# CHAPTER 5.
# Beam Theory

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# 5.1 The Euler-Bernoulli assumptions

- **One of its dimensions much large than the other two**
  - Civil engineering structure – assembly on grid of beams with cross-sections having shapes such as T's on I's
  - Machine parts – beam-like structures lever arms, shafts, etc.
  - Aeronautic structures – wings, fuselages → can be treated as thin-walled beams

- **Beam theory**

  – important role, simple tool to analyze numerous structures
  valuable insight at a pre-design stage

- **Euler-Bernoulli beam theory** – simplest, must be useful
  - Assumption

    ① Cross-section of the beam is infinitely rigid in its own plane

    $\rightarrow$ in-plane displacement field $\rightarrow \begin{cases} \text{2 rigid body translations} \\ \text{1 rigid body rotation} \end{cases}$

    ② The cross-section is assumed to remain plane

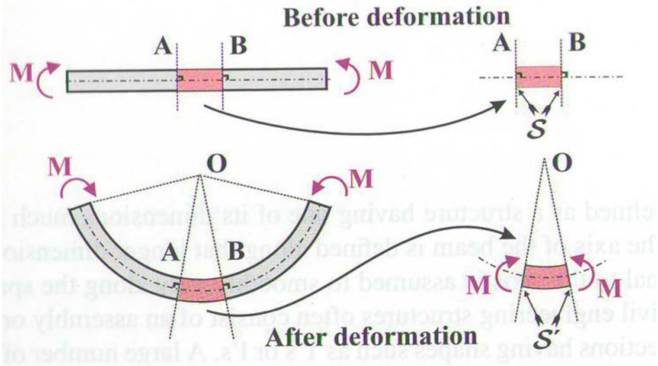
    ③ The cross-section is assumed to normal to the deformed axis

# 5.1 The Euler-Bernoulli assumptions

- **Fig. 5.1**

  "pure bending" beam deforms into a curve of constant curvature

  → a circle with center O, symmetric w.r.t. any plane perpendicular to its deformed axis



- **Kinematic assumptions "Euler-Bernoulli"**

  ① Cross-section is infinitely rigid in its own plane

  ② Cross-section remains plane after deformation

  ③ Cross-section remains normal to the deformed axis of the beam

  → valid for long, slender beams made of isotropic materials with solid cross-sections

# 5.2 Implication of the E-B assumption

❖ $\begin{Bmatrix} u_1(x_1,x_2,x_3) \\ u_2(x_1,x_2,x_3) \\ u_3(x_1,x_2,x_3) \end{Bmatrix}$ **displacement of an arbitrary point of the beam**

- E-B assumption

① Displacement field in the plane of cross-section consists solely of 2 rigid body translations $\bar{u}_2(x_1),\ \bar{u}_3(x_1)$

$$u_2(x_1,x_2,x_3)=\bar{u}_2(x_1)\ ,\ \ u_3(x_1,x_2,x_3)=\bar{u}_3(x_1) \qquad (5.1)$$

② Axial displacement field consists of
- rigid body translation $\bar{u}_1(x_1)$
- 2 rigid body rotation $\Phi_2(x_1),\Phi_3(x_1)$

$$u_1(x_1,x_2,x_3)=\bar{u}_1(x_1)+x_3\Phi_2(x_1)-x_2\Phi_3(x_1) \qquad (5.2)$$

③ Equality of
- the slope of the beam
- the rotation of the section

$$\Phi_3=\frac{d\bar{u}_2}{dx_1} \qquad \Phi_2=\underset{\uparrow}{-}\frac{d\bar{u}_3}{dx_1} \qquad (5.3)$$

consequence of the sign convention

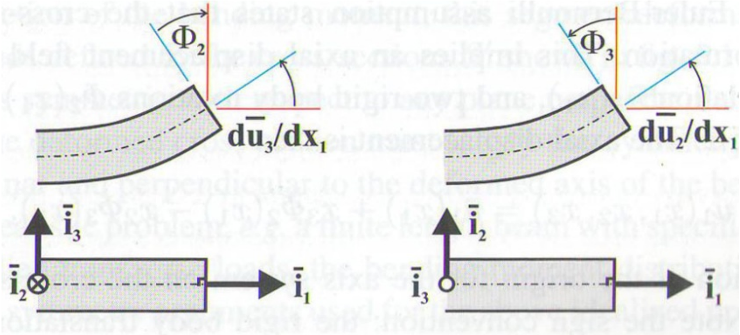


**Fig. 5.4.** Beam slope and cross-sectional rotation.

# 5.2 Implication of the E-B assumption

- To eliminate the sectional rotation from the axial displacement field

$$u_1\left(x_1,x_2,x_3\right)=\bar{u}_1(x_1)-x_3\frac{d\bar{u}_3(x_1)}{dx_1}-x_2\frac{d\bar{u}_2(x_1)}{dx_1} \qquad (5.4.a)$$

→ Important simplification of E-B : unknown displacements are functions of the span-wise coord, $x_1$ ,alone

## ❖ Strain field

$$\varepsilon_2=0,\ \ \varepsilon_3=0,\ \ \gamma_{23}=0 \qquad (5.5.a) \leftarrow \text{E-B(1)}$$

$$\gamma_{12}=0,\quad \gamma_{13}=0 \qquad (5.5.b) \leftarrow \text{E-B(2)}$$

$$\varepsilon_1=\frac{\partial u_1}{\partial x_1}=\frac{d\bar{u}_1(x_1)}{dx_1}-x_3\frac{d^2\bar{u}_3(x_1)}{dx_1{}^2}-x_2\frac{d^2\bar{u}_2(x_1)}{dx_1{}^2} \qquad (5.5.c)$$

$$\bar{\varepsilon}_1(x_1)=\frac{d\bar{u}_1(x_1)}{dx_1},\qquad \kappa_2(x_1)=-\frac{d^2\bar{u}_3(x_1)}{dx_1{}^2},\quad \kappa_3(x_1)=\frac{d^2\bar{u}_2(x_1)}{dx_1{}^2}. \qquad (5.6)$$

Sectional axial strain

Sectional curvature about $\bar{i}_2, \bar{i}_3$ axes

➡ $\varepsilon_1\left(x_1,x_2,x_3\right)=\bar{\varepsilon}_1(x_1)+x_3\kappa_2(x_1)-x_2\kappa_3(x_1)$ (5.7) ← E-B(2)

- Assuming a strain field of the form Eqs (5.5.a),(5.5.b),(5.7)
  → Math. Expression of the E-B assumptions

# 5.3 Stress resultants

- **3-D stress field ⇨ described in terms of sectional stresses called "stress resultants"**

  → equipollent to specified components of the stress field

- 3 force resultants $\begin{cases} N_1(x_1) & \text{axial force} \\ V_2(x_1), V_3(x_1) & \text{transverse shearing forces} \end{cases}$

$$N_1(x_1) = \int_A \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.8)$$

$$V_2(x_1) = \int_A \tau_{12}(x_1, x_2, x_3)\, dA, \quad V_3(x_1) = \int_A \tau_{13}(x_1, x_2, x_3)\, dA \qquad (5.9)$$

- 2 moment resultants :$M_2(x_1), M_3(x_1)$ bending moments

$$M_2(x_1) = \int_A x_3 \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.10a)$$

$$M_3(x_1) = \underset{\uparrow}{-}\int_A x_2 \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.10b)$$

(+) equipollent bending moment about $\bar{i}_3$ (Fig 5.5)

bending moments computed about point $P(x_{2p}, x_{3p})$

$$M_2{}^p(x_1) = \int_A (x_3 - x_{3p}) \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.11a)$$

$$M_3{}^p(x_1) = \int_A (x_2 - x_{2p}) \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.11b)$$

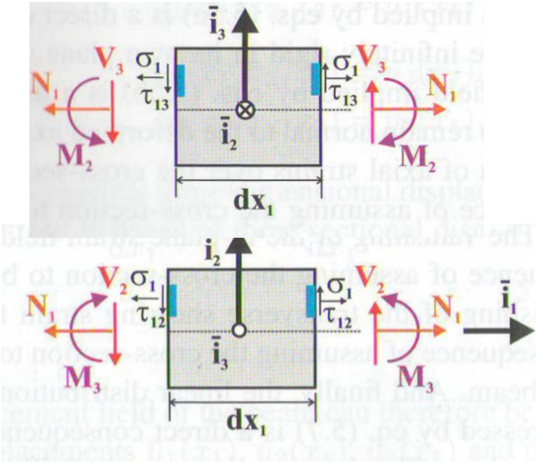


**Fig. 5.5. Sign convention for the sectional stress resultans**

# 5.4 Beams subjected to axial loads

- **Distributed axial load $p_1(x_1)$ [N/m], concentrated axial load $P_1$[N]**

  → axial displacement field $\bar{u}_1(x_1)$ ➡ 'bar' rather than 'beam'

## 5.4.1 Kinematic description

- **Axial loads causes only axial displacement of the section**

Eq.(5.4) → $u_1(x_1, x_2, x_3) = \bar{u}_1(x_1)$ (5.12a) → uniform over the x-s (Fig. 5.7)

$$u_2(x_1, x_2, x_3) = 0 \quad (5.12b)$$

$$u_3(x_1, x_2, x_3) = 0 \quad (5.12c)$$

Axial strain field $\varepsilon_1(x_1, x_2, x_3) = \bar{\varepsilon}_1(x_1)$ (5.13)

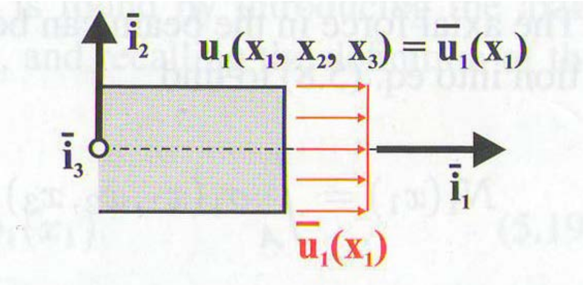


**Fig. 5.7.** Axial displacement distribution.

# 5.4 Beams subjected to axial loads

## 5.4.2 Sectional constitutive law

- **$\sigma_2 << \sigma_1$ , $\sigma_3 << \sigma_2$ ⇨ transverse stress components ≈ 0, $\sigma_2 \approx 0$, $\sigma_3 \approx 0$**
  - Generalized Hooke's law → $\sigma_1(x_1, x_2, x_3) = E\varepsilon_1(x_1, x_2, x_3)$ (5.14)

    ↑ At the "infinitesimal" level

- **Inconsistency in E-B beam theory**

Eq. (5.5a) → $\varepsilon_2 = 0, \varepsilon_3 = 0$

Hooke's law → if $\sigma_2 = \sigma_3 = 0$ , then $\varepsilon_2 = -\nu\sigma_1 / E, \quad \varepsilon_3 = -\nu\sigma_1 / E$

(Poisson's effect) → very small effect, and assumed to vanish

Eq.(5.13) → (5.14) : $\sigma_1(x_1, x_2, x_3) = E\varepsilon_1(x_1, x_2, x_3)$ (5.15)

- Axial force

$$N_1(x_1) = \int_A \sigma_1(x_1, x_2, x_3)\, dA = \Big[\int_A E dA\Big]\bar{\varepsilon}_1(x_1) = S\bar{\varepsilon}(x_1) \qquad (5.16)$$

↑ Axial stiffness ↑

$S = EA$ for homogeneous material

→ constitutive law for the axial behavior of the beam at the sectional level

# 5.4 Beams subjected to axial loads

## 5.4.3 Equilibrium eqns

- **Fig. 5.8** → infinitesimal slice of the beam of length $dx_1$

force equilibrium in axial dir. → $$\frac{dN_1}{dx_1} = -p_1 \qquad (5.18)$$

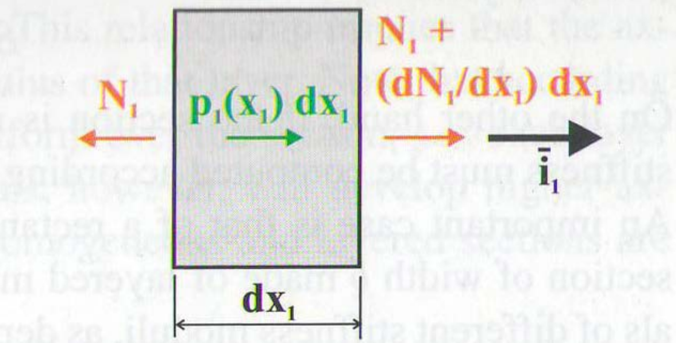


**Fig. 5.8.** Axial forces acting on an infinitesimal slice of the beam.

Eq. (1.4) → equilibrium condition for a differential element of a 3-D solid

Eq. (5.18) → equilibrium condition of a slice of the beam of differential length $dx_1$

# 5.4 Beams subjected to axial loads

## 5.4.4 Governing eqns

- **Eq (5.16) Eq. (5.18) and using Eq. (5.6)**

$$\frac{d}{dx_1}\left[S\frac{d\bar{u}_1}{dx_1}\right]=-p_1(x_1) \qquad (5.19)$$

- **3 B.C**

① Fixed(clamped) : $\bar{u}_1=0$

② Free (unloaded) : $N_1=0 \rightarrow \frac{d\bar{u}_1}{dx_1}=0$

③ Subjected to a concentrated load $P_1$ : $N_1=P_1 \rightarrow S\frac{d\bar{u}_1}{dx_1}=P_1$

## 5.4.5 The sectional axial stiffness

- **Homogeneous material**

$$S=EA \qquad (5.20)$$

# 5.4 Beams subjected to axial loads

- ❖ **Rectangular section of width b made of layered material of different moduli(Fig. 5.9)**

$$S=\int_A E dA=\sum_{i=1}^{n}E^{[i]}\int_{A^{[i]}}dA^{[i]}=\sum_{i=1}^{n}E^{[i]}b(x_3^{[i+1]}-x_3^{[i]})$$

↑

weighting factor thickness

"weighted average" of the Young's modulus

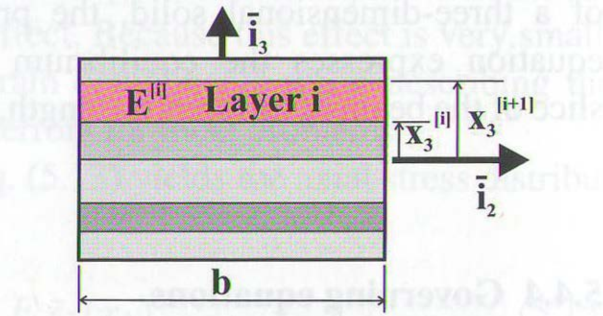


**Fig. 5.9.** Cross-section of a beam with various layered materials.

## 5.4.6 The axial stress distribution

- ❖ **Eliminating the axial strain form Eq.(5.15) and (5.16)**

$$\sigma_1(x_1,x_2,x_3)=\frac{E}{S}N_1(x_1) \tag{5.21}$$

- ➢ Homogeneous material

$$\sigma_1(x_1,x_2,x_3)=\frac{N_1(x_1)}{A} \tag{5.22}$$

→ Uniformly distributed over the section

- ➢ Sections made of layers presenting different moduli

$$\sigma_1^{[i]}(x_1,x_2,x_3)=E^{[i]}\frac{N_1(x_1)}{S} \tag{5.23}$$

→ Stress in layer I is proportional to the modulus of the layer

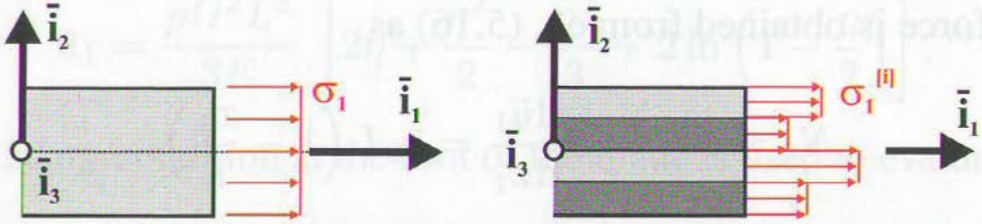
# 5.4 Beams subjected to axial loads

- **Eq (5.13) ⇨ axial strain distribution is uniform over the section, i.e. each layer is equally strained (Fig. 5.10)**

  - Strength criterion

$$\frac{E}{S}\left|N_{1\max}^{tens}\right| \leq \sigma_{allow}^{tens}, \quad \frac{E}{S}\left|N_{1\max}^{comp}\right| \leq \sigma_{allow}^{comp} \qquad (5.24)$$

in case compressive, buckling failure mode may occur → Chap. 14



**Section made of homogeneous material** **Section made of layered material**

**Fig. 5.10.** Axial stress distribution for sections made of homogeneous and layered materials

# 5.5 Beams subjected to transverse loads

- **Fig. 5.14** → "transverse direction" distributed load, $p_2(x_1)$ [N/m]
  concentrated load, $P_2$ [N]

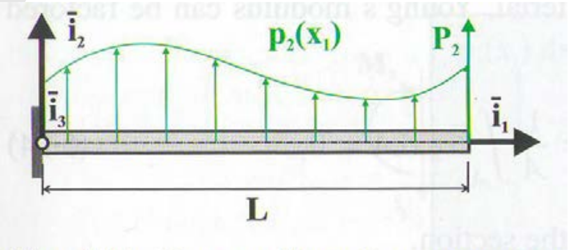


Fig. 5.14. Beam subjected to transverse loads.

➡ bending moments, transverse shear forces,

and { axial
transverse shearing } stresses will be generated

## 5.5.1 Kinematic description

- **Assumption** → transverse loads only cause { transverse displacement
  curvature of the section }
  - General displacement field (Eq.(5.6))

$$u_1(x_1,x_2,x_3) = -x_2 \frac{d\bar{u}_2(x_1)}{dx_1} \qquad (5.29a)$$

$$u_2(x_1,x_2,x_3) = \bar{u}_2(x_1) \qquad (5.29b)$$

$$u_3(x_1,x_2,x_3) = 0 \qquad (5.29c)$$

➡

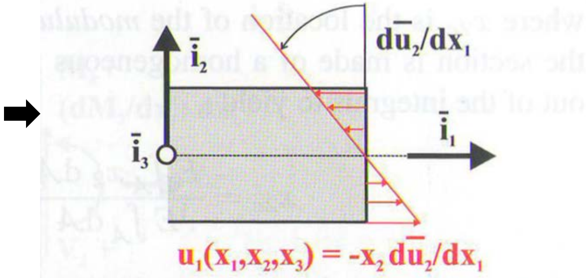


Fig. 5.15. Axial displacement distribution on cross-section.

→ linear distribution of the axial displacement component over the x-s

# 5.5 Beams subjected to transverse loads

- Only non-vanishing strain component

$$\varepsilon_1\left(x_1,x_2,x_3\right)=-x_2\kappa_3(x_1)$$ (5.36) → linear distribution of the axial strain

## 5.5.2 Sectional constitutive law

- Linearly elastic material, axial stress distribution

$$\sigma_1\left(x_1,x_2,x_3\right)=-Ex_2\kappa_3(x_1) \qquad (5.31)$$

- Sectional axial force by Eq. (5.8)

$$N_1(x_1)=\int_A\sigma_1\left(x_1,x_2,x_3\right)dA=-\left[\int_A Ex_2dA\right]\kappa_3(x_1) \qquad (5.32)$$

- Axial force =0 since subjected to transverse loads only

$$\kappa_3\neq 0, \quad then \quad \left[\int_A Ex_2dA\right]=0$$

$$\Rightarrow \underset{\uparrow}{x_{2c}}=\frac{1}{S}\int_A Ex_2dA=\frac{S_2}{S}=0 \qquad (5.33)$$

Location of the “modulus-weighted centroid” of the x-s

# 5.5 Beams subjected to transverse loads

- If homogeneous material

$$x_{2c} = \frac{E\int_A x_2 dA}{E\int_A dA} = \frac{1}{A}\int_A x_2 dA = 0 \qquad (5.34)$$

→ $x_2$ is simply the area center of the section

➡ The axis system is located at the { modulus-weighted centroid
area center if homogeneous material
↑
center of mass – 3 coincide

✓ Center of mass

$$x_{2nn} = \frac{\rho\int_A x_2 dA}{\rho\int_A dA} = \frac{\int_A x_2 dA}{\int_A dA} = x_{2c}$$

- Bending moment by Eq.(5.31)

$$M_3(x_1) = \left[\int_A E x_2{}^2 dA\right]\kappa_3(x_1) = H_{33}^c \kappa_3(x_1) \qquad (5.35)$$

↑
“centroid bending stiffness” about axis $\bar{i}_3$

Constitutive law for the bending behavior of the beam bending moment ∝ the curvature

↑
Bending stiffness
(“flexural rigidity”)

➡ $$M_1\left(x_1\right) = H_{33}^c \kappa_3(x_1) \qquad (5.37)$$

“moment-curvature” relationship

# 5.5 Beams subjected to transverse loads

## 5.5.3 Equilibrium eqns

- **Fig. 5.16** → infinitesimal slice of the beam of length $dx_1$
  $M_3(x_1), V_2(x_1)$ acting at a face at location $x_1$

  @ $x_1 + dx_1$ , evaluated using a Taylor series expansion, and H.O terms ignore

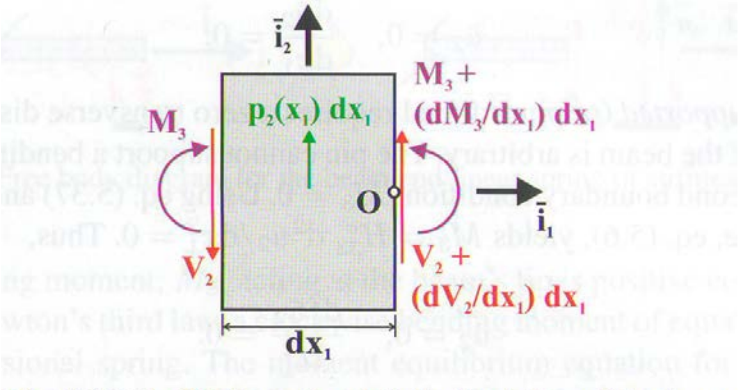


**Fig. 5.16.** Equilibrium of an infinitesimal slice of the beam.

➡ 2 equilibrium eqns

vertical force → $$\frac{dV_2}{dx_1} = -p_2(x_1) \qquad (5.38a)$$

moment about O → $$\frac{dM_3}{dx_1} + V_2 = 0 \qquad (5.38b)$$

$$\frac{d^2 M_3}{dx_1^{\;2}} = p_2(x_1) \qquad (5.39)$$

# 5.5 Beams subjected to transverse loads

## 5.5.4 Governing eqns

❖ **Eq. (5.37) Eq. (5.39), and recalling Eq. (5.6)**

$$\frac{d^2}{dx_1{}^2}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1{}^2}\right] = p_2(x_1) \qquad (5.40)$$

4th order DE

- 4 B.C
  - ① Clamped end $\bar{u}_2 = 0, \ \frac{d\bar{u}_2}{dx_1} = 0$
  - ② Simply supported(pinned) $\bar{u}_2 = 0, \ \frac{d^2\bar{u}_1}{dx_1{}^2} = 0$
  - ③ Free(or unloaded) end $\frac{d^2\bar{u}_2}{dx_1{}^2} = 0, \ -\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1{}^2}\right] = 0$
  - ④ End subjected to a concentrated transverse load $P_2$ $\quad P_2 = V_2 = -\frac{dM_3}{dx_1}$

$$\frac{d^2\bar{u}_2}{dx_1{}^2} = 0, \ -\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1{}^2}\right] = P_2$$

# 5.5 Beams subjected to transverse loads

⑤ Rectilinear spring (Fig. 5.17) $-V_2(L) = k\bar{u}_2(L)$

↑ sign convention

$$\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right]_{x_1=L} - k\bar{u}_2(L) = 0, \quad \frac{d^2\bar{u}_2}{dx_1^2} = 0$$

↑ (+) when the spring is located at the left end

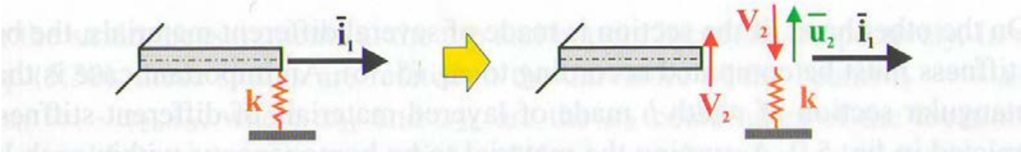


**Fig. 5.17.** Free body diagram for the beam end linear spring of stiffness constant $k$.

⑥ Rotational spring (Fig. 5.18) $-M_3(L) = k\Phi_3(L)$

$$\left|H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right|_{x_1=L} + k\frac{d^2\bar{u}_2}{dx_1^2} = 0, \quad -\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^2}\right]_{x_1=L} = 0$$

↑ (-) when at the left end

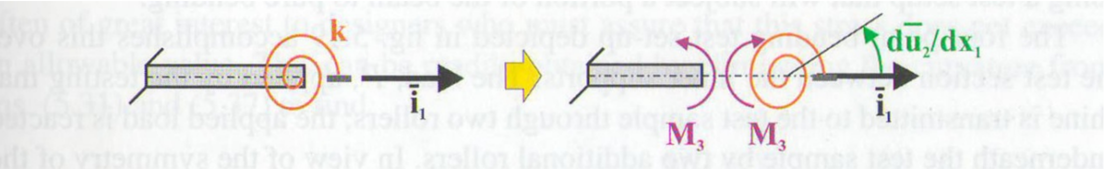


**Fig. 5.18.** Free body diagram for a beam with end rotational spring of stiffness constant $k$.

# 5.5 Beams subjected to transverse loads

## 5.5.5 The sectional bending stiffness

- **Homogeneous material**

$$H_{33}^{c} = EI^{c}{}_{33} \qquad (5.41)$$

$$I^{c}{}_{33} = \int_{A} x_2{}^{2} dA \qquad (5.42)$$

: purely geometric quantity, the area second moment of the section computed about the area center

- Rectangular section of width b made of layered materials (Fig. 5.9)

$$H_{33}^{c} = \int_{A} E x_2{}^{2} dA = \sum_{i=1}^{n} E^{[i]} \int_{A^{[i]}} x_2{}^{2} dA^{[i]} = \frac{b}{3} \sum_{i=1}^{n} E^{[i]} \left[ (x_2{}^{[i+1]})^3 - (x_2{}^{[i]})^3 \right] \qquad (5.43)$$

"weighted average" of the Young's moduli

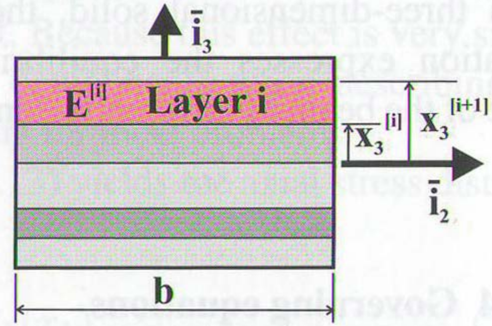


**Fig. 5.9.** Cross-section of a beam with various layered materials.

# 5.5 Beams subjected to transverse loads

## 5.5.6 The axial stress distribution

❖ **Local axial stress**→ eliminating the curvature from Eq.(5.3), (5.37)

$$\sigma_1(x_1,x_2,x_3) = -Ex_2\frac{M_3(x_1)}{H_{33}^c} \qquad (5.44)$$

- ➢ homogeneous material

$$\sigma_1(x_1,x_2,x_3) = -x_2\frac{M_3(x_1)}{I_{33}} \qquad (5.45)$$

→ linearly distributed over the section, independent of Young's modulus

- ➢ various layer of materials

$$\sigma_1^{[i]}(x_1,x_2,x_3) = -E^{[i]}x_2\frac{M_3(x_1)}{H_{33}^c} \qquad (5.46)$$

→ axial STRAIN distribution is linear over the section ← Eq.(5.30)

axial stress distribution → piecewise linear (Fig. 5.20)

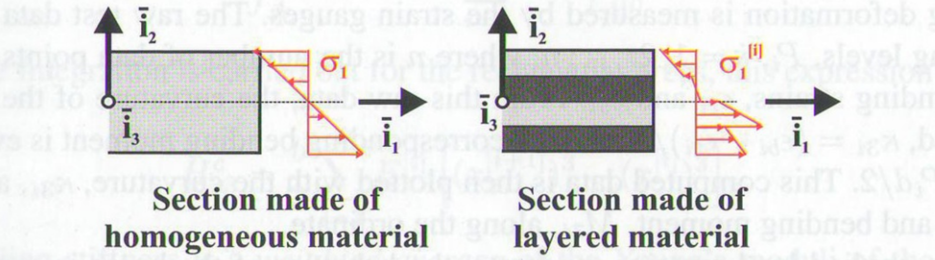


**Fig. 5.20.** Axial stress distributions in homogeneous and layered sections.

# 5.5 Beams subjected to transverse loads

- **Strength criterion**

$$\frac{\left|x_2^{\max}\right|}{H_{33}^{c}} E \left|M_3^{\max}\right| \leq \sigma_{allow}^{comp}, \quad \frac{\left|x_2^{\max}\right|}{H_{33}^{c}} E \left|M_3^{\max}\right| \leq \sigma_{allow}^{tens},$$

↑ Maximum (+) bending moment in the beam

- Layers of various material
  → must be computed at the $\left\{\begin{matrix}\text{top}\\ \text{bottom}\end{matrix}\right\}$ locations of each ply

## 5.5.7 Rational design of beams under bending

- **"Neutral axis"** → along axis $\bar{i}_3$ which passes through the section's centroid
  - Material located near the N.A carries almost no stress
  - Material located near the N.A contributes little to the bending stiffness

➡ Rational design → removal of the material located at and near the N.A and relocation away from that axis

# 5.5 Beams subjected to transverse loads

- **Fig. 5.21** → { rectangular / ideal } section, same mass $m = bh\rho$

  ↑ a thin web would be used to keep the 2 flanges (under "ideal")

- Ratio of bending stiffness

$$\frac{H_{ideal}}{H_{rect}} = \frac{E\bullet 2\left[\frac{b(h/2)^2}{12}+\frac{bh}{2}d^2\right]}{E\frac{bh^3}{12}} = \frac{1}{4}+12\left(\frac{d}{h}\right)^2$$

For $d/h = 10,\quad \frac{H_{ideal}}{H_{rect}} \cong 1200$

- Ratio of max. axial stress

$$\frac{\sigma_{ideal}}{\sigma_{rect}} = \frac{E\frac{h}{2}M_3 I_{ideal}}{I_{ideal}E\left(d+\frac{h}{4}\right)M_3} = \frac{\frac{1}{4}+12\left(\frac{d}{h}\right)^2}{\frac{1}{2}+2\left(\frac{d}{h}\right)}$$

For $d/h = 0,\quad \frac{\sigma_{rect}^{\max}}{\sigma_{ideal}^{\max}} \cong 6(d/h) = 60$

→ ideal section can carry a 60 times larger bending moment

- Ideal section ➡ "I beam," but prone to instabilities of web and flange buckling

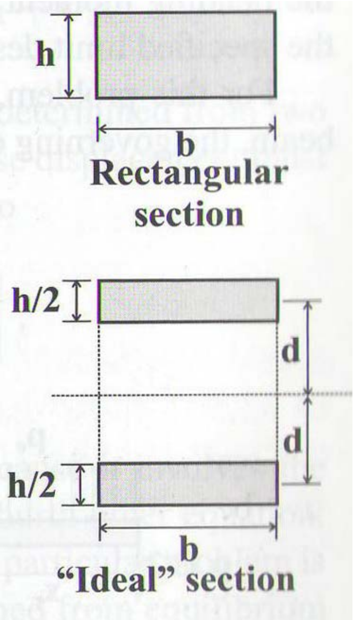


**Fig. 5.21.** A rectangular section, and the ideal section.

# 5.6 Beams subjected to combined and transverse loads

- Sec. 5.4, 5.5 → convenient to locate the origin of the axes system at the centroid of the beam's x-s

## 5.6.1 Kinematic description

$$u_1\left(x_1,x_2,x_3\right)=\bar{u}_1(x_1)-(x_2-\overset{\substack{\text{location of centroid}\\ \downarrow}}{x_{2C}})\frac{d\bar{u}_2(x_1)}{dx_1} \qquad (5.73a)$$

$$u_2\left(x_1,x_2,x_3\right)=\bar{u}_2(x_1) \qquad (5.73b)$$

$$u_3\left(x_1,x_2,x_3\right)=0 \qquad (5.73c)$$

- Strain field

$$\varepsilon_1\left(x_1,x_2,x_3\right)=\bar{\varepsilon}_1(x_1)-(x_2-x_{2C})\kappa_3(x_1) \qquad (5.74)$$

# 5.6 Beams subjected to combined and transverse loads

## 5.6.2 Sectional constitutive law

❖ **Axial stress distribution**

$$\sigma_1\left(x_1, x_2, x_3\right) = E\bar{\varepsilon}_1(x_1) - E(x_2 - x_{2C})\kappa_3(x_1) \qquad (5.75)$$

- ➢ Axial force

$$N_1 = \int_A \left[E\bar{\varepsilon}_1(x_1) - E(x_2 - x_{2C})\kappa_3(x_1)\right] dA$$

$$= \underbrace{\left[\int_A E dA\right]}_{\uparrow\; S \text{ (axial stiffness)}} \bar{\varepsilon}_1(x_1) + \underbrace{\left[\int_A E(x_2 - x_{2C}) dA\right]}_{\uparrow\; = \int_A E x_2 dA - x_{2C}\int_A E dA = S_2 - S x_{2C} = 0} \kappa_3(x_1)$$

$$\Rightarrow \quad N_1 = S\bar{\varepsilon}_1$$

- ➢ Bending moment

$$M_3{}^C = -\int_A (x_2 - x_{2C})\left[E\bar{\varepsilon}_1(x_1) - E(x_2 - x_{2C})\kappa_3(x_1)\right] dA$$

$$= -\underbrace{\left[\int_A E(x_2 - x_{2C}) dA\right]}_{\uparrow\; = 0} \bar{\varepsilon}_1(x_1) + \underbrace{\left[\int_A E(x_2 - x_{2C})^2 dA\right]}_{\uparrow\; H^c{}_{33} \text{ (bending stiffness)}} \kappa_3(x_1)$$

$$\Rightarrow \quad M_3{}^C = H^c_{33}\kappa_3$$

➡ "decoupled sectional constitutive law"

↑ 2 crucial steps {
① Displacement field must be in the form of Eq.(5.73)
② Bending moment must be evaluated w.r.t. the centroid

→ Thus, centroid plays a crucial rule

# 5.6 Beams subjected to combined and transverse loads

## 5.6.3 Equilibrium eqns

❖ **Fig. 5.47** → infinitesimal slice of the beam of length $dx_1$

- ➢ Force equilibrium in horizontal dir. $$\frac{dN_1}{dx_1} = -p_1$$ ➡ (5.18)
- ➢ Vertical equilibrium $$\frac{dV_2}{dx_1} = -p_2$$ ➡ (5.38a)
- ➢ Equilibrium of moments about the centroid

$$\frac{dM_3}{dx_1} + V_2 = \underbrace{(x_{2a} - x_{2C})}_{\uparrow} p_1 \qquad (5.77)$$

Moment arm of the axial load w.r.t the centroid

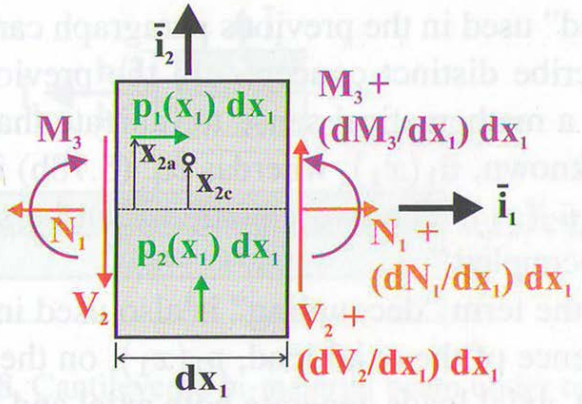


**Fig. 5.47.** Axial forces acting on an infinitesimal slice of the beam.

# 5.6 Beams subjected to combined and transverse loads

## 5.6.4 Governing eqns

$$\begin{cases} \dfrac{d}{dx_1}\left[S\dfrac{d\bar{u}_1}{dx_1}\right] = -p_1(x_1) & \text{(5.78a)} \quad ➡ \quad \text{(5.19)} \\[2ex] \dfrac{d^2}{dx_1{}^2}\left[H_{33}^c\dfrac{d^2\bar{u}_2}{dx_1{}^2}\right] = p_2(x_1) + \underline{\dfrac{d}{dx_1}\left[(x_{2a}-x_{2C})p_1(x_1)\right]} & \text{(5.78b)} \quad ➡ \quad \text{almost similar to (5.19) concept} \end{cases}$$

$$\rightarrow \text{“decoupled” eqns} \begin{cases} \text{(5.78a)} \rightarrow \bar{u}_1(x_1) \\ \text{(5.78b)} \rightarrow \bar{u}_2(x_1) \end{cases} \text{ can be independently solved}$$

If axial loads are applied @ centroid, extension and bending are “decoupled”

If axial loads are **not** applied @ centroid, extension and bending are “coupled”