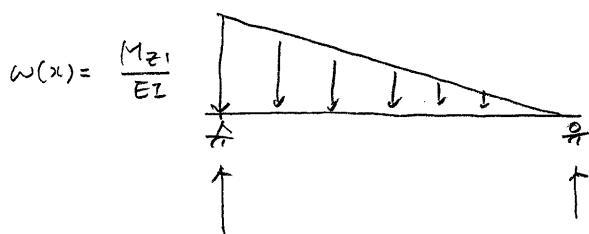
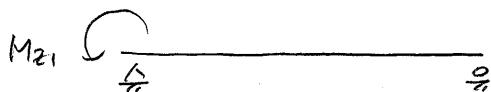
$$\begin{aligned} \frac{dM}{dx} &= V \\ \frac{dV}{dx} &= w \end{aligned} \quad \longleftrightarrow \quad \begin{cases} \frac{dv}{dx} = \theta \\ \frac{d\theta}{dx} = \phi \quad (= \frac{M}{EI}) \end{cases}$$



conjugate beam



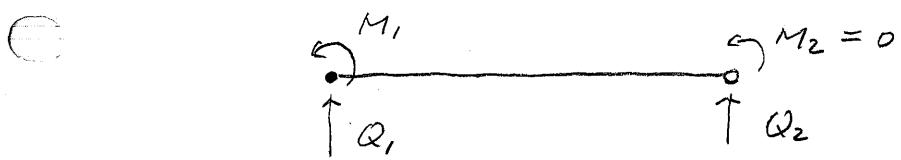
actual beam



$$\begin{aligned} w(x) &= \frac{M_{z1}}{EI} \\ V_1 &= \underbrace{\frac{M}{EI} \cdot \frac{L}{3}}_{\Downarrow} \\ \theta_{z1} &= \underbrace{\frac{M}{EI} \cdot \frac{L}{6}}_{\Downarrow} \end{aligned}$$

Example 4.14

Stiffness of element with a hinge



$$EI \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{bmatrix} = 0$$

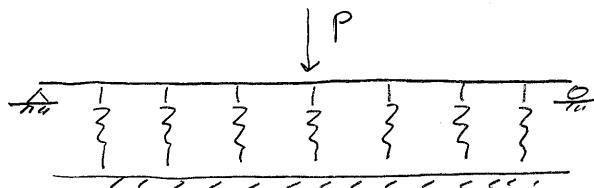
④

$$EI_{\text{min. note}} \quad \text{Eq. ④} \quad \theta_2 = -\frac{3}{2L} v_1 - \frac{1}{2} \theta_1 + \frac{3}{2L} v_2$$

$$EI \begin{bmatrix} \frac{3}{L^3} & \frac{3}{L^2} & -\frac{3}{L^3} \\ \frac{3}{L^2} & \frac{3}{L} & -\frac{3}{L^2} \\ -\frac{3}{L^3} & -\frac{3}{L^2} & \frac{3}{L^3} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \end{bmatrix}$$

Example 4.15

Q



$$M \left(\begin{array}{c} \uparrow \\ V \\ \downarrow \\ kV \end{array} \right) M + dM \quad \left\{ \begin{array}{l} \frac{dV}{dx} = -kV \\ \frac{dM}{dx} = V \end{array} \right.$$

$$(EI) \frac{d^2V}{dx^2} = \frac{M}{EI}, \quad \frac{d^2M}{dx^2} = -kV$$

$$\Rightarrow EI \frac{d^4V}{dx^4} = -kV$$

~~~~~

O

$$B/C, \quad x=0 \quad V(0)=0$$

$$M(0) = \frac{d^2V}{dx^2}(0) = 0$$

$$x = l \quad V(l) = 0$$

$$M(l) = \frac{d^2V}{dx^2}(l) = 0$$

G