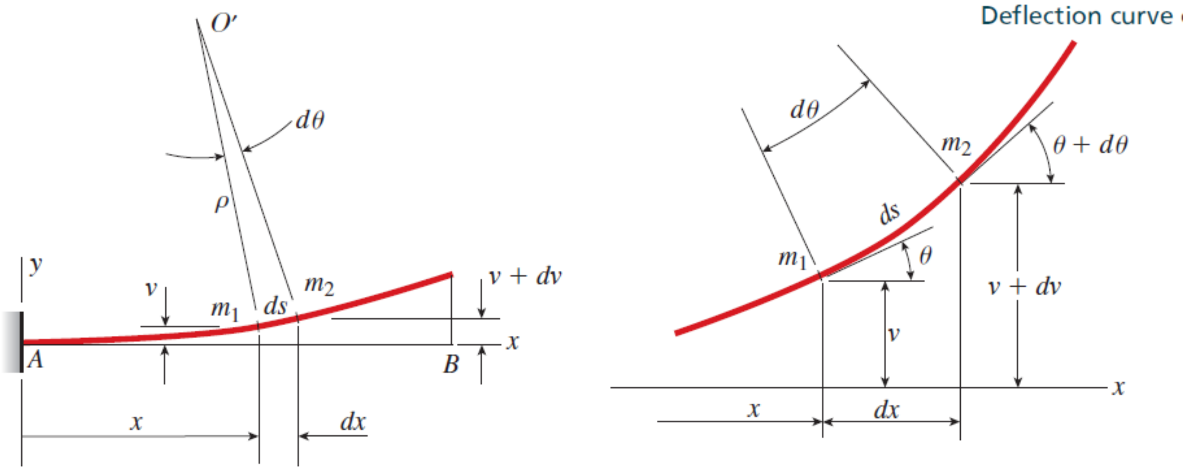
Chapter 9 Deflections of Beams

9.2 Differential Equations of the Deflection Curve



⦿ Sign Conventions and Main Concepts

1. **Deflection** v: Displacement in *y*-direction at a point (upward positive)
2. **Angle of rotation** θ: Angle between *x*-axis and t________ to the deflection curve (counterclockwise positive)
3. **Center of curvature** O': Intersection of the orthogonal lines at two points on the curve
4. **Radius of curvature** ρ: Distance between O' and the deflection curve (Recall $\rho d\theta = ds$)
5. **Curvature** κ: Reciprocal of the radius of curvature, i.e. $\kappa = 1/\rho = d\theta/ds$ (∪ positive)
6. **Slope of the deflection curve**

$$\frac{dv}{dx} = \tan\theta \qquad \theta = \arctan\frac{dv}{dx}$$

   In a similar manner

$$\cos\theta = \frac{dx}{ds} \qquad \sin\theta = \frac{dv}{ds}$$

7. These are all based only on geometric considerations, and thus valid for beams of any material. No restrictions on the magnitudes of the slopes and deflections.

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⦿ Beams with Small Angles of Rotation

1. Very small angle of rotations and deflections → approximation $ds \approx dx$ greatly simplify beam analysis

2. Curvature

$$\kappa = \frac{1}{\rho} \approx \frac{d\theta}{d}$$

3. Angle of rotation (equal to the s________; angle should be given in r_________)

$$\theta \approx \tan\theta = \frac{d}{dx}$$

4. Curvature (in terms of deflection curve):

$$\kappa = \frac{1}{\rho} \approx \frac{d\theta}{dx} = \frac{d^2v}{dx^2}$$

Note: Exact curvature (See textbook)

$$\kappa = \frac{d\theta}{ds} = \frac{v''}{[1 + (v')^2]^{3/2}}$$

5. For a **linearly elastic** beam (i.e. following Hooke's law), $\kappa = 1/\rho = M/EI$, we now obtain **differential equation of the deflection curve** as

$$\frac{d^2v}{dx^2} = v'' = \frac{M}{EI}$$

⦿ Nonprismatic Beams

1. From above, but with flexural rigidity varying over x, i.e. EI_x,

$$EI_x \frac{d^2v}{dx^2} = M$$

2. From $dM/dx = V$ and $dV/dx = -q$

$$\frac{d}{dx}\left(EI_x \frac{d^2v}{dx^2}\right) = \frac{dM}{dx} = V$$

$$\frac{d^2}{dx^2}\left(EI_x \frac{d^2v}{dx^2}\right) = \frac{d^2M}{dx^2} = \frac{dV}{dx} = -q$$

⦿ Prismatic Beams

1. Differential equations for prismatic beams, i.e. $EI_x = EI$

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1) Bending-moment equation $EIv'' = M$

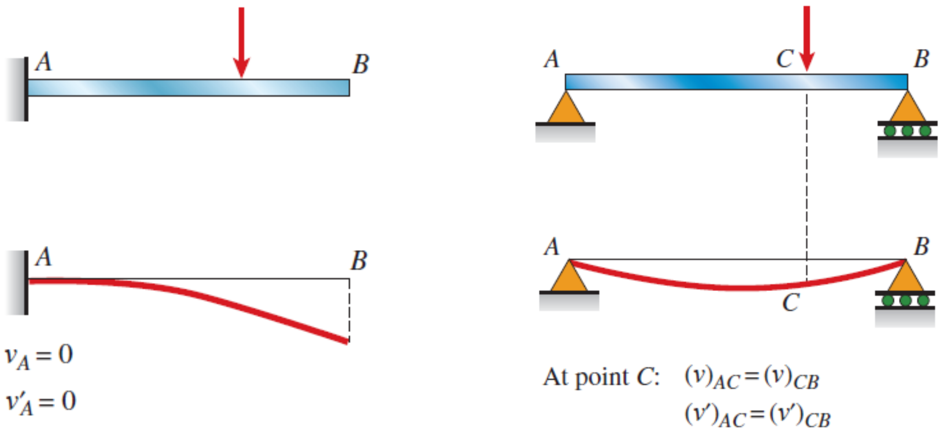
2) Shear-force equation $EIv''' = V$

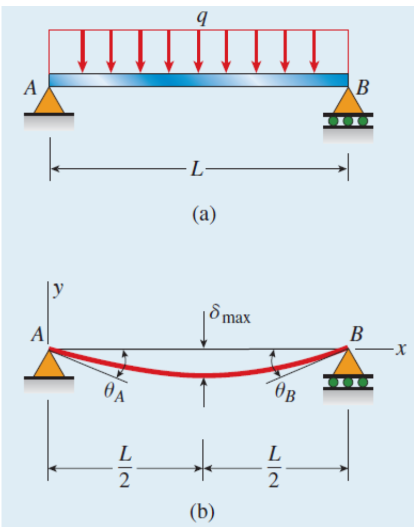
3) Load equation $EIv'''' = -q$

9.3 Deflections by Integration of the Bending-Moment Equation

⦿ Conditions needed for Solving Bending-Moment Equations by **Method of Successive Integrations** (i.e. integrate the differential equation and find the undetermined coefficients by given conditions)

1. **Boundary conditions**: Deflection and slope at boundaries
2. **Continuity conditions:** At a given point, the deflections (or slopes) obtained for the left- and right-hand parts should be equal
3. **Symmetry conditions**: For example, the slope of the deflection curve at the midpoint is zero (for a symmetric beam under symmetric loads)





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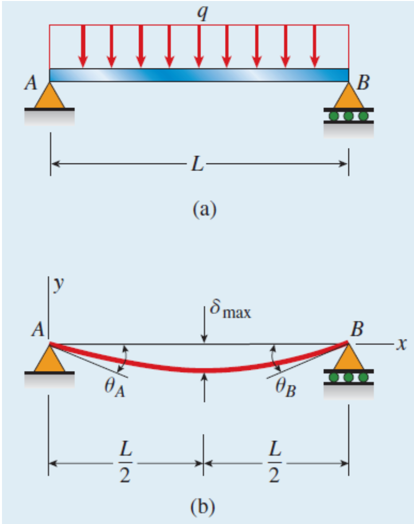
⦿ **Example 9-1**: Determine the equation of the deflection curve for a simple beam supporting a uniform load of intensity q. Also determine the maximum deflection $\delta_{\max}$ at the midpoint of the beam, and the angles of rotation at the supports, i.e. θ_A and θ_B. The beam has length L and constant flexural rigidity EI.



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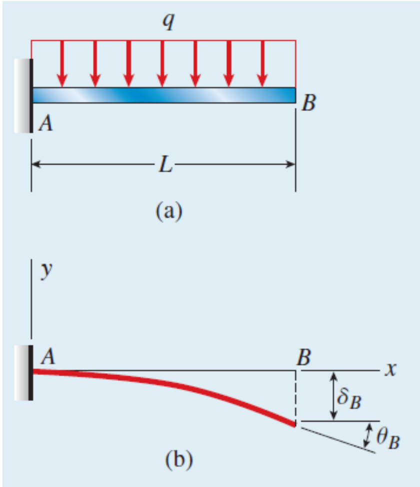
⦿ **Example 9-2**: Determine the equation of the deflection curve for a cantilever beam subjected to a uniform load of intensity q. Also determine the angle of rotation θ_B and the deflection δ_B at the free end. The beam has length L and constant flexural rigidity EI.



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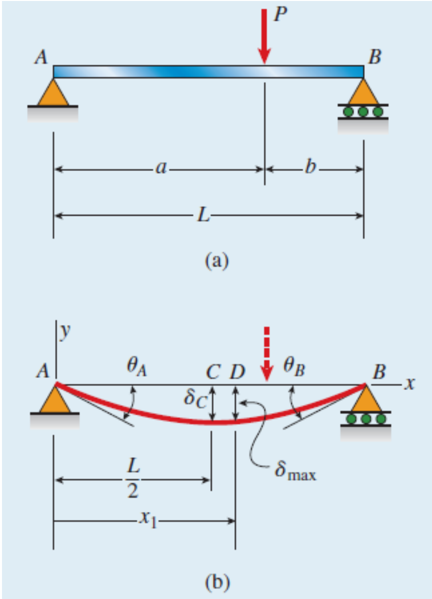
⦿ **Example 9-3**: Determine the equation of the deflection curve for a simple beam subjected to a concentrated load P acting at distances a and b from the left- and right-had supports, respectively. Also determine the angles of rotation θ_A and θ_B, the maximum deflection $\delta_{\max}$, and the deflection δ_C at the midpoint of the beam. The beam has length L and constant flexural rigidity EI.



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9.4 Deflections by Integration of the Shear-Force and Load Equations

⦿ Solving Differential Equations for Deflections

1. Among three differential equations for deflections, i.e.

   1) Bending-moment equation $EIv'' = M$ (______ unknowns)

   2) Shear-force equation $EIv''' = V$ (_______ unknowns)

   3) Load equation $EIv'''' = -q$ (_______ unknowns)

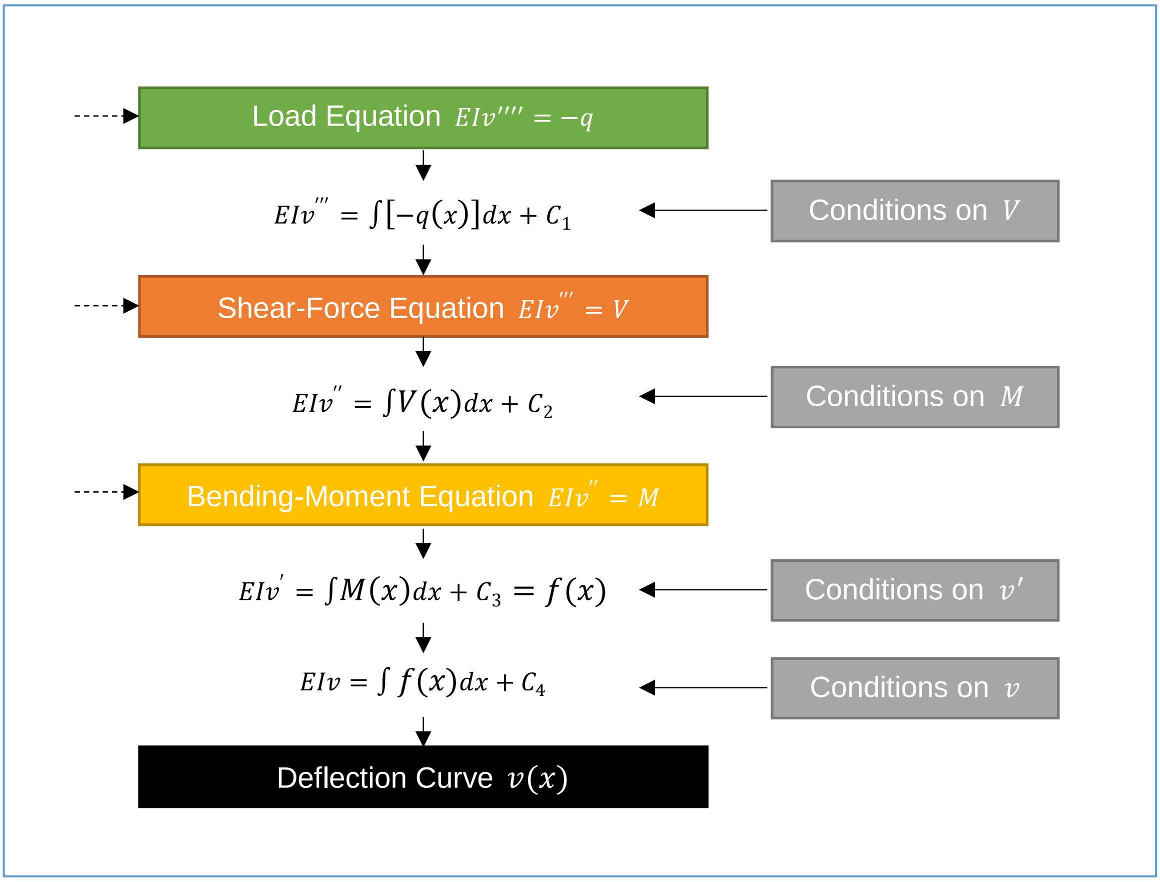
   One can solve a differential equation based on available conditions

2. Boundary, continuity and symmetry conditions are available in terms of

   1) Deflection (v) and slope (v') for bending-moment equation (Section 9.3)

   2) In addition, moment (M) conditions can be used for the shear-force equation because $EIv'' = M$

   3) In addition, shear force (V) conditions can be used for solving the load equation because $EIv''' = V$



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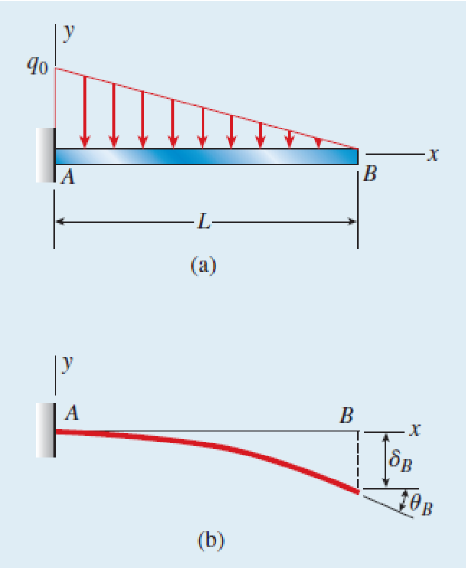
⦿ **Example 9-4**: Determine the equation of the deflection curve for a cantilever beam AB supporting a triangularly distributed load of maximum intensity q_0. Also determine the deflection δ_B and angle of rotation θ_B at the free end. Use the load equation.



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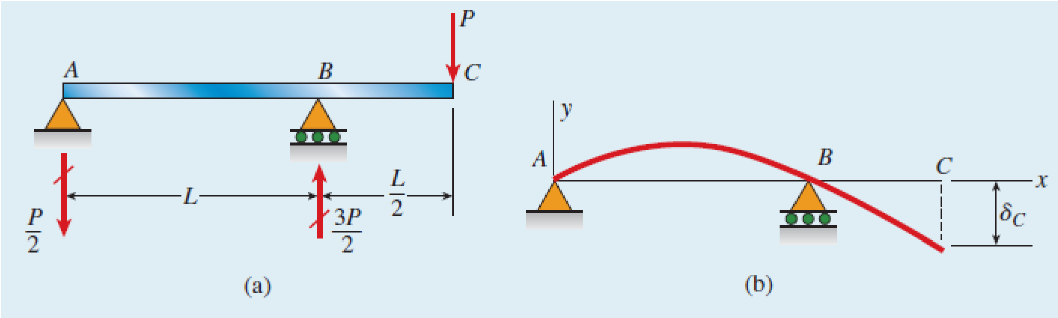
⦿ **Example 9-5**: Determine the equation of the deflection curve for a simple beam with an overhang under concentrated load P at the end. Also determine the deflection δ_C. Use the third-order differential equation, i.e. shear-force equation. The beam has constant flexural rigidity EI.



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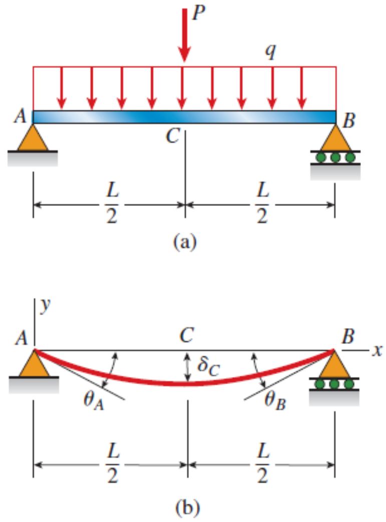
9.5 Method of Superposition

⊙ Method of Superposition



1. Under suitable conditions, the deflection of a beam produced by several different loads acting simultaneously can be found by s___________ the deflections produced by the same loads acting separately.

2. At a given location, the deflection v_1 by a load q_1 (alone) and v_2 by a load q_2 (alone) → the deflection under loads q_1 and q_2 is _____________

3. Justification: the differential equation for deflection is l__________ ~ "Principle of Superposition" (mathematics)

4. "Principle of Superposition" is valid under the following conditions:

   (1) H_______'s law for the material

   (2) Deflections and rotations are s_________

   (3) Presence of deflection does not alter the actions of the applied loads

   → Three sources of n____________ in general structural engineering problems: nonlinear material property (constitutive law, large deformation, load-displacement interactions)

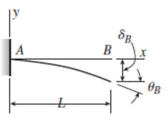
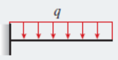
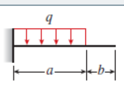
⊙ Tables of Beam Deflections

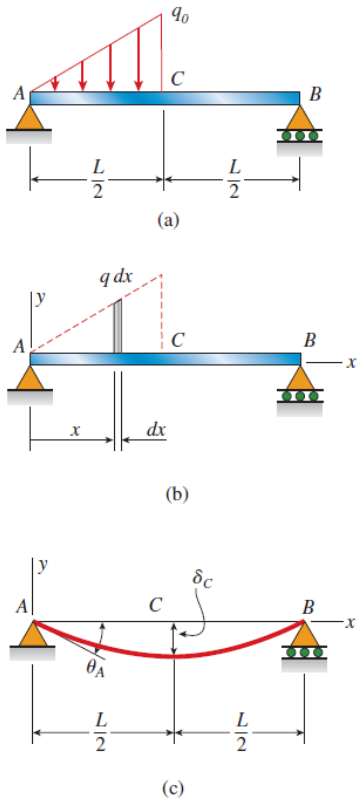
1. Superpose the deflection equations or deflections and slopes at specific locations determined for each type of the loads

2. If a given pattern of distributed loads is not available in the table, one can use the results about concentrated loads as shown in the following example

Table G-1

Deflections and Slopes of Cantilever Beams

		v = deflection in the y direction (positive upward) $v' = dv/dx$ = slope of the deflection curve $\delta_B = -v(L)$ = deflection at end B of the beam (positive downward) $\theta_B = -v'(L)$ = angle of rotation at end B of the beam (positive clockwise) EI = constant
1		$v = -\frac{qx^2}{24EI}(6L^2 - 4Lx + x^2) \qquad v' = -\frac{qx}{6EI}(3L^2 - 3Lx + x^2)$ $\delta_B = \frac{qL^4}{8EI} \qquad \theta_B = \frac{qL^3}{6EI}$
2		$v = -\frac{qx^2}{24EI}(6a^2 - 4ax + x^2) \qquad (0 \le x \le a)$ $v' = -\frac{qx}{6EI}(3a^2 - 3ax + x^2) \qquad (0 \le x \le a)$ $v = -\frac{qa^3}{24EI}(4x - a) \qquad v' = -\frac{qa^3}{6EI} \qquad (a \le x \le L)$ At $x = a$: $v = -\frac{qa^4}{8EI} \qquad v' = -\frac{qa^3}{6EI}$ $\delta_B = \frac{qa^3}{24EI}(4L - a) \qquad \theta_B = \frac{qa^3}{6EI}$



3. From Appendix G, the midpoint deflection caused by a concentrated load P at $x = a$ $(a \le b)$ is

$$\frac{Pa}{48EI}(3L^2 - 4a^2)$$

4. The midpoint deflection caused by qdx at location x is thus

$$\frac{(qdx)x}{48EI}(3L^2 - 4x^2)$$

5. In the example, the distributed load is given as

$$q(x) = \frac{2q_0 x}{L}$$

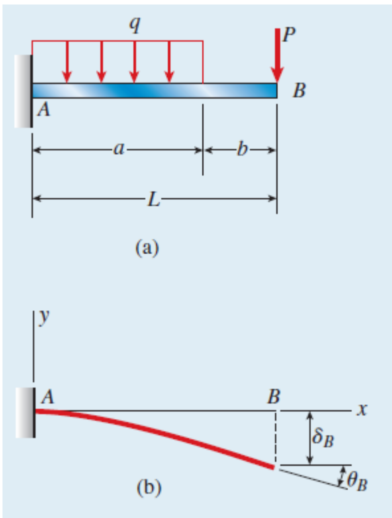
6. Substituting this into the equation in "4",

$$\frac{q_0 x^2}{24EI}(3L^2 - 4x^2)dx$$

7. Finally, the deflection at C is obtained by the integral

$$\delta_C = \int_0^{\frac{L}{2}} \frac{q_0 x^2}{24EI}(3L^2 - 4x^2)dx = \frac{q_0 L^4}{240EI}$$

⦿ **Example 9-6**: A cantilever beam *AB* supports a uniform load of intensity q acting over part of the span and a concentrated load P acting at the free end. Determine the deflection δ_B and the angle of rotation θ_B at end *B*.



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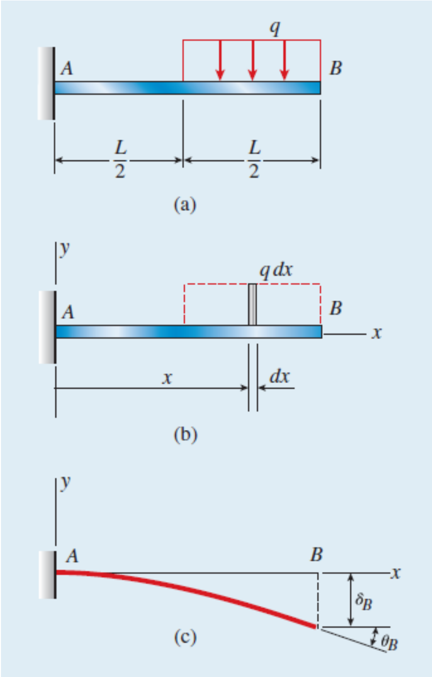
⦿ **Example 9-7**: A cantilever beam *AB* supports a uniform load of intensity q acting on the right-hand half of the beam. Determine the deflection δ_B and the angle of rotation θ_B at the free end.



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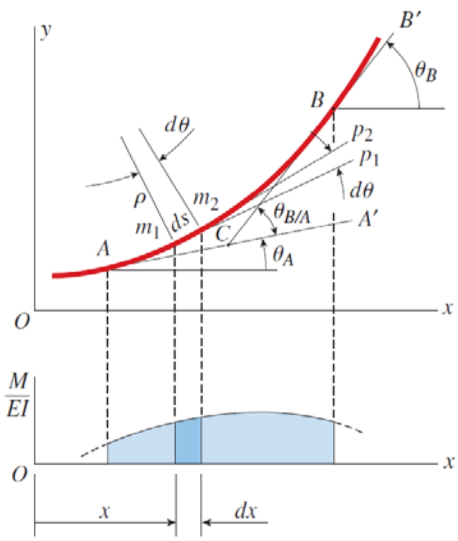
9.6 Moment-Area Method

⦿ Moment-Area Method

1. Refers to two theorems related to the area of the b_______-m_______ diagram $M(x)$ → Find the angle and deflection of a beam

2. Assumptions: L________ e________ materials and s_____ deformation

⦿ First Moment-Area Theorem



1. The angle between the two tangents at points A and B

$$\theta_{B/A} = \theta_B - \theta_A$$

2. The angle between two orthogonal lines at m_1 and m_2

$$d\theta = \frac{ds}{\rho} \cong \frac{dx}{\rho}$$

3. For a beam consisting of a linear elastic material $1/\rho = M/EI$

$$d\theta = \frac{Mdx}{EI}$$

4. Integrating the change in the angle from point A to point B,

$$\int_A^B d\theta = \theta_{B/A}$$

$$\boldsymbol{\theta_{B/A} = \int_A^B \frac{Mdx}{EI}}$$

= Area of the M/EI diagram between points A and B

5. **First moment-area theorem:** The angle $\theta_{B/A}$ between the tangents to the deflection curve at two points A and B is equal to the area of the M/EI diagram between those points

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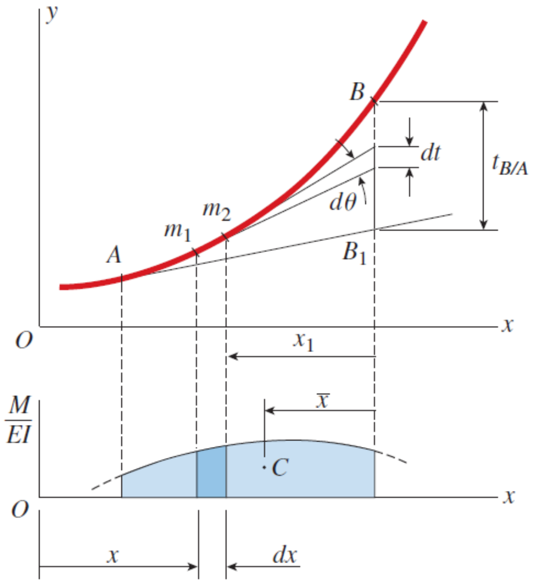
⦿ Second Moment-Area Theorem

1. Tangent deviation of point B with respect to A: $t_{B/A}$



2. Tangent deviation of point B made by the element m_1m_2:

$$dt = x_1 d\theta = x_1 \frac{Mdx}{EI}$$

3. This is the contribution of the element m_1m_2 to the total tangent deviation. Thus, the total tangent deviation is

$$\boldsymbol{t_{B/A} = \int_A^B dt = \int_A^B x_1 \frac{Mdx}{EI}}$$

4. This is <u>the first moment of the area</u> of the M/EI diagram between points A and B, evaluated <u>with respect to point B</u>

5. **Second moment-area theorem:** The tangential deviation $t_{B/A}$ of point B from the tangent at point A is equal to the first moment of the area of the M/EI diagram between A and B, evaluated with respect to B.

6. **Note:** The first moment can be computed alternative by (   )×(   )

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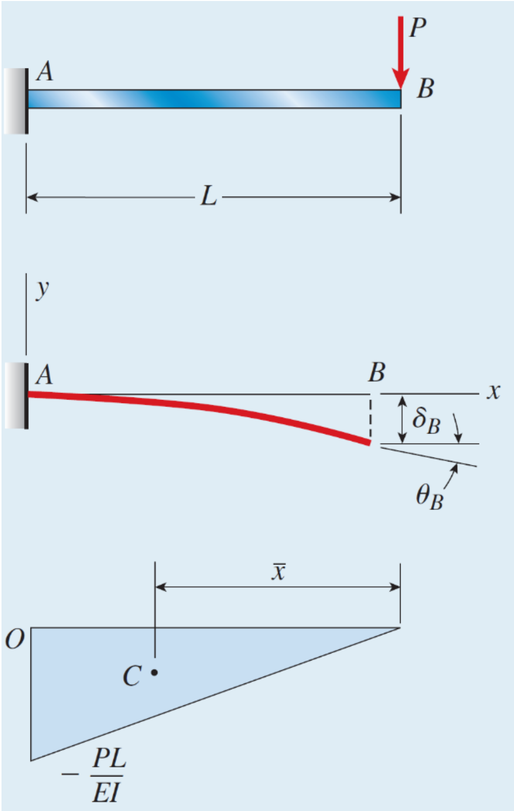
⦿ **Example 9-10**: Determine the angle of rotation θ_B and deflection δ_B at the free end B of a cantilever beam AB supporting a concentrated load P using the moment-area method.



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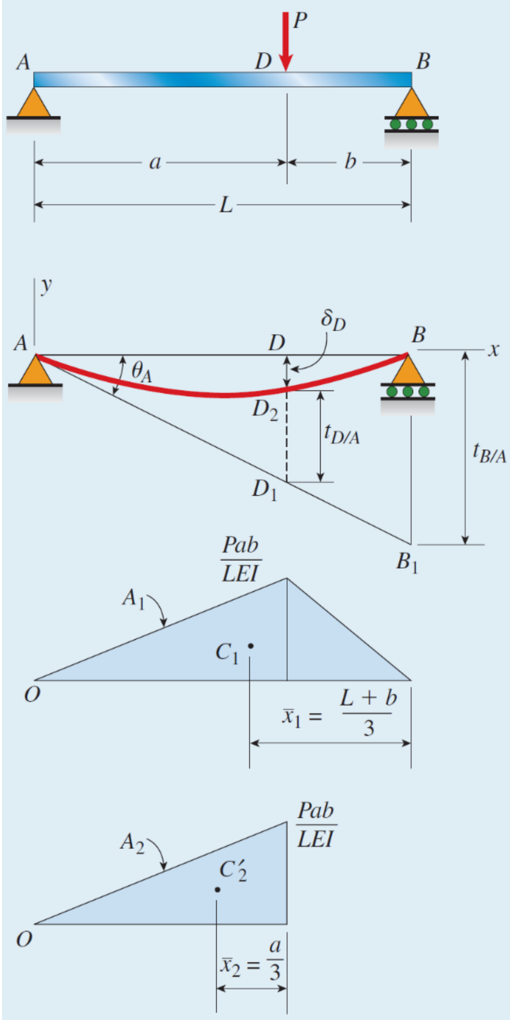
⦿ **Example 9-12**: A simple beam ADB supports a concentrated load P acting at the position shown in the figure. Determine the angle of rotation θ_A at support A and the deflection δ_D under the load P using the moment area method.



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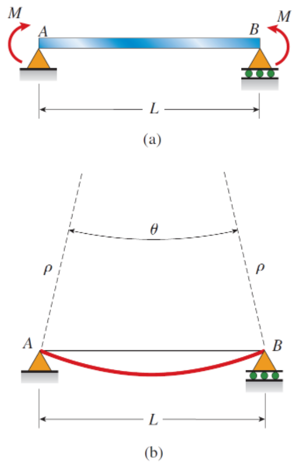
9.8 Strain Energy of Bending



⦿ Strain Energy of a Beam under Bending

1. Consider a linear elastic beam under pure bending showing small deformation, i.e. the bending moment is constant and Hooke's law works

2. The angle of the circular arc is

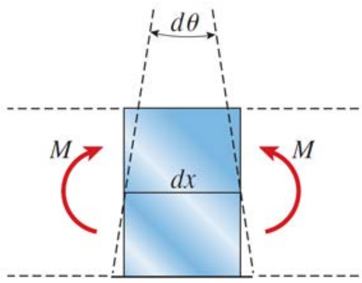
$$\theta = \frac{L}{\rho} = \kappa L = \frac{ML}{EI}$$

3. Linear relationship between M and θ → Therefore, when the bending moment gradually increases, the work and the stored strain energy are

$$W = U = \frac{M\theta}{2}$$

4. Using the linear relationship in Item 3, the strain energy can be expressed as

$$U = \frac{M^2 L}{2EI} \quad \text{or} \quad U = \frac{EI\theta^2}{2L}$$



5. If the bending moment varies, the angle between the side faces of a small element with length dx is

$$d\theta = \kappa dx = \frac{d^2v}{dx^2}dx$$

6. In analogy to Item 4, the strain energy in the element is

$$dU = \frac{M^2 dx}{2EI} \quad \text{or} \quad dU = \frac{EI(d\theta)^2}{2dx} = \frac{EI}{2dx}\left(\frac{d^2v}{dx^2}dx\right)^2 = \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx$$

7. By integrating the strain energy along the beam, the strain energy is derived as

$$U = \int \frac{M^2 dx}{2EI} \quad \text{or} \quad U = \int \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx$$

⦿ Deflections Caused by a Single Load

1. Strain energy U by a concentrated load P and a couple M_0 is $\frac{P\delta}{2}$ and $\frac{M_0\theta}{2}$

2. Deflection and the rotation can be computed by

$$\delta = \frac{2U}{P} \quad \text{and} \quad \theta = \frac{2U}{M_0}$$

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- **Example 9-6**: Cantilever beam AB is subjected to three different loading conditions: (a) a concentrated load P at its free end, (b) a couple M_0 at its free end, and (c) both loads acting simultaneously. For each loading condition, determine the strain energy of the beam. Also, determine the vertical deflection δ_A at end A of the beam due to the load P acting alone, and determine the angle of rotation θ_A at end A due to the moment M_0 acting alone.