CHAPTER 4. PROBABILISTIC ENGINEERING ANALYSIS – TIME-INDEPENDENT PERFORMANCE

4.1 Motivation

Many system failures can be traced back to various difficulties in evaluating and designing complex systems under highly uncertain manufacturing and operational conditions and our limited understanding of physics-of-failures (PoFs). One of the greatest challenges in engineered systems design is how to evaluate the probability of an engineering event accurately before prototyping or actual testing. One way to evaluate the probability of an engineering event is known as *Monte Carlo simulation*, based on random sampling. Due to inefficiency of Monte Carlo when data is not given sufficiently, many "efficient" methods have been devised to alleviate the need for Monte Carlo simulation. These methods included the first and second-order reliability method (FORM and SORM), the response surface method (RSM), and the Bayesian inference.

4.2 Probabilistic Description of System Performance

Uncertainty affects the entire lifecycle of engineered systems from the impurity of the resources to the assembly of the finished goods. No matter manufacturer design the product perfectly, there is always errors or imperfection in manufacturing and operation. It is extremely difficult to predict engineering performances precisely due to substantial uncertainty in engineering design, manufacturing and operation. For example, engineers cannot predict how much engine mount bushing transmits engine noise and vibration to drivers and passengers; how much head/neck/chest/femur injury occurs during a car crash; what is a critical height for a drop test that breaks the display of a smartphone. Thus, we should define engineering performances as a function of uncertainty as shown below.

Probabilistic performance
$= Y(\mathbf{X})$; $\mathbf{X}$ is a vector of uncertainty that affects system performance

Engineering systems have specifications in terms of systems' performances. The specification can set a threshold in a quantitative scale. Therefore we can set a probability of safety which is under the pre-determined threshold, say Y_U.

$$\begin{aligned} \text{Probability } (Y(\mathbf{X}) \le Y_U) &= \text{Probability of safety (=success)} \\ &= \text{Reliability} \\ &= 1 - \text{Probability of failure} \end{aligned}$$

On the other hand, our system is now reliable—it meets our design goals or specifications—but it may not be robust. Operation of the system is affected by variabilities of the inputs. To be robust, a system performance must be insensitive to input variabilities. In other words, the performance thus possesses a narrow distribution subject to input variabilities as shown below

Probability ($Y_L \le Y(\mathbf{X}) \le Y_U$) = Robustness

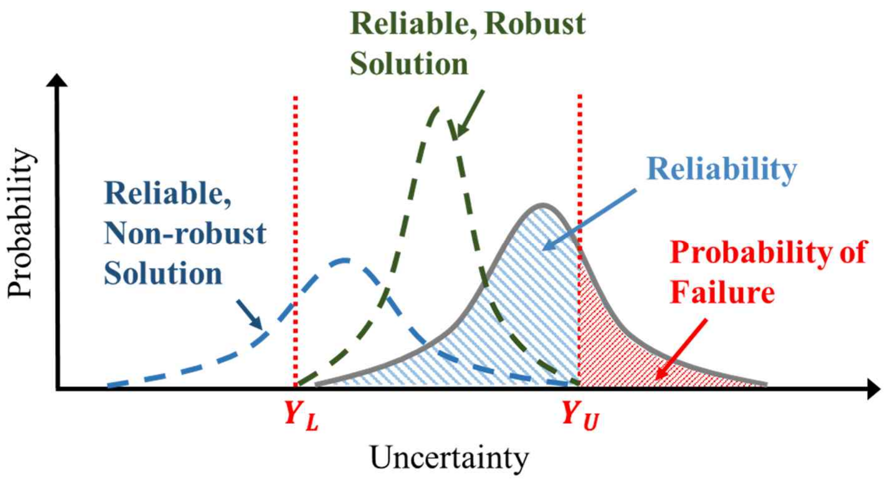


Figure 4.1: Fundamentals of Probabilistic Performance Analysis

4.3 Probabilistic Description of System Performance – Reliability

A system performance is defined in many different ways as practiced in different applications; say the electronics, civil structures, nuclear/chemical plants, and aerospace industries. In some instances, system performances can be treated time-independently due to their characteristics. Other instances situate the performances time-dependently.

4.3.1 Time-Independent Performance:

The probability that the actual performance of a particular system will meet the required or specified design performance without considering the degradation of system performances over time. It is often found in mechanical and civil structural systems.

$$R(\mathbf{X}) = P(Y(\mathbf{X}) > Y_c) = 1 - P(Y(\mathbf{X}) \le Y_c) \quad \text{for} \quad \text{larger-the-better performances}$$

where the safety of the system is defined as $Y > Y_c$ and Y_c is the critical value for Y. Y_c can be either deterministic or random. Examples include natural frequency, engine power, energy efficiency, etc.

$$R(\mathbf{X}) = P(Y(\mathbf{X}) < Y_c) = 1 - P(Y(\mathbf{X}) \ge Y_c) \quad \text{for} \quad \text{smaller-the-better performances}$$

where the safety of the system is defined as $Y < Y_c$. Examples include stress, strain, crack size, etc.

4.3.2 Time-Dependent Performance:

The probability that the actual life of a particular system will exceed the required or specified design life.

$$R_T(t) = P\left[T(\mathbf{X}) > t\right] = 1 - P\left[T(\mathbf{X}) \le t\right] = 1 - F_T(t)$$

where the time-to-failure (TTF) of a system is defined as a time that a system health condition, $G(\mathbf{X})$, is worse than its critical value, G_c, and $\mathbf{X}$ is the random vector representing engineering uncertain factors.

4.3.3 Challenges:

1. Modeling random variables ($\mathbf{X}$) for future loading, material property, and manufacturing tolerances (section 3).
2. Analyzing how input uncertainties propagate to those of system performances (section 4.4-4.7)
3. Extending the ideas of probabilistic analysis to the case with a lack of data (section 4.8)
4. Identification of the probability distribution for a reliability function (sections 5).
5. Predicting the failure time or performance failure when designing a system or component (section 5).
6. A long-time failure or lack of failure in test-based reliability assessment (section 5).
7. Consideration of performance degradation in time-dependent reliability (sections 5).

4.4 Probabilistic Description of Time-Independent Performance

- Structural reliability is defined in many different ways as practiced in different applications; say the electronics, civil structures, nuclear/chemical plants, and aero-space industries.
- Most electrical, electronic and mechanical components and systems deteriorate during use as a result of elevated operating temperatures, chemical changes, mechanical wear, fatigue, overloading, and for a number of other reasons. Failure of a particular component may eventually occur for one of these reasons, or it may be caused indirectly as a result of the deterioration of some other parts of the system. However, it is very difficult to estimate TTF distribution precisely.
- In contrast to electronic/mechanical systems, structural systems tend not to deteriorate, except by the mechanical corrosion and fatigue, and in some cases may even get stronger, for example, the increase in the strength of concrete with time, and the increase in the strength of soils as a result of consolidation.
- In other cases, engineers are interested in initial performances.

For a simple structural member, the strength R and load S of the structure can describe the probability of failure or reliability. Suppose the strength R and load S to be random with the known distributions, $F_R(r)$ and $F_S(s)$. The probability of failure is defined as

$$P_f = P(R-S \le 0) = \int_{-\infty}^{\infty} F_R(s) f_S(s)ds \tag{30}$$

Then, the reliability can be defined as

$$R = 1 - P_f = 1 - \int_{-\infty}^{\infty} F_R(s) f_S(s)ds \tag{31}$$

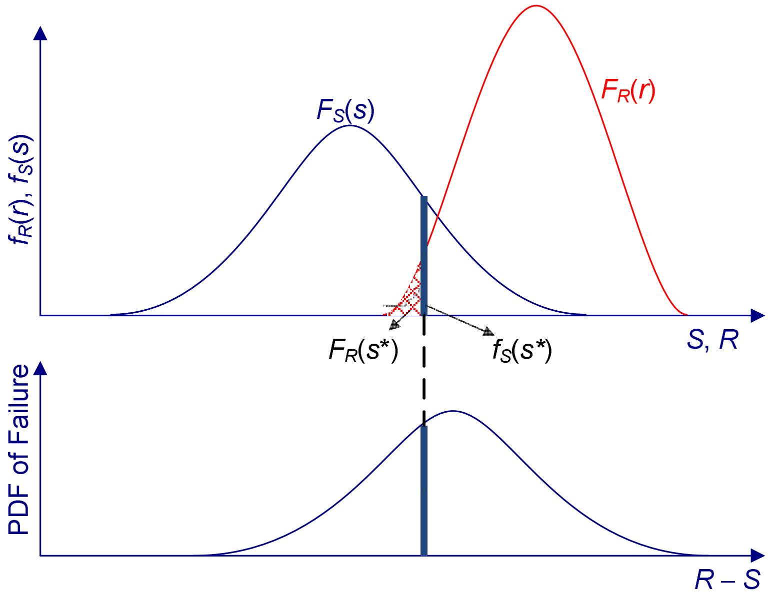


Figure 4.2: A Simple Case of Reliability (= 1–P_f): Strength-Load

4.5 General Description of Time-Independent Performance

The reliability is defined as the probability that the performance of a system exceeds the required or specified design limit over operating time *t*.

$$R(t) = P(Y(\mathbf{X},t) \ge Y_c) = 1 - P(Y(\mathbf{X},t) < Y_c) \quad \text{for} \quad \text{Larger-the-better type}$$
$$R(t) = P(Y(\mathbf{X},t) \le Y_c) = 1 - P(Y(\mathbf{X},t) > Y_c) \quad \text{for} \quad \text{Smaller-the-better type}$$

where the failure of the system is defined as $Y \ge Y_c$ for L-Type (or $Y \le Y_c$) and Y_c is the required design limit for Y. Y_c can be either deterministic or random.

4.6 Probabilistic Engineering Analysis Using Simulation Models

For probabilistic engineering analysis, uncertainty in engineered system performances (or outputs) must be understood by taking into account various uncertainties in engineered system inputs. As shown Fig. 4.3, input uncertainties are propagated through the system to those in outputs (e.g., natural frequency, fuel consumption, energy conversion efficiency, vibration, transmission error, temperature distribution, head injury).

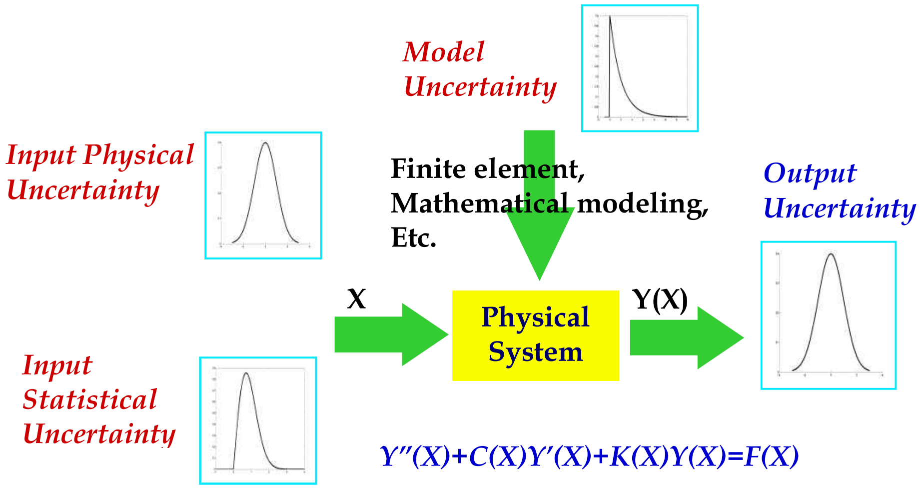


Figure 4.3: Uncertainty Propagation through Physical System

Then, the probability of safety (L-Type) can be estimated by integrating the PDFs of the system performances over the safety region.

$$R = P\left\{Y(\mathbf{X},t) \geq Y_c\right\} = \int_{Y_c}^{\infty} f_Y(y)dy = \int \cdots \int_{Y(\mathbf{X}) \geq Y_c} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{32}$$

4.7 Methods for Probabilistic Performance Analysis (Frequentist)

4.7.1 General Model of Design under Uncertainty

The design under uncertainty can generally be defined as:

$$\begin{array}{ll} \text{Minimize} & \text{Cost}(\mathbf{d}) \\ \text{subject to} & P\left\{G_i\left\{\mathbf{X};\mathbf{d}(\mathbf{X})\right\} > 0\right\} < P_{f_i},\ i = 1,\cdots,nc \\ & \mathbf{d}_{\mathrm{L}} \leq \mathbf{d} \leq \mathbf{d}_{\mathrm{U}},\quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr} \end{array} \tag{33}$$

where nc is the number of probabilistic constraints; nd is the number of design parameters; nr is the number of random variables; $\mathbf{d} = [d_i]^T = \mu(\mathbf{X})$ is the design vector; $\mathbf{X} = [X_i]^T$ is the random vector; and the probabilistic constraints are described by the performance function $G_i\left\{\mathbf{X};\mathbf{d}(\mathbf{X})\right\}$, their probabilistic models, and the probability of failure. The probability of failure is defined as $P_f \equiv \Phi(-\beta_t)$ with a target reliability index β_t where the failure is defined as $G_i\left\{\mathbf{X};\mathbf{d}(\mathbf{X})\right\} = Y_c - Y_i(\mathbf{X};\mathbf{d}(\mathbf{X})) > 0$ for L-type. The design procedure under uncertainty is graphically illustrated in Fig. 4.9.

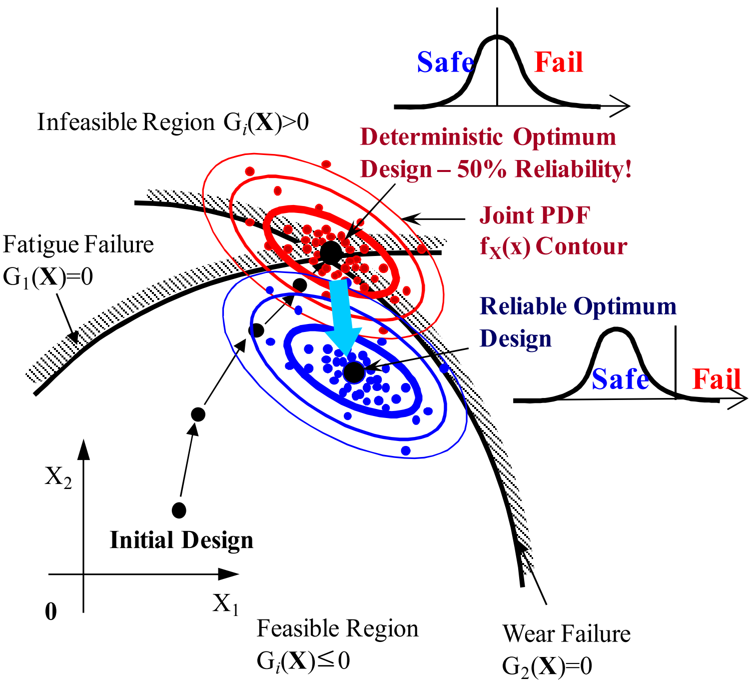


Figure 4.4: Design under Uncertainty

The probability of failure is defined as

$$\begin{aligned} P(G(\mathbf{X})>0) &= 1-P(G(\mathbf{X})\le 0) \\ &= 1-F_G(0) \\ &= \int_{G(\mathbf{X})>0}\cdots\int f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}, \quad \mathbf{X}\in R^{nv} \end{aligned} \tag{34}$$

The reliability (or the probability of safety) is inversely defined as

$$\begin{aligned} P(G(\mathbf{X})\le 0) &= F_G(0) \\ &= \int_{G(\mathbf{X})\le 0}\cdots\int f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}, \quad \mathbf{X}\in R^{nr} \end{aligned} \tag{35}$$

Figure 4.5 explains both the probability of failure and reliability.

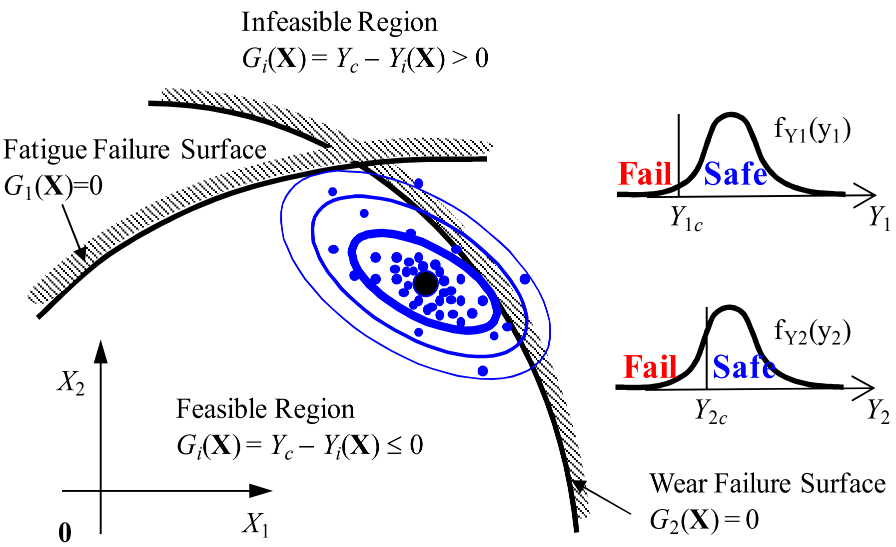


Figure 4.5: Reliability or Probability of Safety

The statistical description of the safety (or failure) of the performance function $G_i(\mathbf{X})$ requires a reliability analysis and is expressed by the CDF $F_{G_i}(0)$ of the constraint as

$$P(G_i(\mathbf{X}) \le 0) = F_{G_i}(0) \ge \Phi(\beta_{t_i}) \text{ or } R_{t_i}$$
$$\text{Time-dependent: } G_i(\mathbf{X},T) = T_d - T_i \le 0 \quad \text{where } T_d \text{ is a designed life.} \tag{36}$$
$$\text{Time-independent: } G_i(\mathbf{X}) = P_i - P_c \le 0 \quad \text{where } P_c \text{ is a critical buckling load.}$$

where the probability of the safety constraint $G_i(\mathbf{X}) \le 0$ is described as

$$F_{G_i}(0) = \int_{-\infty}^{0} f_{G_i}(g_i)dg_i = \int_{G_i(\mathbf{X})\le 0} \cdots \int f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}, \quad i = 1, \cdots, nc \text{ and } \mathbf{x} \in R^{nr} \tag{37}$$

In Eq. (37), $f_{\mathbf{X}}(\mathbf{x})$ is the joint PDF of all random parameters and the evaluation of Eq. (37) involves multiple integration. Neither analytical multi-dimensional integration nor direct numerical integration is possible for large-scale engineering applications. Existing approximate methods for probability analysis can be categorized into four groups: 1) sampling method; 2) expansion method; 3) the most probable point (MPP)-based method; and 4) stochastic response surface method.

4.7.2 Random sampling techniques (Monte Carlo simulation)

Let us recall the reliability or the probability of safety as

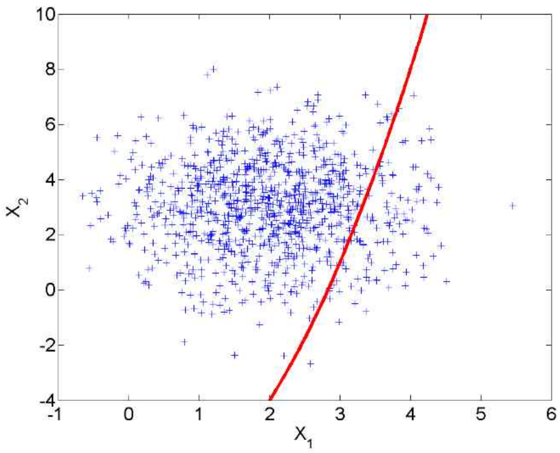
$$P\{G_i(\mathbf{X};\mathbf{d}) \le 0\} = F_{G_i}(0) = \int_{G_i(\mathbf{X})\le 0} \cdots \int f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \approx \frac{\text{Number of safe trials}}{\text{Number of total trials}} \tag{38}$$

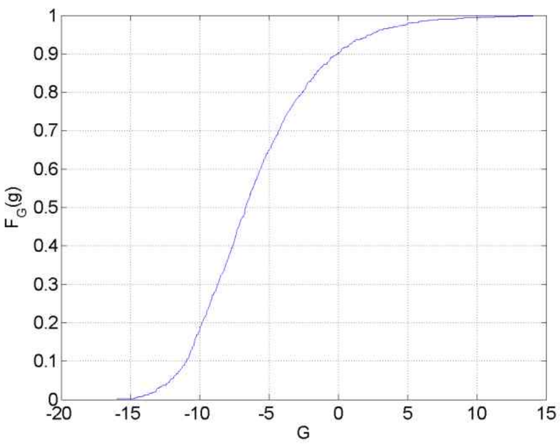
Or, inversely, the probability of failure can be obtained as

$$P\{G_i(\mathbf{X};\mathbf{d}) > 0\} = 1 - F_{G_i}(0) = \int_{G_i(\mathbf{X})> 0} \cdots \int f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \approx \frac{\text{Number of failure trials}}{\text{Number of total trials}} \tag{39}$$

- Simple but extremely expensive
- Seldom used due to its computational intensiveness, but used for a benchmark study
- To estimate a failure rate,

$$p_f = \frac{N_f}{N}, \quad N_f: \text{Number of failed samples} \quad N: \text{Number of total samples}$$






```
% generate random samples
>> m=[2 3];
>> s=[1 0;0 3];
>> n=1000;
>> x=mvnrnd(m,s,n);
>> plot(x(:,1),x(:,2),'+')

% plot a failure surface
>> [x1,x2] = meshgrid(-1:.1:6,-4:.2:10);
>> gg=x1.^2-x2-8;
>> v=[0 0];
>> [C,h]=contour(x1,x2,gg,v)
```

```
% calculate reliability
>> ns=0;
>> for i = 1:1000
       g(i) = x(i,1)^2-x(i,2)-8;
       if g(i) <= 0
           ns = ns + 1
       end
   end
>> rel = ns/n
>> cdfplot(g)
```

Homework 12: *Monte Carlo Simulation*

Consider the following simply supported beam subject to a uniform load, as illustrated in Figure below. Suppose L = 5 m and w=10 kN/m.

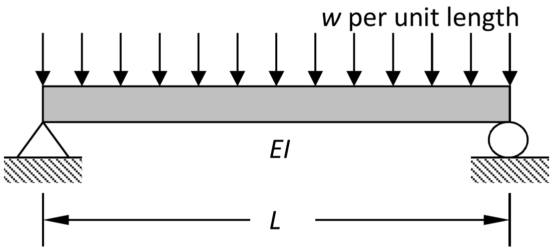


Figure 4.6: Simply Supported Beam

Random Vector:

$$EI = X_1 \sim N(\mu_{X_1} = 3\times 10^7, \sigma_{X_1} = 10^5)$$
$$w = X_2 \sim N(\mu_{X_2} = 10^4, \sigma_{X_2} = 10^3)$$

The maximum deflection of the beam is shown as

$$Y = g(X_1, X_2) = -\frac{5X_2L^4}{384X_1}$$

Determine the PDF (or CDF) of the maximum deflection and estimate its reliability using the MC simulation when the failure is defined as $Y < y_c = -3\times 10^{-3}$m.

4.7.3 Expansion methods

First-order method

Any nonlinear function (Y) can be linearized in terms of an input random vector $\mathbf{X} = \{X_1, \cdots, X_n\}^T$, i.e.,

$$\begin{aligned} Y(\mathbf{X}) &= Y(\boldsymbol{\mu}_\mathbf{X}) + \sum_{i=1}^{n} \frac{\partial Y(\boldsymbol{\mu}_\mathbf{X})}{\partial X_i}\left(X_i - \mu_{X_i}\right) + h.o.t. \\ &\approx a_1 X_1 + \cdots + a_n X_n + b \\ \text{or} \quad Y &\approx \mathbf{a}^T \mathbf{X} + b \end{aligned} \tag{40}$$

where $\mathbf{a} = \{a_1, \cdots, a_n\}^T$ is a sensitivity vector of Y.

- Mean of Y

$$\begin{aligned} E[Y] = \mu_Y &\approx E[\mathbf{a}^T \mathbf{X} + b] \\ &= E[\mathbf{a}^T \mathbf{X}] + E[b] \\ &= \mathbf{a}^T E[\mathbf{X}] + b \\ &= \mathbf{a}^T \mu_{\mathbf{X}} + b \end{aligned}$$

- Variance of Y

$$\begin{aligned} Var[Y] = \sigma_Y^2 &= E[(Y-\mu_Y)^2] \\ &= E[(Y-\mu_Y)(Y-\mu_Y)^T] \\ &\approx E[(\mathbf{a}^T\mathbf{X} + b - \mathbf{a}^T\boldsymbol{\mu}_{\mathbf{X}} - b)(\mathbf{a}^T\mathbf{X} + b - \mathbf{a}^T\boldsymbol{\mu}_{\mathbf{X}} - b)^T] \\ &= \mathbf{a}^T E[(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})^T]\mathbf{a} \\ &= \mathbf{a}^T \Sigma_{\mathbf{X}} \mathbf{a} \end{aligned}$$

- Generalization

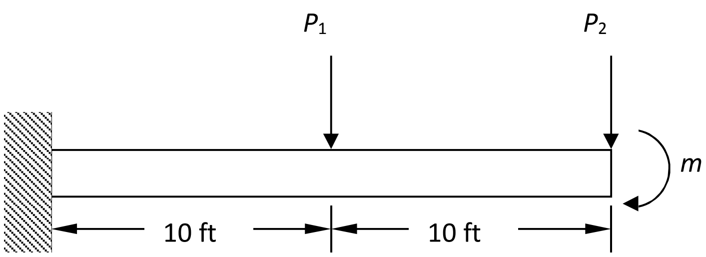
Let $\mathbf{Y} \in R^m$ be a random response vector of interest, which is related to input $\mathbf{X} \in R^n$. The linear system is given in the following equation.

$$\mathbf{Y} \approx \mathbf{A}^T \mathbf{X} + \mathbf{B}$$

where $\mathbf{A} \in R^n \times R^m$ and $\mathbf{B} \in R^m$ are coefficient matrix and vector, respectively. Let $\boldsymbol{\mu}_{\mathbf{Y}} \in R^m$ and $\boldsymbol{\Sigma}_{\mathbf{Y}} \in R^m \times R^m$ be the mean vector and covariance matrix of output $\mathbf{Y}$. Then,

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{Y}} &\approx \mathbf{A}^T \boldsymbol{\mu}_{\mathbf{X}} + \mathbf{B} \\ \boldsymbol{\Sigma}_{\mathbf{Y}} &\approx \mathbf{A}^T \boldsymbol{\Sigma}_{\mathbf{X}} \mathbf{A} \end{aligned} \tag{41}$$

Exercise: Cantilever Beam



Given that:

$P_1 \sim N\left[1000 \text{ lb}, 100^2 \text{ lb}^2\right], P_2 \sim N\left[500 \text{ lb}, 50^2 \text{ lb}^2\right], m = 10000 \text{ lb-ft, deterministic}$

Assume P_1 and P_2 are uncorrelated. Calculate mean, standard deviation, and coefficient of variation (COV) of maximum moment at the fixed end. Estimate the reliability when an allowable bending moment (M_a) is 33,000 lb-ft.

Solution: At fixed end, the maximum moment is expressed as

$$M_{\max} = 10P_1 + 20P_2 + 10000 = \mathbf{a}^T\mathbf{X} + b$$

where $\mathbf{a}^T = \{10 \quad 20\}$, $b = 10000$ lb-ft,

$$\mathbf{X} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \text{lb}, \quad \mu_X = \begin{Bmatrix} 1000 \\ 500 \end{Bmatrix} \text{lb}, \quad \mathbf{\Sigma_X} = \begin{bmatrix} 100^2 & 0 \\ 0 & 50^2 \end{bmatrix} \text{lb}^2$$

Mean of $M_{\max}$

$$\mu_{M_{\max}} = \mathbf{a}^T\mu_{\mathbf{X}} + b = \{10 \quad 20\}\begin{Bmatrix} 1000 \\ 500 \end{Bmatrix} + 10000$$
$$= 30,000 \text{ lb-ft}$$

Variance of $M_{\max}$

$$\sigma^2_{M_{\max}} = \mathbf{a}^T\Sigma_{\mathbf{X}}\mathbf{a} = \{10 \quad 20\}\begin{bmatrix} 100^2 & 0 \\ 0 & 50^2 \end{bmatrix}\begin{Bmatrix} 10 \\ 20 \end{Bmatrix}$$
$$= 2\times10^6 \ [\text{lb-ft}]^2$$
$$\sigma_{M_{\max}} = \sqrt{2\times10^6} = 1414.2 \text{ lb-ft}$$

Coefficient of Variation (COV) of $M_{\max}$

$$\text{COV} \equiv \frac{\text{standard deviation}}{\text{mean}} = \frac{1414.2}{30000} = 0.047 \text{ or } 4.7\%$$

Reliability with an allowable moment M_a = 33,000 lb-ft

$$\begin{aligned} R &= P(M_{\max} \le M_a) \\ &= P\left(\frac{M_{\max} - \mu_{M_{\max}}}{\sigma_{M_{\max}}} \le \frac{M_a - \mu_{M_{\max}}}{\sigma_{M_{\max}}}\right) \\ &= P\left(z \le \frac{33,000 - 30,000}{1414} = 2.12\right) \\ &= \Phi(2.12) = 98.3\% \end{aligned}$$

Homework 13: *Expansion method*

Recall Homework 12. Estimate its reliability using the expansion method when the failure is defined as $Y < y_c = -3\times10^{-3}$m.

Second-order method

Second-order approximation of any nonlinear function (Y) can be used for the second-order method as

$$Y(\mathbf{X}) \approx Y(\boldsymbol{\mu}_\mathbf{X}) + \sum_{i=1}^{n} \frac{\partial Y(\boldsymbol{\mu}_\mathbf{X})}{\partial X_i}\left(X_i - \mu_{X_i}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 Y(\boldsymbol{\mu}_\mathbf{X})}{\partial X_i \partial X_j}\left(X_i - \mu_{X_i}\right)\left(X_j - \mu_{X_j}\right) \tag{42}$$

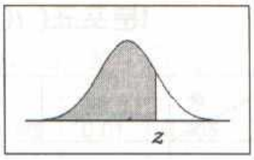
- Mean of Y

$$E[Y] = \mu_Y \sim Y(\mu_X) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 Y(\mu_X)}{\partial X_i^2}\sigma_{X_i}^2$$

- Variance of Y

$$Var[Y] = \sigma_Y^2 \cong \sum_{i=1}^{n}\left(\frac{\partial Y(\boldsymbol{\mu}_\mathbf{X})}{\partial X_i}\right)^2 \sigma_X^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 Y(\boldsymbol{\mu}_\mathbf{X})}{\partial X_i \partial X_j}\sigma_{X_i}^2\sigma_{X_j}^2$$

3. 확률분포표



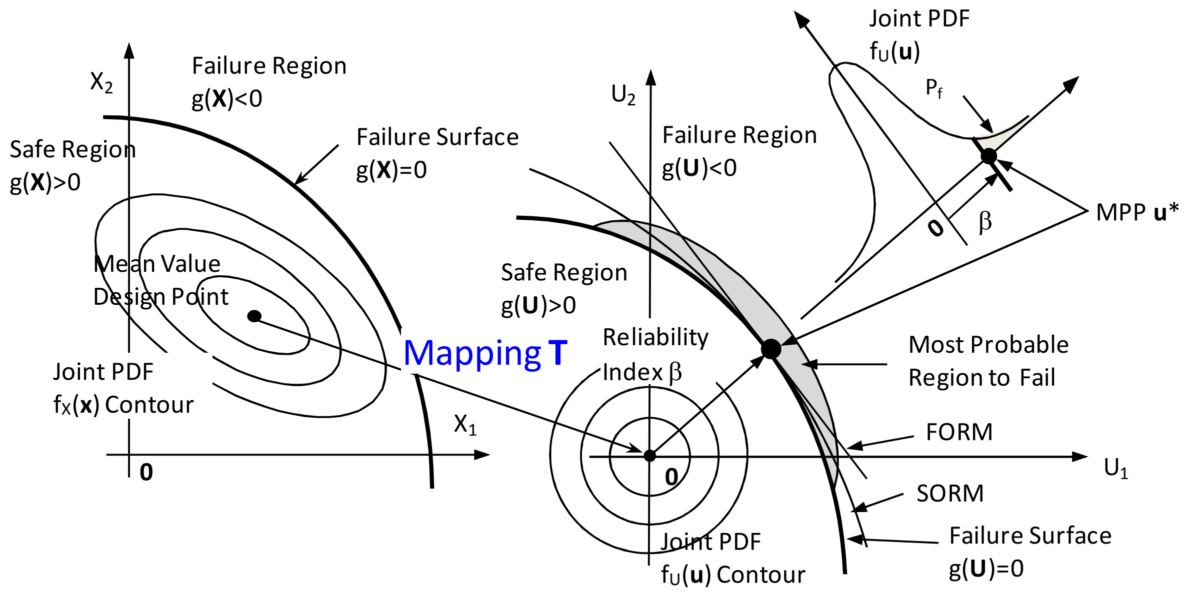
표준정규분포표

$(\Pr(Z \leq z) = \Phi(z),\ Z \sim N(0, 1))$

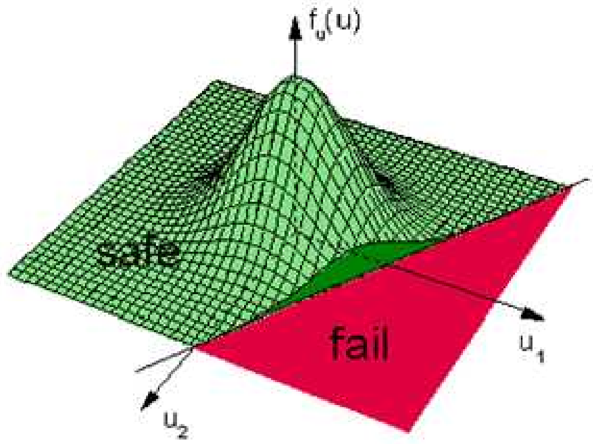
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

4.7.4 Most Probable Point (MPP) based methods

Most probable point (MPP) based methods include the first order reliability method (FORM) and second order reliability method (SORM). Instead of approximating a response Y at the mean of $\mathbf{X}$, it approximates the function at the most probable point in either a linear or quadratic manner. This is illustrated in Figure 4.6. The MPP is a pointwise representation of the failure surface and normally computed in a transformed space (or standard Gaussian space). In the MPP based methods, the reliability analysis requires a transformation **T** from the original random parameter **X** to the independent and standard normal parameter **U**. The constraint function $G(\mathbf{X})$ in X-space can then be mapped onto $G(\mathbf{T}(\mathbf{X})) \equiv G(\mathbf{U})$ in U-space. Rosenblatt transformation is most widely used for transforming any non-normally distributed random vector to standard normal random vector.



(a) Nonlinear Transformation of Non-normal Distributions



(b) First-Order Reliability Method

Figure 4.6: Nonlinear Transformation of Non-normal Distributions

Table 4.1: Nonlinear Transformation, T: X → U

	Parameters	PDF	Transformation
Normal	μ = mean, σ = standard deviation	$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-0.5[(x-\mu)/\sigma]^2}$, $-\infty \le x \le \infty$	$X=\mu+\sigma U$
Lognormal	μ = mean, σ = standard deviation $\bar{\sigma}^2=\ln\left[1+(\sigma/\mu)^2\right]$ $\bar{\mu}=\ln(\mu)-0.5\bar{\sigma}^2$	$f(x)=\frac{1}{\sqrt{2\pi}x\bar{\sigma}}e^{-0.5[(\ln x-\bar{\mu})/\bar{\sigma}]^2}$, $x>0$	$X=e^{\bar{\mu}+\bar{\sigma}U}$
Weibull	$k>0,\ \mu=v\Gamma(1+1/k)$ $\sigma^2=v^2\left[\Gamma(1+2/k)-\Gamma^2(1+1/k)\right]$	$f(x)=\frac{k}{v}\left(\frac{x}{v}\right)^{k-1}e^{-(x/v)^k}$, $x>0$	$X=v[-\ln(\Phi(-U))]^{\frac{1}{k}}$
Gumbel	$\mu=v+(0.577/\alpha),\ \sigma=\pi/\sqrt{6\alpha}$	$f(x)=\alpha e^{-\alpha(x-v)-e^{-\alpha(x-v)}}$, $-\infty \le x \le \infty\sqrt{2}$	$X=v-\frac{1}{\alpha}\ln[-\ln(\Phi(U))]$
Uniform	$\mu=(a+b)/2,\ \sigma=(b-a)/\sqrt{12}$	$f(x)=\frac{1}{b-a}$, $a \le x \le b$	$X=a+(b-a)\Phi(U)$

where $\Phi(U)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{U}e^{-u^2/2}du$.

$P(G_i(\mathbf{X})\le 0)=F_{G_i}(0)\ge\Phi(\beta_{t_i})$ or R_{t_i}

Time-dependent: $G_i(\mathbf{X},T)=T_d-T_i\le 0$ where T_d is a designed life. (43)

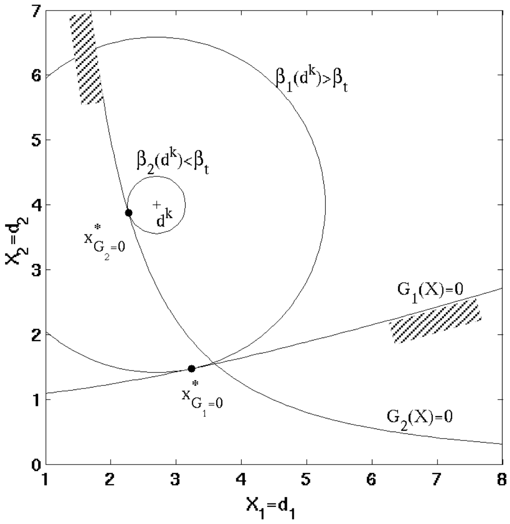
Time-independent: $G_i(\mathbf{X})=P_i-P_c\le 0$ where P_c is a critical buckling load.

The probabilistic constraint in Eq. (36) can be further expressed in two different ways through inverse transformations as (see Fig. 4.7):

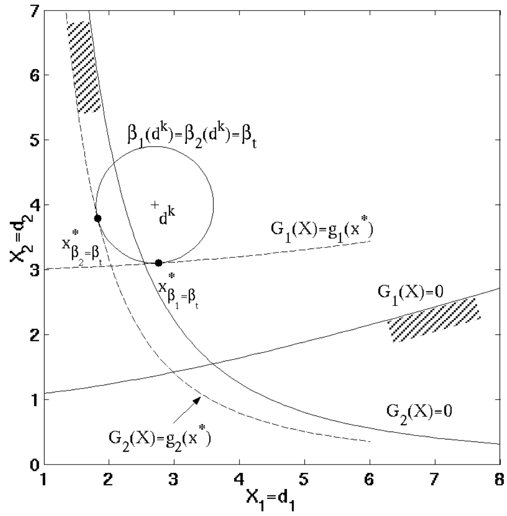
$$\text{RIA:}\quad \beta_{s_i}=\Phi^{-1}\left\{F_{G_i}(0)\right\}\ge\beta_{t_i} \tag{44}$$

$$\text{PMA:}\quad G_{p_i}=F_{G_i}^{-1}\left\{\Phi(\beta_{t_i})\right\}\le 0 \tag{45}$$

where β_{s_i} and G_{p_i} are respectively called the safety reliability index and the probabilistic performance measure for the i^{th} probabilistic constraint. Equation (44) is employed to prescribe the probabilistic constraint in Eq. (33) using the reliability measure, i.e. the so-called Reliability Index Approach (RIA). Similarly, Eq. (45) can replace the same probabilistic constraint in Eq. (33) with the performance measure, which is referred to as the Performance Measure Approach (PMA).



(a) MPP Search Space in RIA



(b) MPP Search Space in PMA

Figure 4.7: Random Search Space in RIA and PMA

Formulation for Reliability Index Approach (RIA)

In RIA, the first-order safety reliability index $\beta_{s,\text{FORM}}$ is obtained using FORM by formulating as an optimization problem with one equality constraint in U-space, which is defined as a limit state function:

$$\begin{array}{ll} \text{minimize} & \|\mathbf{U}\| \\ \text{subject to} & G(\mathbf{U})=0 \end{array} \tag{46}$$

where the optimum point on the failure surface is called the Most Probable Failure Point (MPFP) $\mathbf{u}^*_{G(\mathbf{U})=0}$, and thus $\beta_{s,\text{FORM}}=\left\|\mathbf{u}^*_{G(\mathbf{U})=0}\right\|$.

Either MPFP search algorithms specifically developed for the first-order reliability analysis, or general optimization algorithms can be used to solve Eq. (46). The HL-RF method is employed to perform reliability analyses in RIA due to its simplicity and efficiency.

HL-RF Method

The HL-RF method is formulated as follows

$$\begin{aligned} \mathbf{u}^{(k+1)} &= \left(\mathbf{u}^{(k)}\bullet\mathbf{n}^{(k)}-\frac{G(\mathbf{u}^{(k)})}{\left\|\nabla_U G(\mathbf{u}^{(k)})\right\|}\right)\mathbf{n}^{(k)} \\ &= \left[\nabla_U G(\mathbf{u}^{(k)})\bullet\mathbf{u}^{(k)}-G(\mathbf{u}^{(k)})\right]\frac{\nabla_U G(\mathbf{u}^{(k)})}{\left\|\nabla_U G(\mathbf{u}^{(k)})\right\|^2} \end{aligned} \tag{47}$$

where the normalized steepest ascent direction of $G(\mathbf{U})$ at $\mathbf{u}^{(k)}$

$$\mathbf{n}^{(k)} = \mathbf{n}(\mathbf{u}^{(k)}) = \frac{\nabla_U G(\mathbf{u}^{(k)})}{\left\|\nabla_U G(\mathbf{u}^{(k)})\right\|}$$

and the second term in Eq. (47) is introduced to account for the fact that $G(\mathbf{U})$ may be other than zero.

```
function [beta,dbeta]=HL_RF(x,kc)
    u=zeros(1,nd); iter=0;  Dif=1;
    while Dif >= 1d-5 & iter < 20
        iter=iter + 1;
        [ceq,GCeq]=cons(u,x,kc);
        u=(GCeq*u'-ceq)/norm(GCeq)^2*GCeq;
        U(iter,:)=u/norm(u);
        if iter>1
            Dif=abs(U(iter-1,:)*U(iter,:)' - 1);
        end
    end
    beta = norm(u);
    dbeta = -u./(beta*stdx);
end
```

Formulation for Performance Measure Approach (PMA)

Reliability analysis in PMA can be formulated as the inverse of reliability analysis in RIA. The first-order probabilistic performance measure $G_{p,\mathrm{FORM}}$ is obtained from a nonlinear optimization problem in U-space defined as

$$\begin{array}{ll} \text{maximize} & G(\mathbf{U}) \\ \text{subject to} & \|\mathbf{U}\| = \beta_t \end{array} \tag{48}$$

where the optimum point on a target reliability surface is identified as the Most Probable Point (MPP) $\mathbf{u}^*_{\beta=\beta_t}$ with a prescribed reliability $\beta_t = \left\|\mathbf{u}^*_{\beta=\beta_t}\right\|$, which will be referred to as MPP. Unlike RIA, only the direction vector $\mathbf{u}^*_{\beta=\beta_t} / \left\|\mathbf{u}^*_{\beta=\beta_t}\right\|$ needs to be determined by exploring the explicit sphere constraint $\|\mathbf{U}\| = \beta_t$.

General optimization algorithms can be employed to solve the optimization problem in Eq. (48). However, the Advanced Mean Value (AMV) method is well suited for PMA due to its simplicity and efficiency.

AMV method

Thus, the AMV method can be formulated as

$$\mathbf{u}^{(1)}_{\mathrm{AMV}} = \mathbf{u}^*_{\mathrm{MV}}, \quad \mathbf{u}^{(k+1)}_{\mathrm{AMV}} = \beta_t \mathbf{n}(\mathbf{u}^{(k)}_{\mathrm{AMV}}) \tag{49}$$

where

$$\mathbf{n}(\mathbf{u}_{\mathrm{AMV}}^{(k)}) = \frac{\nabla_U G(\mathbf{u}_{\mathrm{AMV}}^{(k)})}{\left\|\nabla_U G(\mathbf{u}_{\mathrm{AMV}}^{(k)})\right\|} \tag{50}$$

```
function [G,DG]=AMV(x,kc)
    u=zeros(1,nd); iter = 0; Dif=1;
    while Dif>1d-5 & iter<20
        iter=iter+1;
        if iter>1
            u=DG*bt/norm(DG);
        end
        [G,DG]=cons(u,x,kc);
        U(iter,:)=u/bt;
        if iter>1
            Dif=abs(U(iter,:)*U(iter-1,:)'-1);
        end
    end
end
```

Table 4.2: Properties of the RIA and PMA

	Properties
RIA	1. Good for reliability analysis 2. Expensive with sampling method and MPP-based method when reliability is high. 3. MPP-based method could be unstable when reliability is high or a performance function is highly nonlinear.
PMA	1. Good for design optimization. 2. Not suitable for assessing reliability. 3. Efficient and stable for design optimization.

Homework 14: *First-Order Reliability Method*

Consider the following simply supported beam subject to a uniform load, as illustrated in Fig. 4.13. Suppose L = 5 m and w=10 kN/m.

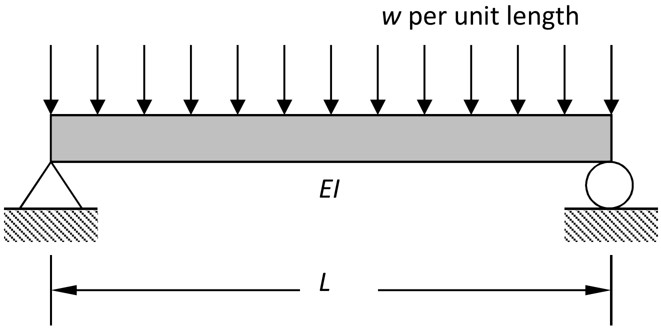


Figure 4.8: Simply Supported Beam

Random Vector:

$$EI = X_1 \sim N(\mu_{X_1} = 3\times10^7, \sigma_{X_1} = 10^5)$$
$$w = X_2 \sim N(\mu_{X_2} = 10^4, \sigma_{X_2} = 10^3)$$

The maximum deflection of the beam is shown as

$$Y = g(X_1, X_2) = -\frac{5X_2L^4}{384X_1}$$

Estimate its reliability using the MPP-based method (HL-RF) when the failure is defined as $Y < y_c = -3\times10^{-3}$m. Make your own discussion and conclusion.

4.7.5 Stochastic response surface method

Dimension reduction family:

Dimension reduction (DR) method simplifies a single multi-dimensional integration to multiple one-dimensional integration or multiple one- and two-dimensional integration using additive decomposition. This section introduces univariate dimension reduction (UDR) method.

For the approximation of the multi-dimensional integration, consider an integration of two dimensional function which can be expressed by the Taylor series expansion by

$$
\begin{aligned}
I\left(Y(x_1,x_2)\right) &= I\left(Y(0,0)\right) \\
&+\sum_{i=1}^{2}\frac{\partial Y}{\partial x_i}(0,0)\,\mathrm{I}\left[x_i\right] \\
&+\frac{1}{2!}\sum_{i=1}^{2}\frac{\partial^2 Y}{\partial x_i^2}(0,0)\,\mathrm{I}\left[x_i^2\right]+\frac{\partial^2 Y}{\partial x_1 \partial x_2}(0,0)\,\mathrm{I}\left[x_1x_2\right] \\
&+\frac{1}{3!}\sum_{i=1}^{2}\frac{\partial^3 Y}{\partial x_i^3}(0,0)\,\mathrm{I}\left[x_i^3\right]+\frac{1}{2!}\frac{\partial^3 Y}{\partial x_1^2 \partial x_2}(0,0)\,\mathrm{I}\left[x_1^2x_2\right]+\frac{1}{2!}\frac{\partial^3 Y}{\partial x_1 \partial x_2^2}(0,0)\,\mathrm{I}\left[x_1x_2^2\right] \\
&+\frac{1}{4!}\sum_{i=1}^{2}\frac{\partial^4 Y}{\partial x_i^4}(0,0)\,\mathrm{I}\left[x_i^4\right] \\
&+\frac{1}{3!}\frac{\partial^4 Y}{\partial x_1^3 \partial x_2}(0,0)\,\mathrm{I}\left[x_1^3x_2\right]+\frac{1}{3!}\frac{\partial^4 Y}{\partial x_1^2 \partial x_2^2}(0,0)\,\mathrm{I}\left[x_1^2x_2^2\right]+\frac{1}{3!}\frac{\partial^4 Y}{\partial x_1 \partial x_2^3}(0,0)\,\mathrm{I}\left[x_1x_2^3\right]+\cdots
\end{aligned}
$$

where integration term can be defined as

$$I\left[Y\left(x_1,x_2\right)\right]=\int_{-a}^{+a}\int_{-a}^{+a}Y(x_1,x_2)dx_1dx_2$$

Because integrations of the odd functions are zero, the integration of Taylor series expansion of the target function (Y) can be expressed as:

$$
\begin{aligned}
I\left[Y\left(x\right)\right]=I\left[Y(0)\right]&+\frac{1}{2!}\sum_{i=1}^{N}\frac{\partial^2 Y}{\partial x_i^2}(0)\,\mathrm{I}\left[x_i^2\right] \\
&+\frac{1}{4!}\sum_{i=1}^{N}\frac{\partial^4 Y}{\partial x_i^4}(0)\,\mathrm{I}\left[x_i^4\right] \\
&+\frac{1}{2!2!}\sum_{i<j}^{N}\frac{\partial^4 Y}{\partial x_i^2 \partial x_j^2}(0)\,\mathrm{I}\left[x_i^2x_j^2\right]+\cdots
\end{aligned}
$$

where $I(\bullet)$ calculates integration over the given space.

This is also computationally expensive because of the terms including multi-dimensional integration such as $\mathrm{I}\left[x_i^2x_j^2\right]$. To effectively remove the terms with multi-dimensional integration, additive decomposition, Y_a, is defined as:

$$
\begin{aligned}
&Y(X_1,\ldots,X_N)\cong Y_a(X_1,\ldots X_N) \\
&=\sum_{j=1}^{N}Y\left(\mu_1,\ldots,\mu_{j-1},X_j,\mu_{j+1},\ldots,\mu_N\right)-(N-1)\,\mathrm{Y}\left(\mu_1,\ldots,\mu_N\right)
\end{aligned}
$$

Integration of Taylor series expansion of the additive decomposition (Y_a) can be expressed as:

$$
\begin{aligned}
I\left[Y_a\left(x\right)\right]=I\left[Y(0)\right]&+\frac{1}{2!}\sum_{i=1}^{N}\frac{\partial^2 Y}{\partial x_i^2}(0)\,\mathrm{I}\left[x_i^2\right] \\
&+\frac{1}{4!}\sum_{i=1}^{N}\frac{\partial^4 Y}{\partial x_i^4}(0)\,\mathrm{I}\left[x_i^4\right] \\
&+\frac{1}{6!}\sum_{i=1}^{N}\frac{\partial^6 Y}{\partial x_i^6}(0)\,\mathrm{I}\left[x_i^6\right]+\cdots
\end{aligned}
$$

This results the largest error at the fourth even-order term, producing negligible error.

$$I[Y(x)] - I\left[Y_a(x)\right] = \frac{1}{2!2!}\sum_{i<j}^{N} \frac{\partial^4 Y}{\partial x_i^2 \partial x_j^2}(0)\, \mathrm{I}\left[x_i^2 x_j^2\right] + \cdots$$

For probabilistic engineering analysis, the m^{th} statistical moments for the responses are considered as

$$\begin{aligned} E\left[Y^m(X)\right] &\cong E\left[Y_a^m(X)\right] \\ &= E\left\{\left[\sum_{j=1}^{N} Y\left(\mu_1,\ldots,\mu_{j-1},X_j,\mu_{j+1},\ldots,\mu_N\right) - (N-1)\,\mathrm{Y}\left(\mu_1,\ldots,\mu_N\right)\right]^m\right\} \end{aligned}$$

Applying the Binomial formula on the right-hand side of the equation above gives

$$\begin{aligned} m_l \cong \sum_{i=0}^{l} \binom{l}{i} \mathscr{E}\left\{\sum_{j=1}^{N} Y(\mu_1,\ldots,\mu_{j-1},X_j,\mu_{j+1},\ldots,\mu_N)\right\}^i \\ [-(N-1)y(\mu_1,\ldots,\mu_N)]^{l-i}. \end{aligned}$$

One-dimensional integration will be performed with integration weights $w_{j,i}$ and points $x_{j,i}$ as

$$\begin{aligned} &E\left[\sum_{j=1}^{N} Y^m\left(\mu_1,\ldots,\mu_{j-1},X_j,\mu_{j+1},\ldots,\mu_N\right)\right] \\ &\cong \sum_{j=1}^{N}\sum_{i=1}^{n} w_{j,i} Y^m\left(\mu_1,\ldots,\mu_{j-1},x_{j,i},\mu_{j+1},\ldots,\mu_N\right) \end{aligned}$$

where N is the number of input random parameters and n is the number integration point along each random variable. An empirical sample point distribution for the UDR when m = 3 is shown in the Fig. 4-DR1. We can see that, compared to the full factorial sample points, the UDR achieves a significant reduction in the number of sample points.

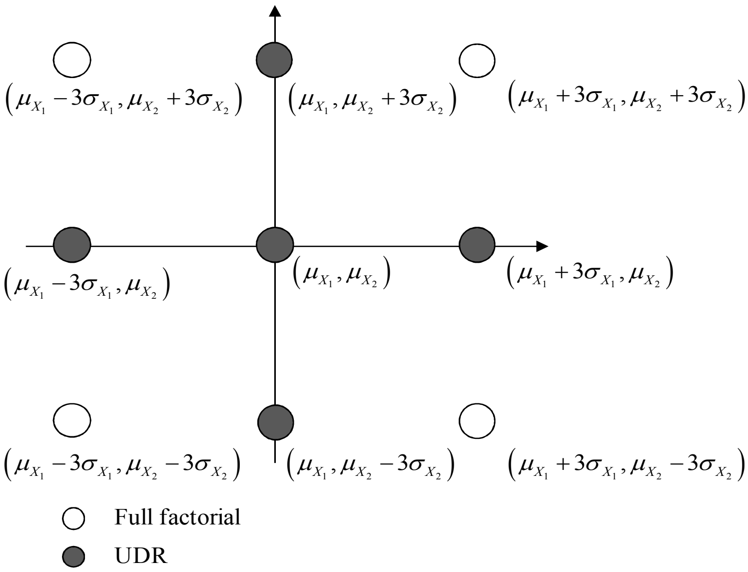


Fig. 4-DR1. Empirical sample point distribution for UDR (m=3)

- Refer to http://www.sciencedirect.com/science?_ob=ArticleURL&_udi=B6V4M-4H74MB0-1&_user=961305&_rdoc=1&_fmt=&_orig=search&_sort=d&view=c&_acct=C000049425&_version=1&_urlVersion=0&_userid=961305&md5=6e56b71561720cfe918f32c3eaa2cf86
- Refer to http://www.springerlink.com/content/416l79447313n8q1

Polynomial Chaos Expansion (PCE) method
Tensor-Product (or Stochastic Collocation) Method

4.8 Bayesian Description of Time-Independent Performance

When modeling uncertainties with insufficient data, the probability of safety (or satisfying a specification), referred to as reliability, must be uncertain and subjective. Because the Bayes theory provides a systematic framework of aggregating and updating uncertain information, reliability analysis based on the Bayes theory, referred to as Bayesian reliability, is employed to deal with subjective and insufficient data sets.

4.8.1 Bayesian binomial inference - reliability

<u>• Bayesian binomial inference</u>

If the probability of a safety event in each sample is r and the probability of failure is (1-r), then the probability of x safety occurrences out of a total of N samples can be described by the probability mass function (PMF) of a Binomial distribution as

$$\Pr\left(X=x,N\middle|r\right)=\binom{N}{x}r^{x}\left(1-r\right)^{N-x},\qquad x=0,1,2,\ldots,N \tag{51}$$

When r is an uncertain parameter and a prior distribution is provided, a Bayesian inference process can be employed to update r based on the outcomes of the sample tests. It is possible to obtain a posterior distribution with any type of a prior distribution. A Bayesian inference model is called a conjugate model if the conjugate prior distribution is used. For conjugate Bayesian inference models, the updating results are independent of the sequence of data sets.

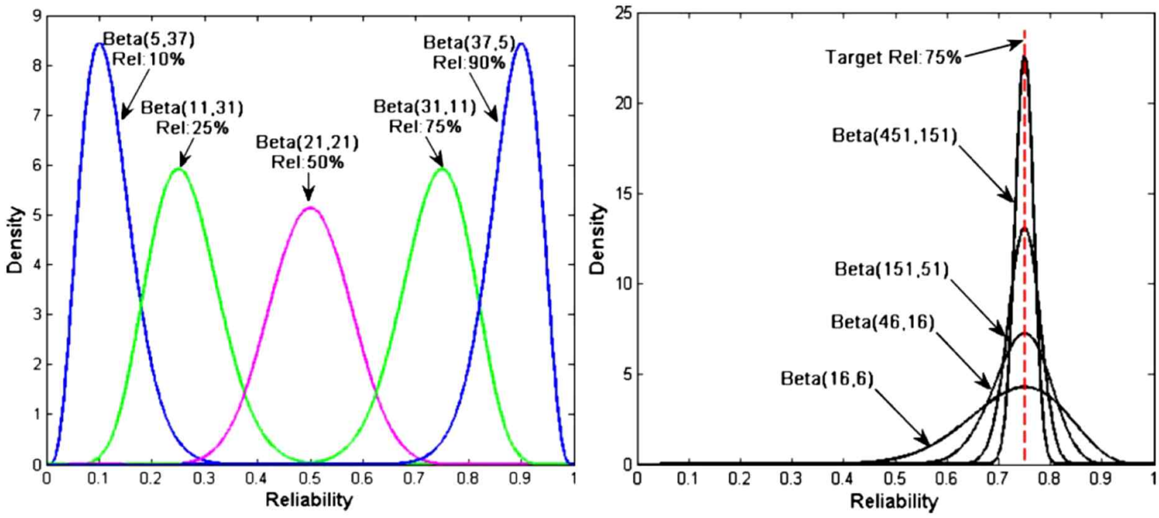
• <u>Conjugate prior reliability distribution</u>

For Bayesian reliability analysis, both prior reliability distribution (r) and the number (x) of safety occurrences out of the total number of test data set N must be known. If prior reliability distribution (r) is unavailable, it will be simply modeled with a uniform distribution, $r \sim U(a, b)$ where $a<b$ and $a, b \in [0, 1]$. In all cases, reliability will be modeled with Beta distribution, the conjugate distribution of the Bayesian binomial inference, because the uniform distribution is a special case of the Beta distribution.

$$f\left(r|x\right)=\frac{1}{B(a,b)}r^{a-1}(1-r)^{b-1}, \quad B(a,b): \text{ Beta function} \tag{52}$$

where $a = x + 1$ and $b = N - x + 1$. The larger the number of safety occurrences for a given N samples, the greater the mean of reliability, as shown in Figure 4.9 (a). As the total number of samples is increased, the variation of reliability is decreased, as shown in Figure 4.9 (b).

In Bayesian inference model, the binomial distribution likelihood function is used for test data, whereas the conjugate prior distribution of this likelihood function is used for reliability (r), which is a beta distribution. However, it is found that the Bayesian updating results often depend on the selection of a prior distribution in the conjugate models. Besides, the available conjugate Bayesian models are limited. To eliminate the dependency and the limitation, a non-conjugate Bayesian updating model can be developed using Markov chain Monte Carlo methods. This is, however, more computationally intensive.



(a) (b)

Figure 4.9: Dependence of the PDF of reliability on the number of safety occurrences, *x* and the total number of samples, *N*

4.8.2 Definition of Bayesian reliability

Bayesian reliability must satisfy two requirements: (a) sufficiency and (b) uniqueness. The sufficiency requirement means that the Bayesian reliability must be no larger than an exact reliability, when it is realized with a sufficient amount of data for input uncertainties. The uniqueness requirement means that the Bayesian reliability must be uniquely defined for the purpose of design optimization. To meet these two requirements, Bayesian reliability is generally defined with a confidence level of reliability prediction where the confidence level C_L of Bayesian reliability is defined as

$$C_L = \Pr(R > R_B) = \int_{R_B}^{1} f(r|\mathbf{\bar{x}})dr = 1 - F_R(R_B) \tag{53}$$

With the predefined confidence level C_L, Bayesian reliability can be defined as

$$R_B = F_R^{-1}[1 - C_L] \tag{54}$$

Therefore, Bayesian reliability can be formulated as a function of a predefined confidence level. Bayesian reliability is desirable since it is defined from the reliability distribution with a corresponding confidence level and accounts for reliability modeling error due to the lack of data.

To guarantee the sufficiency requirement, extreme distribution theory for the smallest reliability value is employed. Based on the extreme distribution theory, the extreme distribution for the smallest reliability value is constructed from the reliability distribution, beta distribution. For random reliability R, which follows the beta distribution, $F_R(r)$, let R_1 be the smallest value among N data points, the CDF of the smallest reliability value, R_1, can be expressed as

$$1 - F_{R_1}(r) = \Pr(R_1 > r) = \Pr(R_1 > r, R_2 > r, \ldots, R_N > r) \tag{55}$$

Since the ith smallest reliability values, $R_i(i=1, \ldots, N)$, are identically distributed and statistically independent, the CDF of the smallest reliability value becomes

$$F_{R_1}(r) = 1 - [1 - F_R(r)]^N \tag{56}$$

Then Bayesian reliability, R_B, is uniquely determined as the median value of the extreme distribution. Based on this definition, Bayesian reliability and its confidence level can be respectively obtained as the solution of the nonlinear equation, by setting $F_{R_1}(R_B) = 0.5$

$$R_B = F_R^{-1}\left[1 - \sqrt[N]{1 - F_{R_1}(R_B)}\right] = F_R^{-1}\left[1 - \sqrt[N]{0.5}\right] \tag{57}$$

$$C_L = 1 - F_R(R_B) = 1 - F_R\left(F_R^{-1}\left[1 - \sqrt[N]{0.5}\right]\right) = \sqrt[N]{0.5} \tag{58}$$

The Beta distribution for reliability, its extreme distribution for the smallest reliability value, and the Bayesian reliability are graphically shown as below.

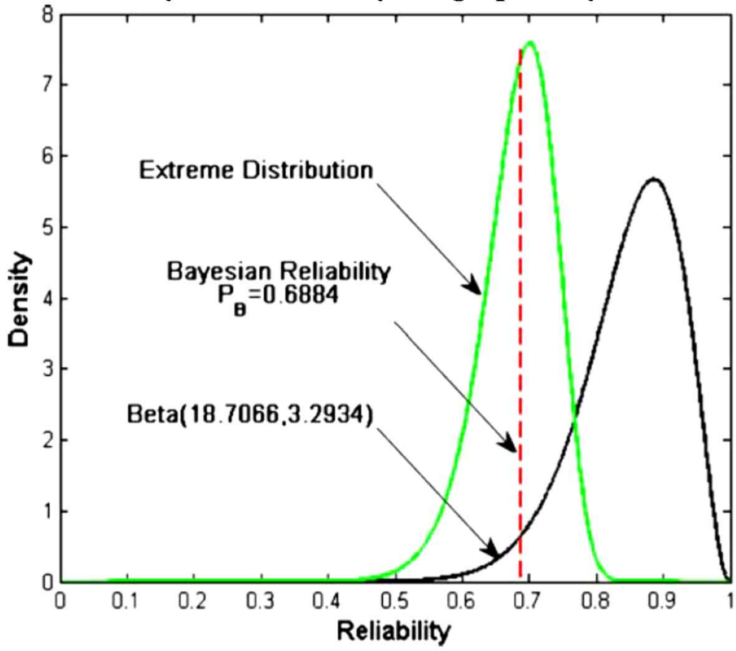


Figure 4.10: Bayesian reliability

4.8.3 Numerical procedure of Bayesian reliability analysis

Bayesian reliability analysis can be conducted using a numerical procedure as follows.

• Step 1: collect a limited data set for epistemic uncertainties where the data size is *N*.

• Step 2: calculate reliabilities (R_k) with consideration of aleatory uncertainties at all epistemic data points.

• Step 3: build a distribution of reliability using the beta distribution with aleatory and/or epistemic uncertainties.

• Step 4: select an appropriate confidence level, C_L, of Bayesian reliability.

• Step 5: determine the Bayesian reliability.

Refer to http://www.springerlink.com/content/u1185070336p4116/fulltext.pdf.

Homework 15: *Bayesian Reliability Analysis*

Consider the following simply supported beam subject to a uniform load, as illustrated in Fig. 4.14. Suppose L = 5 m and w=10 kN/m.

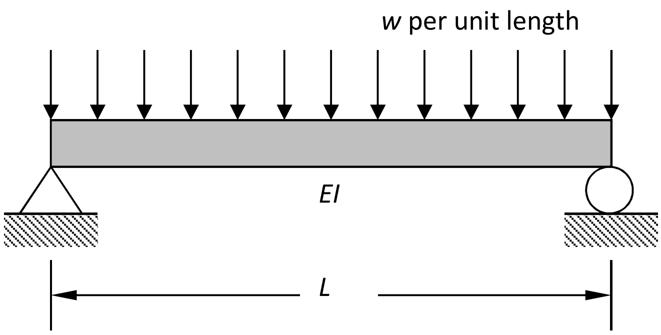


Figure 4.11: Simply Supported Beam

Random Vector:

$$EI = X_1 \sim N(\mu_{X_1} = 3\times10^7, \sigma_{X_1} = 10^5)$$
$$w = X_2 \sim \text{epitemic}$$

The maximum deflection of the beam is shown as

$$Y = g(X_1, X_2) = -\frac{5X_2L^4}{384X_1}$$

The X_2 is an epistemic uncertainty. For X_2, it is assumed that 10 data sets are gradually obtained at different times. Using MPP-based method (HL-RF), determine the reliability of the maximum deflection constraint, $P(Y(X_1) \geq y_c = -3\times10^{-3}\text{m})$, at all individual X_2 points in the table. Predict the PDF of reliability in a Bayesian sense using the first 10 data set and gradually update the PDFs of reliability using the second and third data sets. Make your own discussion and conclusion, and attach your code used for Bayesian reliability analysis.

Table 4.3 Three sets of 10 data for X_2 ($\times10^4$)

Set1	1.0000	0.8126	1.0731	1.0677	0.9623	0.9766	1.1444	1.0799	1.0212	0.9258
Set2	0.9682	1.0428	1.0578	1.0569	0.9704	1.0118	0.9649	1.0941	1.0238	1.1082
Set3	1.1095	1.0896	1.0040	0.9744	0.8525	1.0315	1.0623	0.9008	0.8992	0.9869