

항공기 구조 역학

CHAPTER 4. Engineering structural analysis

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4.3 Hyperstatic systems

❖ “Statically determinate” or “Isostatic”

Total No. of unknown internal forces = No. of equilibrium equations

- Reaction forces
- Forces acting in the members

- Unknown forces can be determined from the equations of equilibrium alone, without using the strain-displacement relation, constitutive laws
- Example 4.1

❖ “Statically indeterminate” or “hyperstatic” systems

Total No. of unknown forces > No. of equilibrium equations

❖ “degree of redundancy” N_R

$N_R =$ Total No. of unknown internal forces - No. of equilibrium equations

- Example 4.2
- Simultaneous solution of the 3 fundamental groups of equations

4.3 Hyperstatic systems

❖ Difference between “isostatic” and “hyperstatic” systems

➤ Solution procedure

- isostatic
 - equations of equilibrium are only needed
- hyperstatic
 - Equilibrium equations cannot be solved independently of the other 2 sets of equations of elasticity
- 2 main approaches
 - the force method
 - the displacement method

➤ Nature of the solution for the unknown internal forces

- isostatic
 - internal forces can be expressed in terms of the externally applied forces.
 - internal force distribution is independent of the stiffness characteristics of the structure
- hyperstatic
 - internal forces depend on the applied loads, but also on the stiffness of the structure
 - internal force distribution depends on the stiffness characteristics of the structure

4.3 Hyperstatic systems

- • Hyperstatic : “dual load paths”
 - Equilibrium equations are not sufficient to determine how much of the load will be carried by load path 1, 2, ...
 - According to their relative stiffness, the stiffer load path will carry more load than the more compliant one
 - More damage tolerant
- Isostatic : “single load path”

4.3 Hyperstatic systems

4.3.2 The displacement or stiffness method

❖ Expressing the governing eqns in terms of displacement

- ① Equilibrium equations of the system : free body diagrams
- ② Constitutive laws : express internal forces in terms of member deformations or strains
- ③ Strain-displacement equations : express system deformation in terms of displacements
- ④ Introduce ③→②: find the internal forces in terms of displacements
- ⑤ Introduce ④→①: yield the equations of equilibrium in terms of displacements
- ⑥ Solve ⑤: find the displacement of the system
- ⑦ Find system deformations : back-substitute the displacements into ③
- ⑧ Find system internal forces : back-substitute the deformations into ②

4.3 Hyperstatic systems

4.3.3 The force or flexibility method

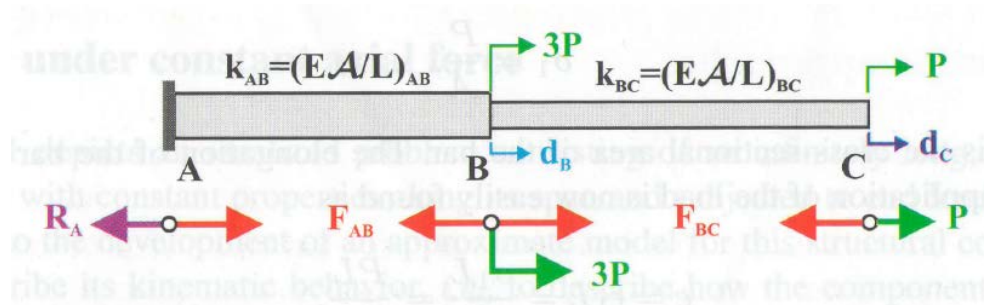
- ❖ **Focuses on the solution of the system internal forces, strains and displacements are then recovered**
 - ① Equilibrium equations of the system
 - ② Determine N_R
 - ③ Cut the system at N_R locations and define a single relative displacement for each of the cuts
 - originally hyperstatic system is transformed into an isostatic system
 - ④ Apply N_R redundant forces, each along the relative displacements. Express all internal forces in terms of the applied loads and N_R redundant forces
 - ⑤ Constitutive laws : express system deformations in terms of N_R redundant forces
 - ⑥ Strain-displacement equations : express the relative displacements at N_R cuts in terms of N_R redundant forces
 - ⑦ Impose vanishing of the relative displacements at N_R cuts
 - ⑧ Recover system deformations and system displacements

4.3 Hyperstatic systems

- Force method → a linear set of equations of size N_R can be applied effectively, using good engineering judgments
- Displacement method → a linear set of equations of size N_D (unknown displacements) more amenable to automated solution processes

Example 4.1

❖ Series connection of axially loaded bars



- Axial force equilibrium condition for each joint - points B and C

$$F_{AB} = 4P \quad F_{BC} = P \quad (4.4)$$

- Constitutive laws for each bar

$$e_{AB} = \frac{4P}{K_{AB}} \quad e_{BC} = \frac{P}{K_{BC}}$$

- Overall extension of the bar, displacement of point C

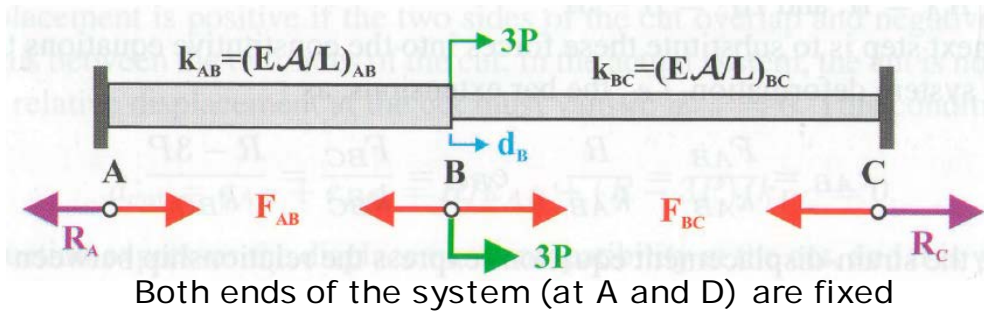
→ compatibility condition, $d_C = e_{AB} + e_{BC}$

$$d_C = e_{AB} + e_{BC} = \left(\frac{4}{K_{AB}} + \frac{1}{K_{BC}} \right) P$$

→ internal forces in the bar and the reaction force at point A can be found from equilibrium conditions alone ("isostatic")

Example 4.2

❖ Series connection of axially loaded bars (displacement approach)



- Unknowns : 4
 - { 2 reaction forces R_A, R_C
 - { 2 bar forces F_{AB}, F_{BC}
- Equilibrium eqns. : 3, one at each of the three joints

$$R_A = F_{AB}$$

$$F_{BC} - F_{AB} + 3P = 0$$

$$R_C = F_{BC}$$

✓ "hyperstatic", "statically indeterminate", "statically redundant"

- Constitutive laws

$$e_{AB} = \frac{F_{AB}}{K_{AB}} \quad e_{BC} = \frac{F_{BC}}{K_{BC}}$$

→ equilibrium eqn. for point B

$$K_{AB}e_{AB} - K_{BC}e_{BC} = 3P$$

Example 4.2

- Kinematics of the system - bar extension in terms of the displacements

$$d_B = e_{AB} \quad d_C = e_{AB} + e_{BC} \quad (4.5)$$

- Displacement at C = 0 $\rightarrow d_C = 0$

$$e_{AB} = -e_{BC} \quad d_B = e_{AB} = -e_{BC}$$

- \rightarrow Eq.(4.5) - single eqn. of the unknown displacement at B

$$(K_{AB} + K_{BC})d_B = 3P$$
$$d_B = e_{AB} = -e_{BC} = \frac{3P}{K_{AB} + K_{BC}}$$

- Back substitution

$$F_{AB} = K_{AB}e_{AB} = \frac{3K_{AB}P}{K_{AB} + K_{BC}} \quad (4.6)$$

$$F_{BC} = -K_{BC}e_{BC} = -\frac{3K_{BC}P}{K_{AB} + K_{BC}}$$

- ✓ "isostatic" problem - internal forces only depend on the externally applied loads.
- ✓ "hyperstatic" problem - internal forces depend on the applied loads, but also on the stiffness of the structure

Example 4.3

❖ Series connection of axially loaded bars (force approach)

✓ Example 4.2 - 4 unknowns, 3 equilibrium eqns.

- If any one of the 4 internal forces is known, the 3 others can be directly determined from equilibrium eqns.

F_{AB} → denoted by R

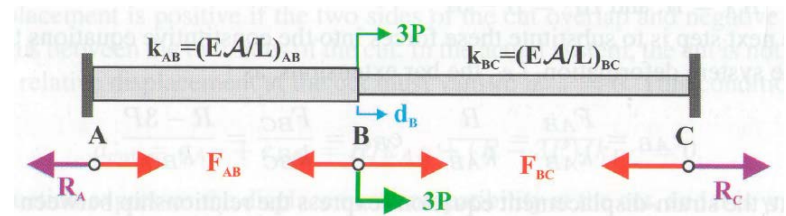
Then, $F_{BC} = R - 3P$, $R_A = R$, $R_C = R - 3P$

- Substitute these forces into the constitutive eqns. to determine the system deformation (bar extensions)

$$e_{AB} = \frac{F_{AB}}{K_{AB}} = \frac{R}{K_{AB}} \quad e_{BC} = \frac{F_{BC}}{K_{BC}} = \frac{R - 3P}{K_{BC}}$$

- Strain-displacement equation - Fig. 4.3

$$d_A = d_C = 0 \quad d_B = e_{AB}$$



compatibility of deformation between A and C → $e_{AB} + e_{BC} = 0$

→ necessary eqn. to solve for R

$$e_{AB} + e_{BC} = \frac{R}{K_{AB}} + \frac{R - 3P}{K_{BC}} = 0 \quad R = \frac{3K_{AB}}{K_{AB} + K_{BC}} P$$

Example 4.3

- Equilibrium eqns.

$$F_{AB} = R = \frac{3K_{AB}}{K_{AB} + K_{BC}} P \quad F_{BC} = R - 3P = -\frac{3K_{AB}}{K_{AB} + K_{BC}} P$$

→ Identical to the previous example (displacement approach)

- “Force” method - determination of the unknown force, R is based on the enforcement of compatibility conditions for system deformations

- ✓ Step 1. - the system is assumed to be “cut” at a location of R

- ✓ Step 2. - think of force R as an externally applied load

$$e_{AB} = \frac{F_{AB}}{K_{AB}} = \frac{R}{K_{AB}} \quad e_{BC} = \frac{F_{BC}}{K_{BC}} = \frac{R - 3P}{K_{BC}}$$

- ✓ Step 3. - compatibility condition, $d_{cut} = e_{AB} + e_{BC}$

in the actual system the cut is not present, $d_{cut} = 0$

$$d_{cut} = e_{AB} + e_{BC} = \frac{R}{K_{AB}} + \frac{R - 3P}{K_{BC}} = 0$$

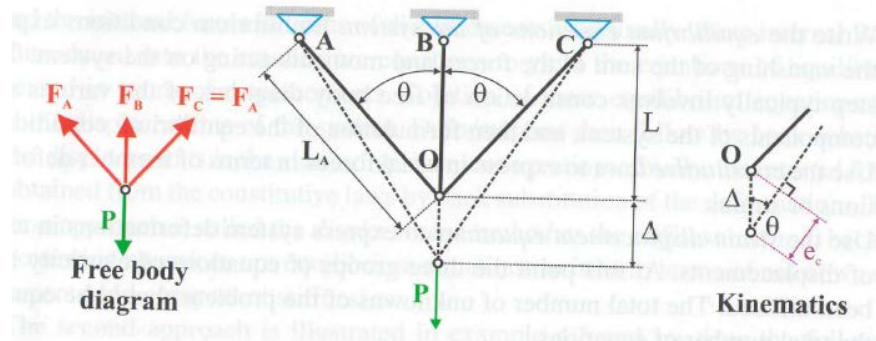
→ expresses the displacement compatibility at the cut, in terms of forces and flexibilities (inverse of stiffness)

- ✓ can be solved for the unknown force, R

$$R = \frac{3/K_{BC}}{1/K_{AB} + 1/K_{BC}} P = \frac{3K_{AB}}{K_{AB} + K_{BC}} P$$

Example 4.4

❖ Hyperstatic 3-bar truss (displacement method)



- 3 bars pinned together, vertical load P applied at O
- Assumption : geometric and material symmetry about vertical axis OB

$$A_A = A_C, \quad E_A = E_C \quad \rightarrow \quad F_A = F_C$$

horizontal displacement component

- ✓ Step 1. - equilibrium eqn.

$$F_B + 2F_A \cos \theta = P \quad (4.8)$$

$$\begin{cases} 2 \text{ unknowns} : F_A, F_B \\ 1 \text{ equilibrium eqn.} \rightarrow \text{hyperstatic system of order 1} \end{cases}$$

- ✓ Step 2. - constitutive laws

$$e_A = e_C = \frac{F_A L_A}{(EA)_A} = \frac{F_A L}{(EA)_A \cos \theta} \quad e_B = \frac{F_B L}{(EA)_B} \quad (4.9)$$

Example 4.4

- ✓ Step 3. - strain-displacement eqn.

If small displacement, i.e., $\Delta \ll L$

→ θ changes little during deformation

$$\rightarrow e_C \approx \Delta \cos \theta \quad (4.10)$$

$$\rightarrow e_A = e_C = \Delta \cos \theta, \quad e_B = \Delta$$

- ✓ Step 4. - express the internal forces in terms of displacements

$$\frac{F_A}{(EA)_B} = \frac{F_C}{(EA)_B} = \frac{\Delta}{L} \bar{k}_A \cos^2 \theta \quad \frac{F_B}{(EA)_B} = \frac{\Delta}{L} \quad (4.11)$$

where $\bar{k}_A = \frac{(EA)_A}{(EA)_B}$: non-dimensioned stiffness of bar A

- ✓ Step 5. - express the single equilibrium condition in terms of the single displacement Δ (substitute Eq. (4.11) into (4.8))

$$\frac{\Delta}{L} + 2 \frac{\Delta}{L} \bar{k}_A \cos^3 \theta = \frac{P}{(EA)_B} \quad (4.12)$$

- ✓ Step 6. - solves for Δ

$$\frac{\Delta}{L} = \frac{1}{1 + 2\bar{k}_A \cos^3 \theta} \frac{P}{(EA)_B} \quad \text{or} \quad \Delta = \frac{P}{k} \quad \text{where} \quad k = \frac{(EA)_B + 2(EA)_A \cos^3 \theta}{L}$$

: equivalent vertical stiffness of the 3-bar truss

Example 4.4

- ✓ Step 7. - deformation recovery, Eq. (4.12) → (4.10)

$$\frac{e_A}{L} = \frac{e_C}{L} = \frac{\cos \theta}{1 + 2\bar{k}_A \cos^3 \theta} \frac{P}{(EA)_B} \quad \frac{e_B}{L} = \frac{1}{1 + 2\bar{k}_A \cos^3 \theta} \frac{P}{(EA)_B} \quad (4.13)$$

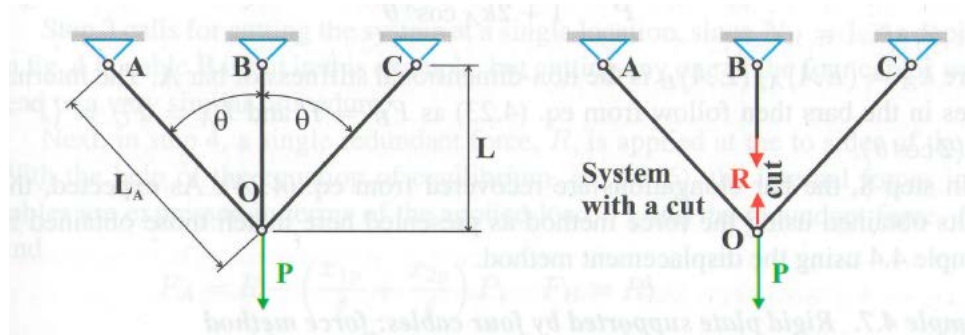
- ✓ Step 8. - internal force, Eq. (4.13) → (4.9)

$$\frac{F_A}{P} = \frac{F_C}{P} = \frac{\bar{k}_A \cos^2 \theta}{1 + 2\bar{k}_A \cos^3 \theta} \quad \frac{F_B}{P} = \frac{1}{1 + 2\bar{k}_A \cos^3 \theta} \quad (4.14)$$

$\frac{F_A}{F_B} = \bar{k}_A \cos^2 \theta$: ratio of the internal forces is in proportion to the ratio of their stiffness

Example 4.6

❖ Hyperstatic 3-bar truss (force method)



- ✓ Step 1. - single equilibrium eqn.

$$F_B + 2F_A \cos \theta = P \quad (4.22)$$

- ✓ Step 2. - system degree of redundancy $N_R = 2 - 1 = 1$

- ✓ Step 3. - cut the system at 1 location since $N_R = 1$

→ bar B is cut

- ✓ Step 4. - single redundant force R is applied at the sides of the cut with R treated as a known load, (4.23) →

$$F_A = F_C = \frac{P - R}{2 \cos \theta} \quad F_B = R \quad (4.23)$$

Example 4.6

- ✓ Step 5. - bar extensions, expressed in terms of R,
Constitutive laws, Eq.(4.9) →

$$\frac{e_C}{L} = \frac{e_A}{L} = \frac{F_A}{(EA)_A \cos \theta} = \frac{P - R}{2(EA)_A \cos^2 \theta}$$
$$\frac{e_B}{L} = \frac{F_B}{(EA)_B} = \frac{R}{(EA)_B}$$
(4.24)

- ✓ Step 6. - determination of the relative displacement at the cut

$$d_{cut} = \frac{e_A}{\cos \theta} - e_B = \frac{(P - R)L}{2(EA)_A \cos^3 \theta} - \frac{RL}{(EA)_B}$$

- ✓ Step 7. - $d_{cut} = 0$, to find the redundant force R

$$\frac{R}{P} = \frac{1}{1 + 2\bar{k}_A \cos^3 \theta}$$

- ✓ Step 8. - recovery of the bar elongation