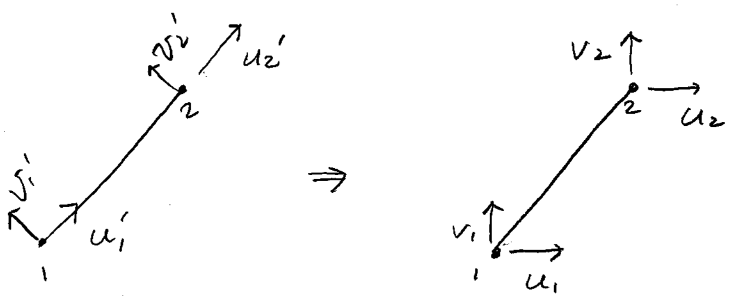
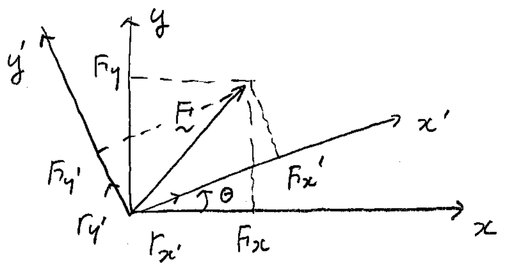
Chapter 5. Stiffness Analysis of Frames - II

- Coordinate transformation
- Equivalent nodal load
  - element load
  - self - straining
  - temperature change

5.1 Coordinate Transformations



Displacement vectors in local coordinates ⇒ displacement vectors in Global coordinates
(force vectors)



$$F_{x'} = \underset{\sim}{F} \cdot \underset{\sim}{r}_{x'} = \underset{\sim}{r}_{x'} \cdot \underset{\sim}{F}$$

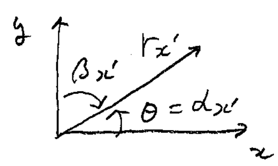
$$= \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

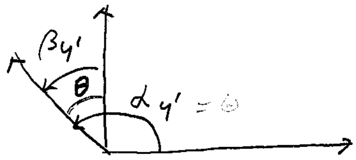
$r_{x'}$ = directional cosine

$$= \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

or

$$\begin{bmatrix} \cos\alpha_{x'} & \cos\beta_{x'} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$



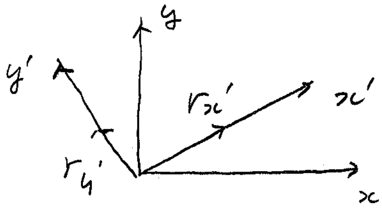


$$F_{y'} = \underset{\sim}{r}_{y'} \cdot \underset{\sim}{F}$$

$$= \begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha_{y'} & \cos\beta_{y'} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$\begin{cases} F_{x'} = F_x \cos\alpha_{x'} + F_y \cos\beta_{x'} \\ F_{y'} = F_x \cos\alpha_{y'} + F_y \cos\beta_{y'} \end{cases}$$



or

$$\begin{cases} F_{x'} = l_{x'} F_x + m_{x'} F_y \\ F_{y'} = l_{y'} F_x + m_{y'} F_y \end{cases}$$

$r_{x'}$ and $r_{y'}$ $\Rightarrow$ unit vector , orthogonality

$$\left.\begin{cases} r_{x'} \cdot r_{x'} = 1 = l_{x'}^2 + m_{x'}^2 \\ r_{y'} \cdot r_{y'} = 1 = l_{y'}^2 + m_{y'}^2 \end{cases}\right\} \Rightarrow \text{unit vector}$$

$$r_{x'} \cdot r_{y'} = 0 = l_{x'} l_{y'} + m_{x'} m_{y'} \Rightarrow \text{orthogonality}$$

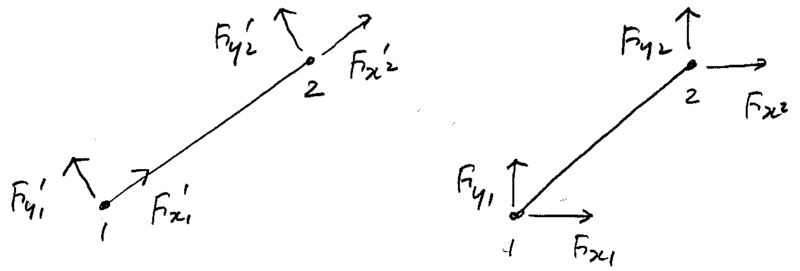
$$\begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} = \begin{bmatrix} l_{x'} & m_{x'} \\ l_{y'} & m_{y'} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$\underset{\sim}{F}' = \underset{\sim}{R} \, \underset{\sim}{F}$$ $\underset{\sim}{R}$ = transformation matrix (rotational matrix)

$$\underset{\sim}{F} = \underset{\sim}{R}^{-1} \underset{\sim}{F}' = \underset{\sim}{R}^T \underset{\sim}{F}'$$

$\underset{\sim}{R}^{-1} = \underset{\sim}{R}^T$: $\underset{\sim}{R}$ = orthogonal matrix

$$\underset{\sim}{u}' = \underset{\sim}{R} \, \underset{\sim}{u}, \quad \underset{\sim}{u} = \underset{\sim}{R}^T \underset{\sim}{u}'$$



$$\begin{bmatrix} F_{x1}' \\ F_{y1}' \\ F_{x2}' \\ F_{y2}' \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} \underset{\sim}{R} & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{R} \end{array}\right]}_{\underset{\sim}{T}} \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{bmatrix}$$

$$\underset{\sim}{F}' = \underset{\sim}{T}\,\underset{\sim}{F}$$

$$\underset{\sim}{u}' = \underset{\sim}{T}\,\underset{\sim}{u}$$

For three dimensional rotational matrix, see Eq. (5.2) ~ (5.7)

5.1.2 Transformation of Degrees of freedom

$$\underset{\sim}{F}' = \underset{\sim}{k}'\,\underset{\sim}{u}'$$

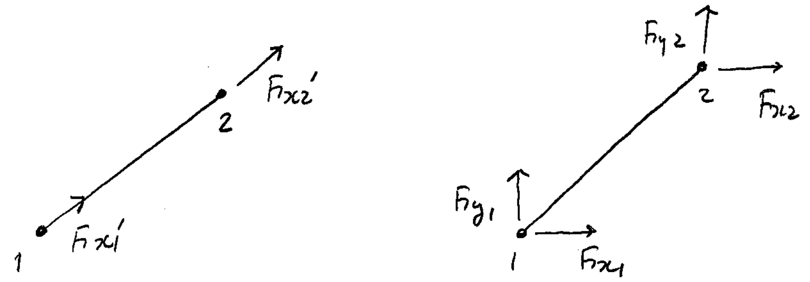
$$\underset{\sim}{T}\,\underset{\sim}{F} = \underset{\sim}{k}'\,\underset{\sim}{T}\,\underset{\sim}{u}$$

$$\underset{\sim}{F} = \underbrace{\underset{\sim}{T}^T\,\underset{\sim}{k}'\,\underset{\sim}{T}}_{\underset{\sim}{k}}\,\underset{\sim}{u}$$

$$\underset{\sim}{k} = \underset{\sim}{T}^T\,\underset{\sim}{k}'\,\underset{\sim}{T}$$ ⇒ stiffness transformation

5.1.3 Transformation and energy

Transformation matrix is not necessarily square matrix.



$$\begin{bmatrix} F_{x1}' \\ F_{x2}' \end{bmatrix} = \underbrace{\left[\begin{array}{cc|cc} l_x & m_x & 0 & 0 \\ \hline 0 & 0 & l_x & m_x \end{array}\right]}_{\underset{\sim}{T}} \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{bmatrix}$$

not square = no inverse matrix exists

Energy conservation

$$\frac{1}{2} \underset{\sim}{F}'^T \underset{\sim}{u}' = \frac{1}{2} \underset{\sim}{F}^T \underset{\sim}{u}$$

$$\frac{1}{2} \underset{\sim}{F}^T \underset{\sim}{T}^T \underset{\sim}{u}' = \frac{1}{2} \underset{\sim}{F}^T \underset{\sim}{u}$$

$$\underset{\sim}{T}^T \underset{\sim}{u}' = \underset{\sim}{u}$$

$$\left.\begin{array}{l} \underset{\sim}{u}' = \underset{\sim}{T}\, \underset{\sim}{u} \\ \underset{\sim}{u} = \underset{\sim}{T}^T \underset{\sim}{u}' \end{array}\right\} \Rightarrow \text{still effective}$$

$$W = \frac{1}{2} \underset{\sim}{F}'^{T} \underset{\sim}{u}' \qquad\qquad W = \frac{1}{2} \underset{\sim}{F}^{T} \underset{\sim}{u}$$

$$= \frac{1}{2} \underset{\sim}{u}'^{T} \underset{\sim}{k}' \underset{\sim}{u}' \qquad\qquad = \frac{1}{2} \underset{\sim}{u}^{T} \underset{\sim}{k} \underset{\sim}{u}$$

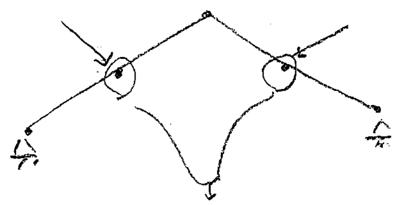
$$= \frac{1}{2} \underset{\sim}{u}^{T} \underline{\underset{\sim}{T}^{T} \underset{\sim}{k}' \underset{\sim}{T}} \underset{\sim}{u}$$

$$\Rightarrow \underset{\sim}{k} = \underset{\sim}{T}^{T} \underset{\sim}{k}' \underset{\sim}{T}$$

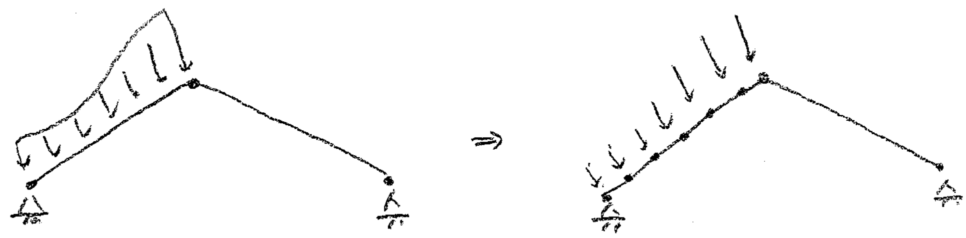
In case of truss element

$$\underset{(4\times4)}{\underset{\sim}{k}} = \underset{(4\times2)}{\underset{\sim}{T}^{T}} \; \underset{(2\times2)}{\underset{\sim}{k}'} \; \underset{(2\times4)}{\underset{\sim}{T}}$$

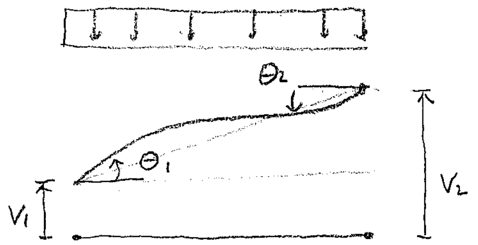
5.2 Loads between nodal loads



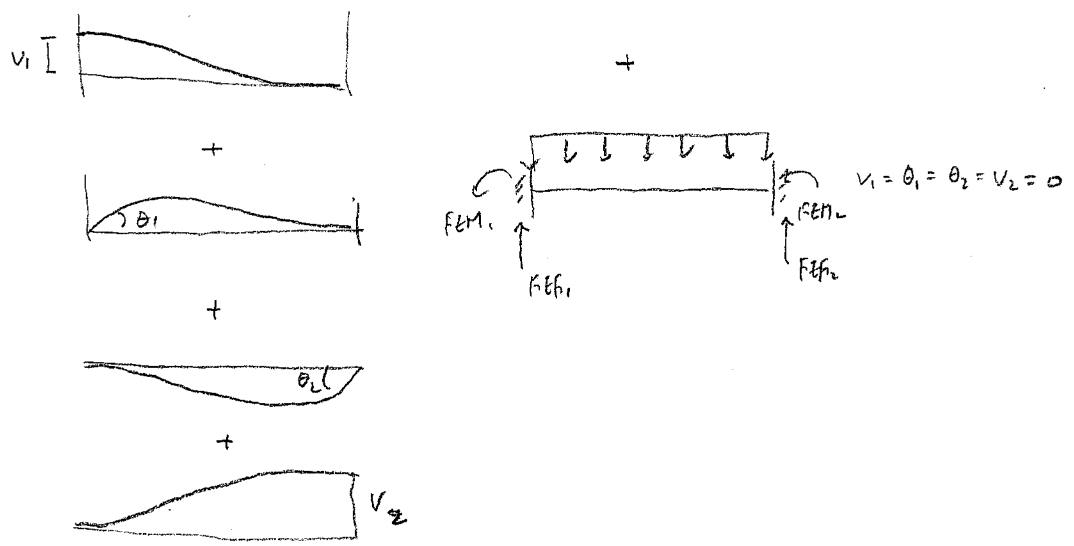
additional nodes for the concentrated load



equivalent concentrated load



$$\begin{bmatrix} Fy_1 \\ M_1 \\ Fy_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} k_{11} \\ k_{21} \\ k_{31} \\ k_{41} \end{bmatrix} V_1 + \begin{bmatrix} k_{12} \\ k_{22} \\ k_{32} \\ k_{42} \end{bmatrix} \theta_1 + \begin{bmatrix} k_{13} \\ k_{23} \\ k_{33} \\ k_{43} \end{bmatrix} V_2 + \begin{bmatrix} k_{14} \\ k_{24} \\ k_{34} \\ k_{44} \end{bmatrix} \theta_2 + \begin{bmatrix} FEf_1 \\ FEM_1 \\ FEf_2 \\ FEM_2 \end{bmatrix}$$



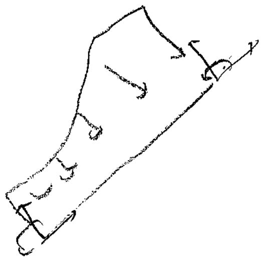
$\underset{\sim}{P} = \underset{\sim}{K}\, \underset{\sim}{\Delta} + \underset{\sim}{P}^F$ $\qquad \underset{\sim}{P}^F$: fixed end forces

or $\quad \underset{\sim}{P} - \underset{\sim}{P}^F = \underset{\sim}{K}\, \underset{\sim}{\Delta}$

$\underset{\sim}{P} + \underset{\sim}{P}^E = \underset{\sim}{K}\, \underset{\sim}{\Delta}$ $\qquad \underset{\sim}{P}^E$: equivalent nodal force due to member forces

$$\begin{cases} \underset{\sim}{K}_{ff}\, \underset{\sim}{\Delta}_f = \underset{\sim}{P}_f - \underset{\sim}{K}_{fs}\, \underset{\sim}{\Delta}_s - \underset{\sim}{P}_f^F & \text{solve } \underset{\sim}{\Delta}_f \\ \underset{\sim}{P}_s = \underset{\sim}{K}_{sf}\, \underset{\sim}{\Delta}_f + \underset{\sim}{K}_{ss}\, \underset{\sim}{\Delta}_s + \underset{\sim}{P}_s^F & \text{solve } \underset{\sim}{P}_s \end{cases}$$

$\underset{\sim}{F} = \underset{\sim}{k}\, \underset{\sim}{\Delta} + \underset{\sim}{F}^F$



$\underset{\sim}{P}' = \underset{\sim}{k}' \underset{\sim}{\Delta}' + \underset{\sim}{P}^{F'}$

$\downarrow$

$\underset{\sim}{P} = \underset{\sim}{k}\, \underset{\sim}{\Delta} + \underset{\sim}{P}^F$

$\underset{\sim}{T}\, \underset{\sim}{P} = \underset{\sim}{k}'\, \underset{\sim}{T}\, \underset{\sim}{\Delta} + \underset{\sim}{P}^{F'}$

$$\underset{\sim}{P} = \underbrace{\underset{\sim}{T}^T \underset{\sim}{k}' \underset{\sim}{T}}_{= \underset{\sim}{k}} \underset{\sim}{\Delta} + \underbrace{\underset{\sim}{T}^T \underset{\sim}{P}^{F'}}_{= \underset{\sim}{P}^F} \qquad \underset{\sim}{P}^F = \underset{\sim}{T}^T \underset{\sim}{P}^{F'}$$

fixed end moments for various loadings

$\Rightarrow$ see Table. 5.1

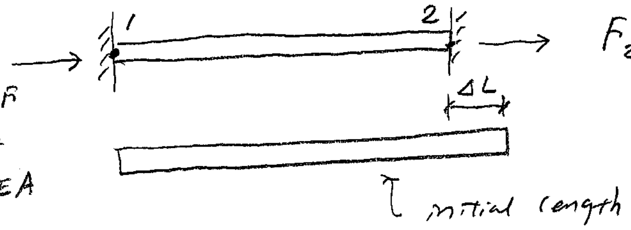
5.3. Self-Straining - Initial and Thermal Strain conditions

Self-straining

$\Rightarrow$ Structure is strained with no external load.

$$\underset{\sim}{F} = \underset{\sim}{k}\,\underset{\sim}{\Delta} + \underset{\sim}{F}^F$$

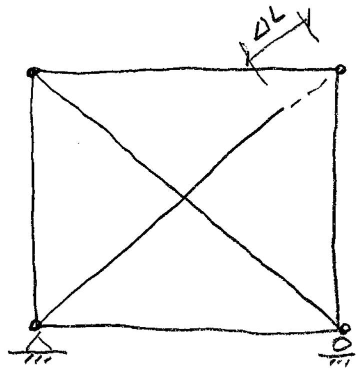
$\underset{\sim}{F}^F$ = fixed end force due to initial strain.

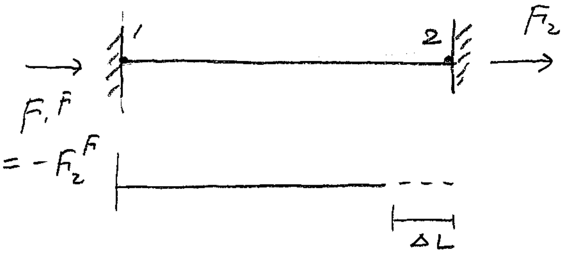


$F_1^F = \frac{\Delta L}{L} EA$

$F_2^F = -\frac{\Delta L}{L} \cdot EA$

$$\underset{\sim}{F} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\frac{\Delta L}{L}EA}_{\underset{\sim}{F}^F}$$

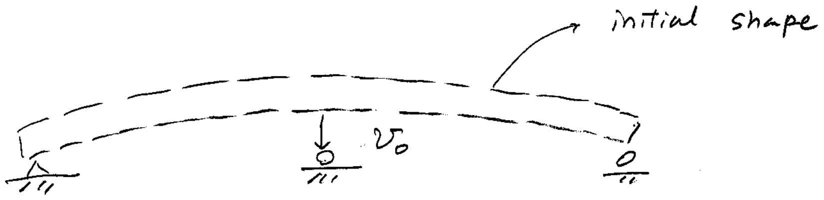


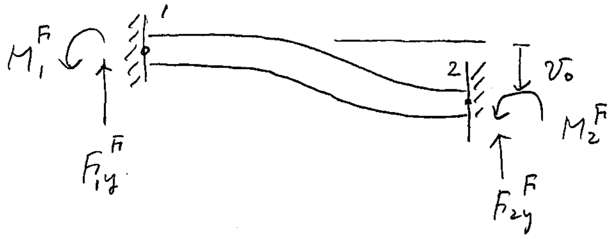


$F_1^F = -F_2^F$

$F_2^F = \frac{\Delta L}{L} \cdot EA$

$$\underset{\sim}{F}^F = \frac{\Delta L}{L} EA \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$





$$\begin{cases} F_{1y}^F = -\dfrac{12EI}{L^3}(-v_0) \\ M_1^F = \dfrac{6EI}{L^2}(-v_0) \\ F_{2y}^F = \dfrac{12EI}{L^3}(-v_0) \\ M_2^F = \dfrac{6EI}{L^2}(-v_0) \end{cases}$$

$$\underset{\sim}{F} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

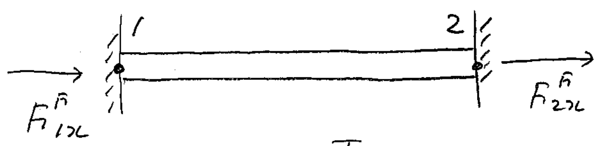
two choices for considering v_0

1) fixed end forces

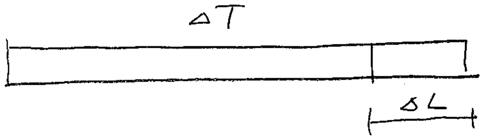
2) support restraints on v_0

Temperature change

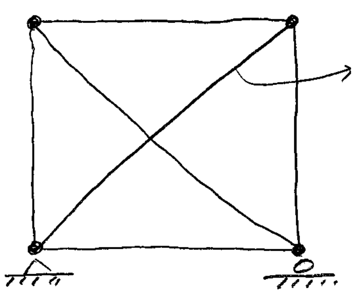
Truss



$$\underset{\sim}{F}^F = \begin{cases} F_{1x}^F = +\alpha \Delta T\, EA \\ F_{2x}^F = -F_{1x}^F = -\alpha \Delta T EA \end{cases}$$



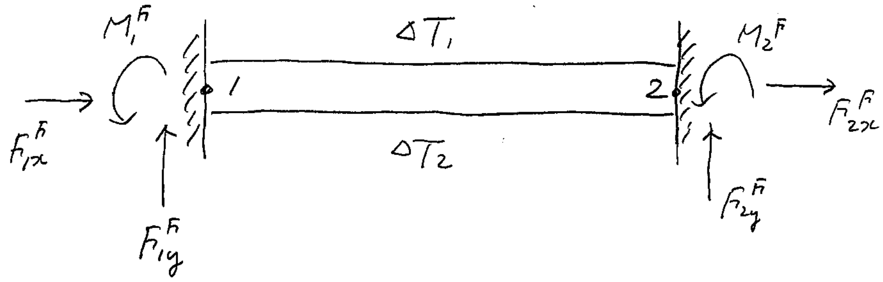
$\Delta L = \alpha \Delta T L$

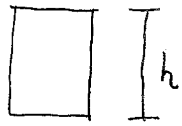


Temperature change

ΔT

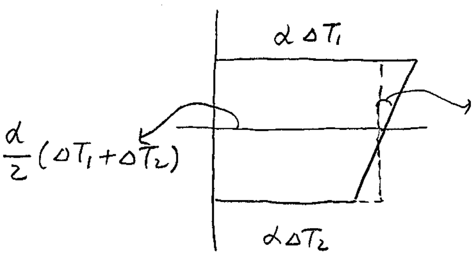
Beam





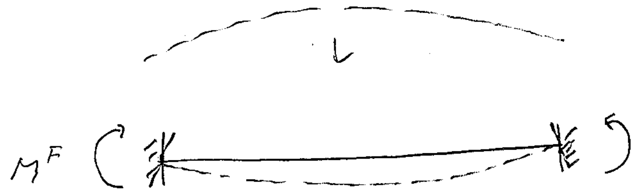
cross-section

Strain (Thermal)



$$\phi_T = \frac{\alpha}{h}(\Delta T_1 - \Delta T_2)$$

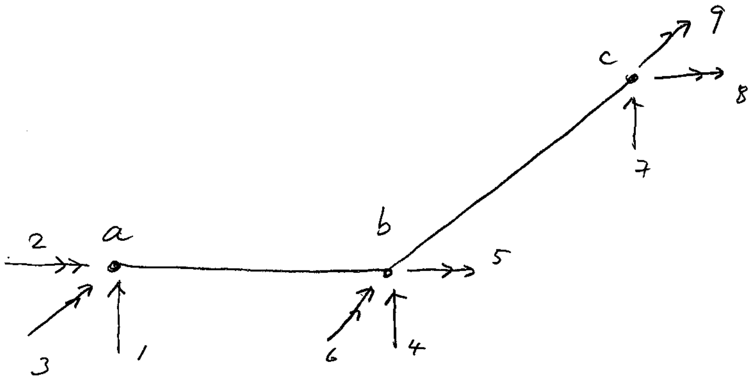
$$\frac{\alpha}{2}(\Delta T_1 + \Delta T_2) \Rightarrow \begin{cases} F_{1x}^F = EA\,\frac{\alpha}{2}(\Delta T_1 + \Delta T_2) \\ F_{2x}^F = -F_{1x}^F = -EA\,\frac{\alpha}{2}(\Delta T_1 + \Delta T_2) \end{cases}$$



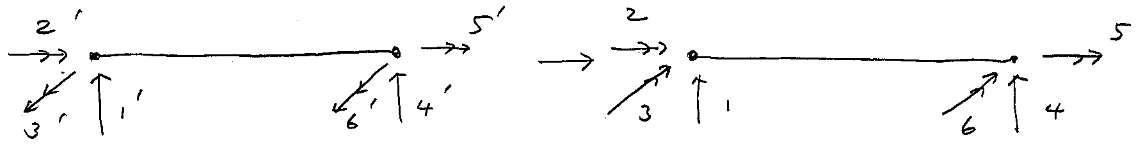
$$M^F = \phi_T \cdot EI$$

$$\begin{cases} F_{1y}^F = 0 \\ M_1^F = -EI\phi_T \\ F_{2y} = 0 \\ M_2^F = EI\phi_T \end{cases} \qquad \phi_T = \frac{\alpha}{h}(\Delta T_1 - \Delta T_2)$$

Example 5.8

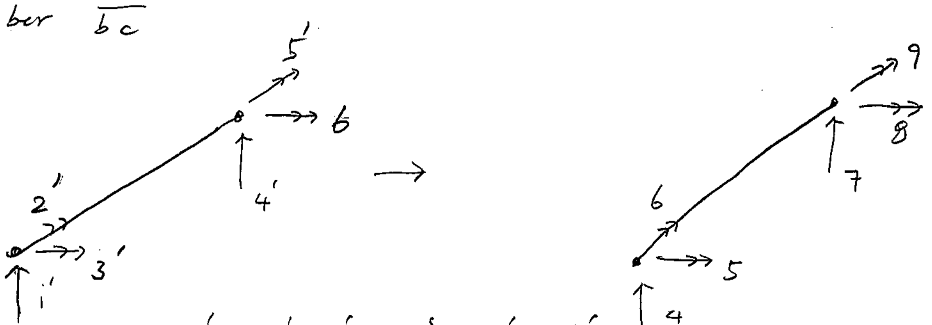


member $\overline{ab}$



$$
\underset{\sim}{T} = \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \left[\begin{array}{c|c|c|c|c|c} 1 & & & & & \\ \hline & 1 & & & & \\ \hline & & -1 & & & \\ \hline & & & 1 & & \\ \hline & & & & 1 & \\ \hline & & & & & -1 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

member $\overline{bc}$



$$
\underset{\sim}{T} = \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \left[\begin{array}{c|c|c|c|c|c} 1 & & & & & \\ \hline & & 1 & & & \\ \hline & 1 & & & & \\ \hline & & & 1 & & \\ \hline & & & & & 1 \\ \hline & & & & 1 & \end{array}\right] \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
$$