"10장 보충자료" → 문제의 풀이 방식 별로

①

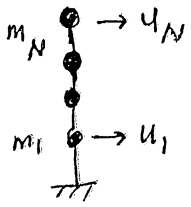
* See Fig. 10.1.1 ~ 10.1.2

10.2 Natural Frequencies and Modes

(Eigenvalue analysis, modal and spectral matrices, orthogonality property, and normalization of modes etc.)

$\boxed{\text{Eigenvalue Analysis}}$

$$\underline{m}\,\ddot{\underline{u}} + \underline{k}\,\underline{u} = \underline{0} \quad \text{---- (1)}$$ ← 비감쇠 자유진동 방정식



$$\underline{u} = \underline{u}(0), \quad \dot{\underline{u}} = \dot{\underline{u}}(0) \quad \text{---- (2)}$$ ← 초기조건

「2계 연립 선형 상미방 풀이의 상황 (일반적으로 coupled)」

"If free vibration is initiated by "appropriate" distributions of displacements" in the various DOFs, an undamped structure undergoes simple harmonic motion without change of deflected shape." ← See Fig. 10.1.1 ~ 2

Observation (?)

↳ natural mode of vibration (일반적으로 DOF수 만큼)

'mode shape'

$$\underline{u} = \underline{\phi}_n\, q_n(t), \quad \ddot{\underline{u}} = -\omega_n^2\, \underline{\phi}_n\, q_n(t) \quad \text{--- (3)}$$

where $\underline{\phi}_n$ = n th mode shape (constant vector)

$q_n(t)$ = amplitude

= simple harmonic function

$= A_n \cos\omega_n t + B_n \sin\omega_n t$

$= C_n \sin(\omega_n t - \alpha)$

"동력학의 근원적 해석"

↳ 「특정 변형 패턴을 유지하면서 조화진동하는 경우를 찾아보자」

(3) → (1)

$$\left[-\omega_n^2\, \underline{m}\, \underline{\phi}_n + \underline{k}\, \underline{\phi}_n\right] \times q_n(t) = 0 \quad \text{for any "t".}$$

$$-\omega_n^2\, \underline{m}\, \underline{\phi}_n + \underline{k}\, \underline{\phi}_n = \underline{0}$$

$$\left[\underline{k} - \omega_n^2\, \underline{m}\right] \underline{\phi}_n = \underline{0} \quad \text{--- (4)}$$ ← Eigenvalue problem

↳ 미지수 (N+1)개, 방정식 N개

By Cramer's rule,

$$\phi_{in} = \frac{\det \begin{bmatrix} & \overset{i^{th}\text{ column}}{\vdots} & \end{bmatrix} \leftarrow [\underline{k} - \omega_n^2 \underline{m}]}{\det [\underline{k} - \omega_n^2 \underline{m}]}$$

$$= \frac{0}{\det [\underline{k} - \omega_n^2 \underline{m}]}$$

Non-trivial solution을 얻기 위해서는 $\left(\frac{0}{0}\right)$ 형태의 不定形이 되어야 하므로

$$\underbrace{\det [\underline{k} - \omega_n^2 \underline{m}]}_{} = 0 \quad \text{----} \quad (5)$$

ω_n^2에 대한 N차 다항식 (특성방정식, characteristic eq.)

Numerical method: See Bathe.

특성방정식의 근을 근의 eigenvalue (고유치)라 하고 일반적으로 N개의 근이 존재한다. 이 값을 (4)식에 대입하여 $\underline{\phi}_n$을 구할 수 있는데 이를 eigenvector (고유벡터)라 한다. 미지수 = N+1, 방정식은 N개인 문제이므로 $\underline{\phi}_n$의 값은 상대비로서만 결정된다.

eigenvalue

$$\omega_1 < \omega_2 < \omega_3 \cdots < \omega_N$$

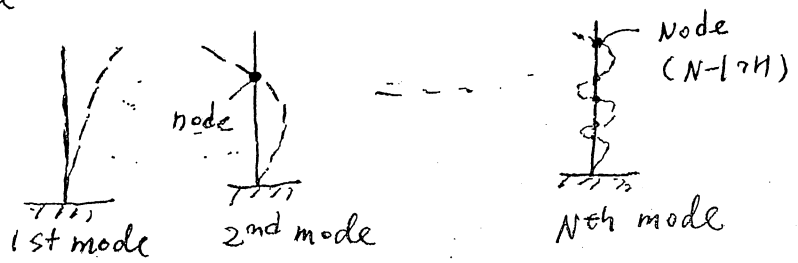
고유원진동수 or 고유주기

$$\underbrace{T_1 = \frac{2\pi}{\omega_1}}_{\text{Fundamental period}} > T_2 = \frac{2\pi}{\omega_2} \cdots > T_N = \frac{2\pi}{\omega_n}$$

← eigenvector

$$\underline{\phi}_1 \quad \underline{\phi}_2 \quad \cdots \quad \underline{\phi}_N$$ 진동모드

Fundamental mode



Modal and Spectral matrices

$$\underline{\Phi} = [\phi_{jn}] = [\underline{\phi}_1, \underline{\phi}_2, \cdots \underline{\phi}_N]$$ ← modal matrix (row vector로 생각)

$$\underline{\Omega}^2 = \mathrm{diag.}(\omega_n^2) = \begin{bmatrix} \omega_1^2 & & \underline{0} \\ & \omega_2^2 & \\ & & \ddots \\ \underline{0} & & & \omega_N^2 \end{bmatrix}$$ ← spectral matrix

$\underline{\Phi}$ - Ω^2의 관계 : "$\underline{k}\,\underline{\Phi} = \underline{m}\,\underline{\Phi}\,\underline{\Omega}^2$" --- (☆)

$$\therefore \underline{k}\,[\underline{\phi}_1, \underline{\phi}_2, \cdots \underline{\phi}_N] = \underline{m}\,[\underline{\phi}_1, \underline{\phi}_2, \cdots \underline{\phi}_N]\,\underline{\Omega}^2$$

$$= \underbrace{[\underline{m}\,\underline{\phi}_1, \underline{m}\,\underline{\phi}_2, \cdots \underline{m}\,\underline{\phi}_N]}_{1 \times N} \underbrace{\begin{bmatrix} \omega_1^2 & & \underline{0} \\ & \ddots & \\ \underline{0} & & \omega_N^2 \end{bmatrix}}_{N \times N}$$

$$[\underline{k}\,\underline{\phi}_1, \underline{k}\,\underline{\phi}_2, \cdots \underline{k}\,\underline{\phi}_N]$$

$$= [\underline{m}\,\underline{\phi}_1 \omega_1^2, \cdots \underline{m}\,\underline{\phi}_N \omega_N^2]$$

$$\therefore \underline{k}\,\underline{\phi}_n = \omega_n^2\,\underline{m}\,\underline{\phi}_n \quad (n = 1, 2, \cdots N) \quad \text{--- (☆☆)}$$

결국 (☆)식은 (☆☆)식으로 표시되는 N개의 matrix equation을 'compact'한 symbol로서 표기한 것이라고 할 수 있음.

Orthogonality of modes

when $\omega_n \neq \omega_r$,

$$\underline{\phi}_n^T \underline{m}\, \underline{\phi}_r = 0 \quad \text{(orthogonal w.r.t mass)}$$

$$\underline{\phi}_n^T \underline{k}\, \underline{\phi}_r = 0 \quad \text{(orthogonal w.r.t stiffness)}$$

↑ 열강방정식을 decouple 시킬 수 있는 조건.

Proof) Maxwell-Betti's principle of reciprocity 에 상당

$$\underline{k}\,\underline{\phi}_n = \omega_n^2 \underline{m}\,\underline{\phi}_n \quad \leftarrow \text{n th mode 에 대한}$$

$$\underline{\phi}_r^T(\underline{k}\,\underline{\phi}_n) = \underline{\phi}_r^T(\omega_n^2 \underline{m}\,\underline{\phi}_n) \quad \text{--- (*)}$$

$$\underline{k}\,\underline{\phi}_r = \omega_r^2 \underline{m}\,\underline{\phi}_r \quad \leftarrow \text{r th mode 에 대한}$$

$$\underline{\phi}_n^T(\underline{k}\,\underline{\phi}_r) = \underline{\phi}_n^T(\omega_r^2 \underline{m}\,\underline{\phi}_r) \quad \text{--- (**)}$$

(*) 식의 좌, 우변을 전치 (모두 scalar 량임), 그리고 $\underline{k} = \underline{k}^T$, $\underline{m} = \underline{m}^T$ 대칭성 반영 :

$$\underline{\phi}_n^T \underline{k}\,\underline{\phi}_r = \omega_n^2 \underline{\phi}_n^T \underline{m}\,\underline{\phi}_r \quad \text{--- (***)}$$

(**) − (***) : $(\omega_r^2 - \omega_n^2)(\underline{\phi}_n^T \underline{m}\,\underline{\phi}_r) = 0$

mass 직교성 → 따라서 $\underline{\phi}_n^T \underline{m}\,\underline{\phi}_r = 0$. when $\omega_r \neq \omega_n$

강성직교성 → 따라서 $\underline{\phi}_n^T \underline{k}\,\underline{\phi}_r = \omega_r^2 \cdot \underbrace{\underline{\phi}_n^T \underline{m}\,\underline{\phi}_r}_{=0} = 0$

↳ Coupled 운동

Decouple 하고 조건

↳ modal superposition method 이용 조건

Note :

$$\underset{N\times N}{\underline{K}} = \underline{\Phi}^T \underline{k}\, \underline{\Phi}$$

$$= [\underline{\phi}_1, \underline{\phi}_2, \cdots \underline{\phi}_N]^T \cdot \underline{k}\, [\underline{\phi}_1, \underline{\phi}_2, \cdots \underline{\phi}_N]$$

$$= \underset{N\times 1}{\begin{bmatrix} \underline{\phi}_1^T \\ \underline{\phi}_2^T \\ \vdots \\ \underline{\phi}_N^T \end{bmatrix}} \underset{1\times N}{[\underline{k}\,\underline{\phi}_1, \underline{k}\,\underline{\phi}_2, \cdots \underline{k}\,\underline{\phi}_N]}$$

비대각성분들은 모두 zero (due to orthogonality)

$$= \underset{N\times N}{\begin{bmatrix} \underline{\phi}_1^T \underline{k}\, \underline{\phi}_1 & & \underline{0} \\ & \ddots & \\ \underline{0} & & \underline{\phi}_N^T \underline{k}\, \underline{\phi}_N \end{bmatrix}} = \underset{N\times N}{\mathrm{diag}\,(\underbrace{\underline{\phi}_m^T \underline{k}\, \underline{\phi}_m}_{K_m})}$$

유사하게 적용하여

$$\underset{N\times N}{\underline{M}} = \underline{\Phi}^T \underline{m}\, \underline{\Phi} = \underset{N\times N}{\mathrm{diag.}\,(\underbrace{\underline{\phi}_m^T \underline{m}\, \underline{\phi}_m}_{M_m})}$$

Note :

$$K_m = \underline{\phi}_m^T (\underline{k}\, \underline{\phi}_m) = \underline{\phi}_m^T (\omega_m^2\, \underline{m}\, \underline{\phi}_m)$$

$$= \omega_m^2\, (\underline{\phi}_m^T \underline{m}\, \underline{\phi}_m)$$

$$\therefore\ K_m = \omega_m^2 M_m$$

Normalization of modes

$$\underline{\phi}_m = \begin{bmatrix} \phi_{1m} \\ \vdots \\ \phi_{im} \\ \vdots \\ \phi_{Nm} \end{bmatrix}$$

← 모든 값이 상대비로만 정의되므로 종종 특정의 취약이 따라 이들 모든 값의 상대크기를 조정할 필요가 있다 (상대크기의 표준화)

Ex. 1) 절대치가 확대된 모드가 1.0 이 되도록 normalize

2) 구조물의 특정 dof (가령 roof 층) 가 1.0 이 되도록 정규화

3) 동상의 컴퓨터 프로그램에서는 mass matrix 에 대하여 ortho-normal 하도록 정규화, 즉

$$\begin{cases} M_m = \underline{\phi}_m^T \underline{m}\, \underline{\phi}_m = 1.0 \quad \text{or} \quad \underline{\Phi}^T \underline{m}\, \underline{\Phi} = \underline{I}_{N\times N} \\ K_m = \omega_m^2 M_m = \omega_m^2 \end{cases}$$

가령 $\hat{\underline{\phi}}_m$ = 비정규 고유벡터라고 할 때

$\sqrt{\hat{\underline{\phi}}_m \underline{m}\, \hat{\underline{\phi}}_m}$ 로써 ... 각 모드를 나누면 될 것임.

실제로 $$\underline{\phi}_m = \frac{1}{\sqrt{\hat{\underline{\phi}}_m \underline{m}\, \hat{\underline{\phi}}_m}}\, \hat{\underline{\phi}}_m$$

$$\underline{\phi}_m^T \underline{m}\, \underline{\phi}_m = \left(\frac{1}{\sqrt{\hat{\underline{\phi}}_m \underline{m}\, \hat{\underline{\phi}}_m}}\, \hat{\underline{\phi}}_m^T \right) \underline{m} \left(\frac{1}{\sqrt{\hat{\underline{\phi}}_m \underline{m}\, \hat{\underline{\phi}}_m}}\, \hat{\underline{\phi}}_m \right)$$

$$= \frac{\hat{\underline{\phi}}_m^T \underline{m}\, \hat{\underline{\phi}}_m}{\left(\sqrt{\hat{\underline{\phi}}_m \underline{m}\, \hat{\underline{\phi}}_m} \right)^2} = 1.0$$

* 예제 10-1 ~ 10-5 :

follow-up 요.

10.7 Modal expansion of displacements (보충)

"Any set of N independent vectors can be used as a basis for representing any other vector of order N." (基)

↑ 「basis는 잘기 나름」

"미지수"

$$\underline{u} = \sum_{r=1}^{N} \underline{\phi}_r \times q_r = \underbrace{\underline{\Phi}}_{1\times N} \times \underline{q}_{N\times 1} = [\underline{\phi}_1 \cdots \underline{\phi}_N] \begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix} \quad \cdots (10.7.1)$$

$\underline{u}$: 임의의 변위 vector (given)

$\underline{\phi}_r$: 고유 vector (known)

$\underbrace{\underline{\Phi}}_{1\times N}$: 실제론 $N\times N$

($\underline{\phi}_r$의 중첩으로 표시되는 vector space로 생각 가능)

"물리적 basis"

Note: ① $\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + u_N \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

② q_r : 새로운 basis에 대응되는 좌표계수 ← change of basis 시 불리 (원래는 연립 방정식 풀기 $\underset{N\times N}{\underline{\Phi}}\ \underset{N\times 1}{\underline{q}} = \underset{N\times 1}{\underline{u}}$ (미지수))

(modal or normal coordinate)

③ $\underline{\phi}$의 직교성을 이용하여 $\underline{q}$를 연립방정식을 풀지 않고 구할 수 있다는 것이 핵심 내용

"mass-orthogonality 이용하면"

$$\underline{\phi}_m^T \underline{m} \times \underline{u} = \underline{\phi}_m^T \underline{m} \times \underline{\Phi}\,\underline{q} = \underline{\phi}_m^T \underline{m} \times [\underline{\phi}_1 \cdots \underline{\phi}_m \cdots \underline{\phi}_N]\,\underline{q}$$

$$= [0 \cdots, \underline{\phi}_m^T \underline{m}\,\underline{\phi}_m, \cdots 0] \begin{bmatrix} q_1 \\ \vdots \\ q_m \\ \vdots \\ q_N \end{bmatrix}$$

$$= \underline{\phi}_m^T \underline{m}\,\underline{\phi}_m \times q_m$$

$$= M_m \cdot q_m$$

$$\therefore\ q_m = \frac{\underline{\phi}_m^T \underline{m}\,\underline{u}}{M_m} \quad \cdots (10.7.2)$$

Note: stiffness-orthogonality를 이용한다면,

$$\underline{\phi}_m^T \underline{k}\,\underline{u} = \underline{\phi}_m^T \underline{k}\,\underline{\Phi}\,\underline{q} \rightarrow q_m = \frac{\underline{\phi}_m^T \underline{k}\,\underline{u}}{K_m} \quad \cdots (10.7.2')$$

☆ [The modal expansion of the displacement $\underline{u}$ plays a central role in the analysis of free, forced vibration response and earthquake response of MDF systems.]

"$\underline{u} = \sum \underline{\phi}_r \cdot q_r$"

실용적에도 약점

10.8 Solution of free vibration equations : undamped case

일반적으로 coupled !

Given : $\underline{m}\,\ddot{\underline{u}} + \underline{k}\,\underline{u} = \underline{0}$ with $\underline{u}_0 = \underline{u}(0)$ and $\dot{\underline{u}}_0 = \dot{\underline{u}}(0)$

Eigen solution : $[\underline{k} - \omega_n^2 \underline{m}]\,\underline{\phi}_n = \underline{0}$ (done) ← 계산과정의 대부분을 차지 !

Solution form : $\underline{u}(t) = \sum_{n=1}^{N} \underline{\phi}_n \times q_n$ ← N개 모드의 중첩

$$= \sum_{n=1}^{N} \underline{\phi}_n \times (A_n \cos\omega_n t + B_n \sin\omega_n t) \quad \text{--- (1)}$$

n차 모드의 비감쇠 자유진동식 $= q_n$

↑ 이렇게 풀어갈 수 있는 이유는 ⑨쪽의 설명참조"

$$\dot{\underline{u}}(t) = \sum_{n=1}^{N} \underline{\phi}_n \times (-A_n \sin\omega_n t + B_n \cos\omega_n t) \quad \text{--- (2)}$$

ω_n

Setting $t=0$ in Eqs (1) & (2),

$$\underline{u}(0) = \sum_{n=1}^{N} \underline{\phi}_n \times \overset{q(0)}{A_n}, \quad \dot{\underline{u}}(0) = \sum_{n=1}^{N} \underline{\phi}_n \times \overset{\dot{q}(0)}{(\omega_n B_n)} \quad \text{--- (3)}$$

Mass-orthogonality 이용하면

$$\underline{\phi}_n^T \underline{m}\, \underline{u}(0) = M_n \times A_n \rightarrow A_n = \frac{\underline{\phi}_n^T \underline{m}\, \underline{u}(0)}{M_n}$$

$$\underline{\phi}_n^T \underline{m}\, \dot{\underline{u}}(0) = M_n \times (\omega_n B_n) \rightarrow B_n = \frac{\underline{\phi}_n^T \underline{m}\, \dot{\underline{u}}(0)}{M_n \omega_n} \qquad \frac{\dot{q}(0)}{\omega_n}$$

$$\therefore \underline{u}(t) = \sum_{n=1}^{N} \underline{\phi}_n \times \left[\overset{q(0)}{\frac{\underline{\phi}_n^T \underline{m}\, \underline{u}(0)}{M_n}} \cos\omega_n t + \frac{\underline{\phi}_n^T \underline{m}\, \dot{\underline{u}}(0)}{M_n \omega_n} \sin\omega_n t \right]$$

SDF system과의 유사성에 주목 (요라서 Eq. 2.1.3)

Note: Decoupling of equations

$$\underset{N\times N}{\underline{m}}\ \underset{N\times 1}{\ddot{\underline{u}}} + \underset{N\times N}{\underline{k}}\ \underset{N\times 1}{\underline{u}} = \underset{N\times 1}{\underline{0}} \quad \text{with } \underline{u}_0 = \underline{u}(0) \ \&\ \dot{\underline{u}}_0 = \dot{\underline{u}}(0)$$

$$\left\{ \begin{array}{l} \underline{u} = \Sigma\, \underline{\phi}_n \times q_n \ , \quad \ddot{\underline{u}} = \Sigma\, \underline{\phi}_n \times \ddot{q}_n \end{array} \right.$$

$$\underline{m}\, \Sigma \underline{\phi}_n \cdot \ddot{q}_n + \underline{k}\ \Sigma \underline{\phi}_n \cdot q_n = \underline{0}$$

$$\underline{\phi}_r^T \left(\underline{m}\, \Sigma \underline{\phi}_n \cdot \ddot{q}_n + \underline{k}\ \Sigma \underline{\phi}_n \cdot q_n \right) = \underline{\phi}_r^T \cdot \underline{0}$$

$$\left(\underline{\phi}_r^T \underline{m}\ \underline{\phi}_r \right) \ddot{q}_r + \underline{\phi}_r^T \underline{k}\ \underline{\phi}_r \cdot q_r = 0$$

← N개의 decoupled 방정식 $(r = 1, 2 \ldots N)$

$$\ddot{q}_r + \frac{\underline{\phi}_r^T \underline{k}\ \underline{\phi}_r}{\underline{\phi}_r \underline{m}\ \underline{\phi}_r} \cdot q_r = 0$$

or $\ddot{q}_r + \omega_r^2 q_r = 0 \xrightarrow[\text{Solution form}]{} q_r = q_r(0) \cos \omega_r t + \dfrac{\dot{q}_r(0)}{\omega_r} \sin \omega_r t$

"② 값을 (1)식에서

$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t$ 의

형태로 가정할 수 있었던 것"

10.9 – 10.10 Solution of free vibration equations with "classical damping"

damping-orthogonality 가정

Given : $\underline{m}\, \ddot{\underline{u}} + \underline{c}\, \dot{\underline{u}} + \underline{k}\ \underline{u} = \underline{0}$ with $\underline{u}_0 = \underline{u}(0)$ & $\dot{\underline{u}}_0 = \dot{\underline{u}}(0)$

Eigen solution : $[\underline{k} - \omega_n^2 \underline{m}]\, \underline{\phi}_n = \underline{0}$ (done)

Solution form : (modal expansion)

$$\underline{u}(t) = \sum_{n=1}^{N} \underline{\phi}_n \times q_n(t) = \underline{\Phi}\, \underline{q}$$

$$\dot{\underline{u}}(t) = \Sigma\, \underline{\phi}_n \dot{q}_n \ , \quad \ddot{\underline{u}}(t) = \Sigma\, \underline{\phi}_n \ddot{q}_n$$

$$= \underline{\Phi}\, \dot{\underline{q}} \qquad\qquad = \underline{\Phi}\, \ddot{\underline{q}}$$

Decoupling of equations :

$$\underline{\phi}_n^T x \left(\underline{m}\,\underline{\Phi}\,\ddot{\underline{q}} + \underline{c}\,\underline{\Phi}\,\dot{\underline{q}} + \underline{k}\,\underline{\Phi}\,\underline{q} \right) = \underline{\phi}_n^T x\, \underline{0}$$

$$\underbrace{(\underline{\phi}_n^T \underline{m}\, \underline{\phi}_n)}_{M_n} \ddot{q}_n + \underbrace{(\underline{\phi}_n^T \underline{c}\, \underline{\phi}_n)}_{C_n} \dot{q}_n + \underbrace{(\underline{\phi}_n^T \underline{k}\, \underline{\phi}_n)}_{K_n} q_n = 0$$

(orthogonality assumed)

$$\ddot{q}_n + \frac{C_n}{M_n} \dot{q}_n + \frac{K_n}{M_n} q = 0 \qquad (n = 1, 2, \cdots, N)$$

$$\left\{ \begin{array}{l} \dfrac{C_n}{M_n} = 2\zeta_n \omega_n \quad (\zeta_n = \text{damping ratio of } n^{th} \text{ mode}) \\ \dfrac{K_n}{M_n} = \omega_n^2 \quad (\text{from eigen property}) \end{array} \right.$$

$$\ddot{q}_n + 2\zeta_n \omega_n + \omega_n^2 q = 0 \qquad (n = 1, 2, \cdots N)$$

↳ N decoupled equation

$$q_n(t) = e^{-\zeta_n \omega_n t} \left[q_n(0) \cos \omega_D t + \frac{\dot{q}_n(0) + \zeta_n \omega_n q_n(0)}{\omega_D} \sin \omega_D t \right]$$

↳ Eq. (2.2.4)

where

$$\underline{u}(0) = \sum \underline{\phi}_n q_n(0) \rightarrow q_n(0) = \frac{\underline{\phi}_n^T \underline{m}\, \underline{u}(0)}{\underline{\phi}_n^T \underline{m}\, \underline{\phi}_n}$$

mass 직교성 이용

$$\dot{\underline{u}}(0) = \sum \underline{\phi}_n \dot{q}_n(0) \rightarrow \dot{q}_n(0) = \frac{\underline{\phi}_n^T \underline{m}\, \dot{\underline{u}}(0)}{\underline{\phi}_n^T \underline{m}\, \underline{\phi}_n}$$

$$\therefore \underline{u}(t) = \sum_{n=1}^{N} \underline{\phi}_n \times q_n(t)$$

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(10.6, 8, 9, 11-12, (5-16)

"EX. 10.9, 10.10, 10.11, 10.12, 10.13 참고"

Substituting $q_n(t)$ and ϕ_n in Eq. (10.8.7) leads to

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \cos \omega_1 t + \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \cos \omega_2 t$$

These solutions for $q_n(t)$ and $u_j(t)$ had been plotted in Fig. 10.1.1c and d. Observe that both natural modes contribute to the response due to these initial displacements.

10.9 FREE VIBRATION OF SYSTEMS WITH DAMPING

When damping is included, the free vibration response of the system is governed by Eq. (9.2.12) with $\mathbf{p}(t) = \mathbf{0}$:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{0} \tag{10.9.1}$$

It is desired to find the solution $\mathbf{u}(t)$ of Eq. (10.9.1) that satisfies the initial conditions

$$\mathbf{u} = \mathbf{u}(0) \qquad \dot{\mathbf{u}} = \dot{\mathbf{u}}(0) \tag{10.9.2}$$

at $t = 0$. A procedure to obtain the desired solution will be developed in Section 10.10 for certain forms of damping that are reasonable models for many real structures. In this section the solution is presented in graphical form for a specific system that enables us to understand qualitatively the effects of damping on the free vibration of MDF systems.

For this purpose we express the displacement $\mathbf{u}$ in terms of the natural modes of the system without damping; thus we substitute Eq. (10.7.1) in Eq. (10.9.1):

$$\mathbf{m}\underbrace{\Phi\ddot{\mathbf{q}}}_{\ddot{\mathbf{u}}} + \mathbf{c}\underbrace{\Phi\dot{\mathbf{q}}}_{\dot{\mathbf{u}}} + \mathbf{k}\underbrace{\Phi\mathbf{q}}_{\mathbf{u}} = \mathbf{0}$$

Premultiplying by Φ^T gives

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{10.9.3}$$

where the diagonal matrices $\mathbf{M}$ and $\mathbf{K}$ were defined in Eq. (10.4.5) and

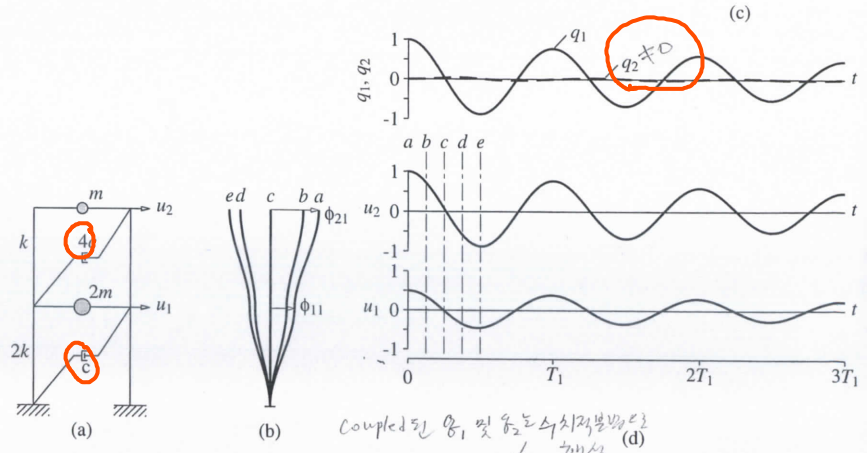
$$\mathbf{C} = \Phi^T\mathbf{c}\Phi \tag{10.9.4}$$

The square matrix $\mathbf{C}$ may or may not be diagonal, depending on the distribution of damping in the system. If $\mathbf{C}$ is diagonal, Eq. (10.9.3) represents N uncoupled differential equations in modal coordinates q_n, and the system is said to have *classical damping* because classical modal analysis (Chapters 12 and 13) is applicable to such systems. These systems possess the same natural modes as those of the undamped system. The general form of the damping matrix that satisfies these properties is presented in Chapter 11. Systems with damping such that $\mathbf{C}$ is nondiagonal are said to have *nonclassical damping*. These systems are not amenable to classical modal analysis, and they do not possess the same natural modes as the undamped system.

10.9.1 Nonclassically Damped System

Figure 10.9.1a shows the two-story system of Fig. E10.4 but now with viscous dampers of coefficients c and $4c$ in the two stories. The equations of motion for this system are given by Eq. (10.9.1) with

$$\mathbf{m} = m\begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \qquad \mathbf{k} = k\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{c} = c\begin{bmatrix} 5 & -4 \\ -4 & 4 \end{bmatrix} \tag{10.9.5}$$



coupled된 q_1 및 q_2도 수치적분방법을 해석

Figure 10.9.1 Free vibration of a nonclassically damped system due to initial displacement in the first natural mode of the undamped system: (a) two-story frame; (b) deflected shapes at time instants a, b, c, d, and e; (c) modal coordinates $q_n(t)$; (d) displacement history.

The $\mathbf{m}$ and $\mathbf{k}$ are available from Example 10.4 and $\mathbf{c}$ is obtained from Eq. (9.1.10) with $c_1 = c$ and $c_2 = 4c$. The natural modes of the system without damping were computed in Example 10.4:

$$\phi_1 = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \tag{10.9.6}$$

Substituting these $\mathbf{m}$, $\mathbf{k}$, $\mathbf{c}$, and ϕ_n in Eqs. (10.4.5), (10.4.6), and (10.9.4) gives the $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ which are inserted in Eq. (10.9.3) to obtain

$$m\begin{bmatrix} 1.5 & \\ & 3.0 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + c\begin{bmatrix} 1.25 & 3.50 \\ 3.50 & 17.00 \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + k\begin{bmatrix} 0.75 & \\ & 6.00 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.9.7}$$

Note that $\mathbf{C}$ is not a diagonal matrix; thus the two differential equations in $q_1(t)$ and $q_2(t)$ are coupled.

Numerical methods are presented in Chapter 15 for solving such coupled differential equations. Using those methods, Eq. (10.9.7) was solved for the system considered, with $\omega = \sqrt{km/200}$ subjected to initial displacements $\mathbf{u}(0)$. The results for initial displacements $\mathbf{u}(0) = \langle \frac{1}{2} \quad 1 \rangle^T$ are presented in Fig. 10.9.1 and for $\mathbf{u}(0) = \langle -1 \quad 1 \rangle^T$ in Fig. 10.9.2. The solutions for $q_1(t)$ and $q_2(t)$ are presented in part (c) of these figures. Substituting $q_n(t)$ and ϕ_n in Eq. (10.7.1) gives the floor displacements $u_1(t)$ and $u_2(t)$, which are plotted

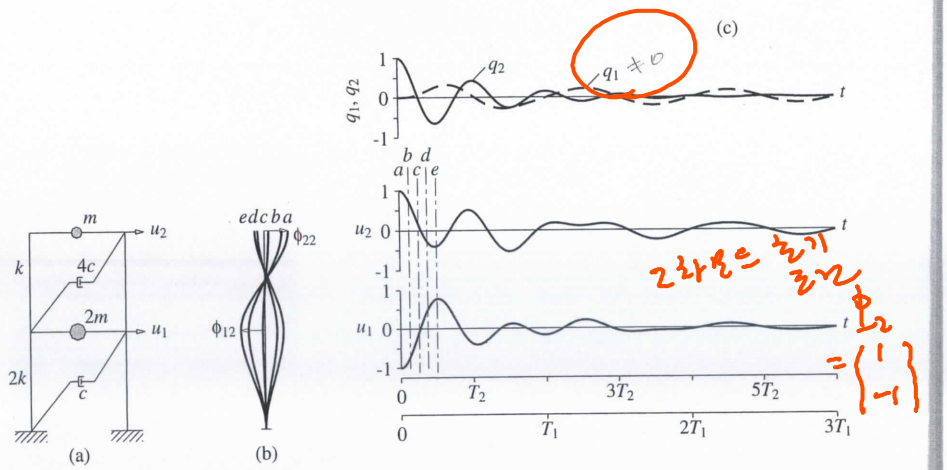


Figure 10.9.2 Free vibration of a nonclassically damped system due to initial displacement in the second natural mode of the undamped system: (a) two-story frame; (b) deflected shapes at time instants a, b, c, d, and e; (c) modal coordinates $q_n(t)$; (d) displacement history.

2차 진동모드 초기조건

in part (d) of these figures; the deflected shapes at selected time instants—a, b, c, d, and e—are plotted in part (b) of these figures.

These results are for a system disturbed from its equilibrium position by imposing displacements—defined by vectors $\mathbf{u}(0)$ in the preceding paragraph—which are proportional to the two natural modes ϕ_n of the undamped system; the results permit three observations. First, contrary to what we observed for undamped systems (Figs. 10.1.2 and 10.1.3), $q_2(t) \neq 0$ in Fig. 10.9.1c and $q_1(t) \neq 0$ in Fig. 10.9.2c. Second, the initial deflected shape is not maintained in free vibration; this is noticeable in Fig. 10.9.2b. Third, unlike for SDF systems with damping (Fig. 2.2.2), the motion in each DOF is not a damped simple harmonic motion with a unique frequency; this is noticeable in Fig. 10.9.2d.

주목사항
① —
② 초기처짐 모드모양에 의해 영향
③ 단일 주기 운동이 아님 (Fig. 10.9.2d)

10.9.2 Classically Damped System

← or, Orthogonal damping

Consider now the system shown in Fig. 10.9.3a, similar to the one considered previously but for a different distribution of damping. In this case the viscous damping coefficients for the first and second stories are $4c$ and $2c$, respectively, and $c = \sqrt{km/200}$. The equations of motion for this system are also given by Eq. (10.9.1), with $\mathbf{m}$ and $\mathbf{k}$ the same as in Eq. (10.9.5) and

$$\mathbf{c} = c\begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}$$

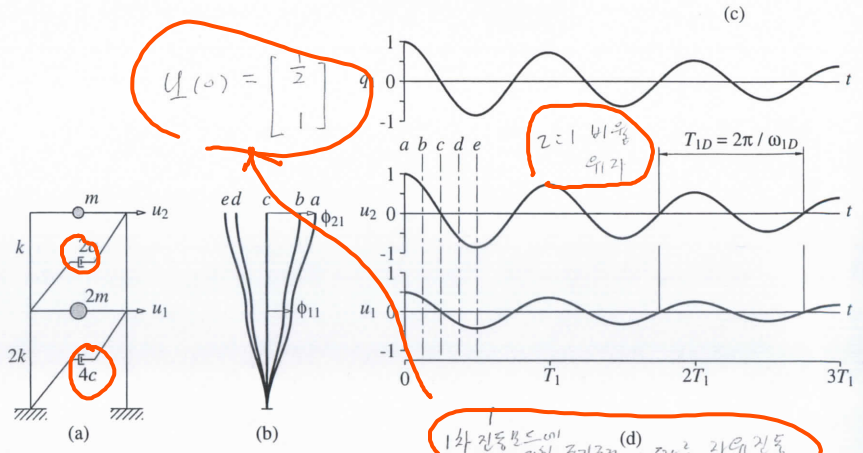


Figure 10.9.3 Free vibration of a classically damped system in the first natural mode of vibration: (a) two-story frame; (b) deflected shapes at time instants a, b, c, d, and e; (c) modal coordinate $q_1(t)$; (d) displacement history.

diagonal matrix

The equations governing the modal coordinates, Eq. (10.9.3), for this system are

$$m\begin{bmatrix} 1.5 & \\ & 3.0 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + c\begin{bmatrix} 1.5 & 0 \\ 0 & 12.0 \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + k\begin{bmatrix} 0.75 & \\ & 6.00 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10.9.8}$$

In this case $\mathbf{C}$ is a diagonal matrix and the two differential equations are uncoupled.

For an N-DOF system with classical damping, each of the N differential equations in modal coordinates is

$$M_n\ddot{q}_n + C_n\dot{q}_n + K_nq_n = 0 \tag{10.9.9}$$

$\leftrightarrow m\ddot{u} + c\dot{u} + ku = 0$ analogy 성립

where M_n and K_n were defined in Eq. (10.4.6) and

$$C_n = \phi_n^T\mathbf{c}\phi_n \tag{10.9.10}$$

즉 2자유도 SDF 계 해석방법과 동일

Equation (10.9.9) is of the same form as Eq. (2.2.1a) for an SDF system with damping. Thus the damping ratio can be defined for each mode in a manner analogous to Eq. (2.2.2) for an SDF system:

$$\zeta_n = \frac{C_n}{2M_n\omega_n} \tag{10.9.11}$$

← n차 모드의 감쇠비 (nth modal damping ratio)

and Eq. (10.9.9) can be solved for $q_n(t)$ by the methods of Chapter 2.

These calculations are implemented for the system of Fig. 10.9.3a: $C_1 = 1.5c$, $M_1 = 1.5m$, $\omega_1 = \sqrt{k/2m}$, and $c = \sqrt{km/200}$, and Eq. (10.9.11) gives $\zeta_1 = 0.05$. → $\zeta_1 = 5\%$ Similar computations lead to $\zeta_2 = 0.10$. Solution of Eq. (10.9.9) for each mode gives $q_n(t)$, and this is substituted in Eq. (10.7.1) to obtain the displacements $u_j(t)$. The results → $\zeta_2 = 10\%$

(10.9.9)에 의해 → (10.7.1)에 대입

$$u = \sum_{r=1}^{N} \phi_r \times q_r = \phi_1 q_1 + \phi_2 \times q_2$$