445.204

Introduction to Mechanics of Materials

(재료역학개론)

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Chapter 10

Deflections in beams

Outline

- The Elastic Curve
- Boundary Conditions
- Method of integration
- (Option) Use of Discontinuity Functions
- Method of Superposition
- Statically Indeterminate Beams
- Statically Indeterminate Beams — Method of Integration
- Statically Indeterminate Beams—Method of Superposition
- (Option) Moment-Area Method
- (Option) Statically Indeterminate Beams—Moment-Area Method

Elastic Curve

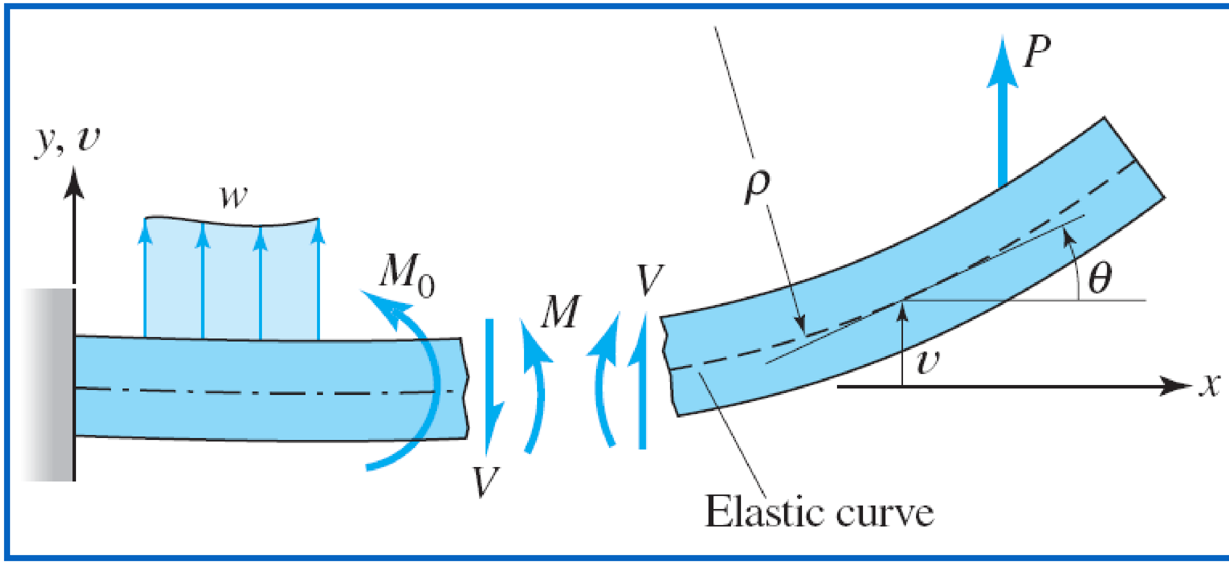


FIGURE 10.1 Positive loads and internal force resultants.

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

Bernoulli-Euler Law

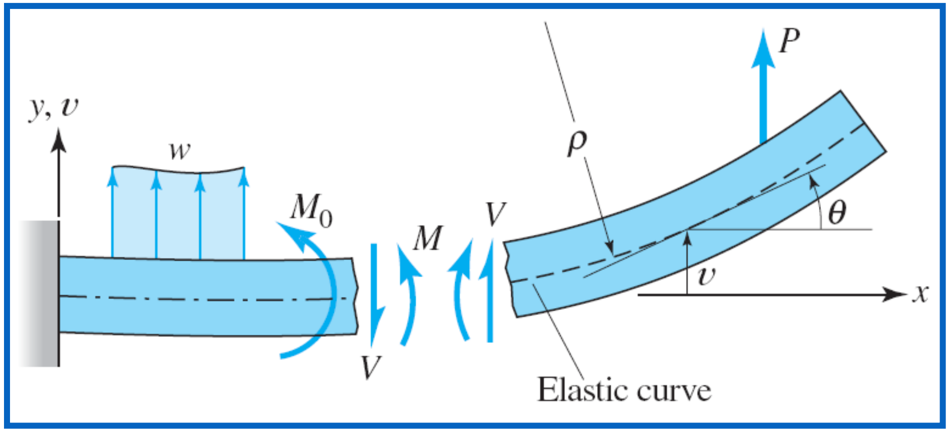
$$\kappa = \frac{1}{\rho} = \frac{d^2v/dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

$$\kappa = \frac{1}{\rho} = \frac{d^2v}{dx^2}$$

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

Sign convention

The deflection v is measured from the *x* axis to the elastic curve. An upward (↑) deflection is therefore positive. The angle of rotation θ (measured in radians) is the angle between the *x* axis and the tangent to the curve at a point. The sense of the θ vector, also of the *M* vector, follows the right-hand rule. That is, θ is positive when counterclockwise, as portrayed in the figure. For small displacements, the <u>angle of rotation is approximately equal to the slope</u> <u>(*dv /dx*) of the elastic curve</u>.



$$\text{Deflection} = v$$

$$\text{Slope} = \theta = \frac{dv}{dx} = v'$$

$$\text{Moment} = M = EI\,\frac{d\theta}{dx} = EIv''$$

$$\text{Shear} = V = \frac{dM}{dx} = (EIv'')'$$

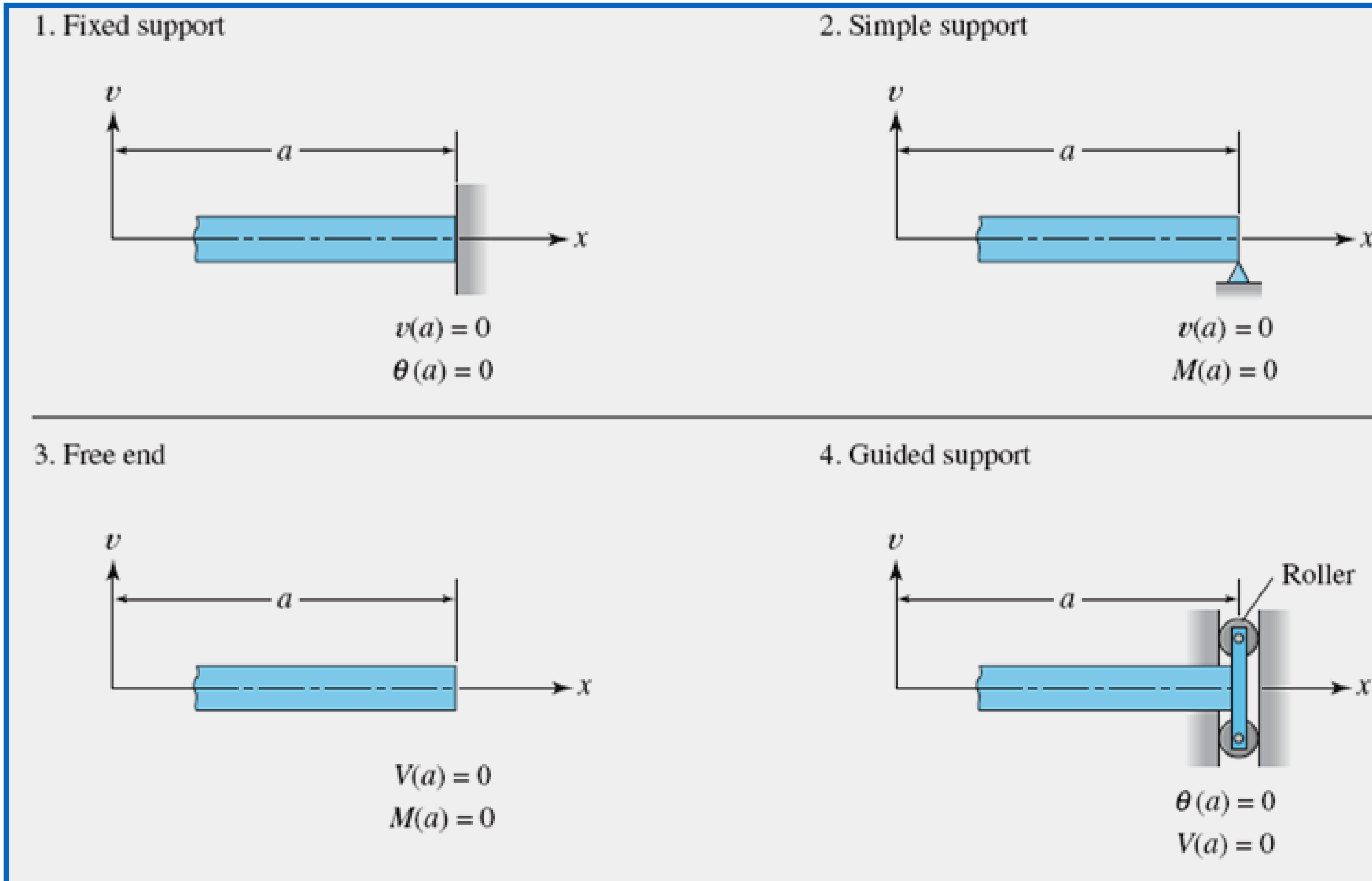
$$\text{Load} = w = \frac{dV}{dx} = (EIv'')''$$

$$\boxed{EI\,\frac{d^2v}{dx^2} = M}$$

$$\boxed{EI\,\frac{d^3v}{dx^3} = V}$$

$$\boxed{EI\,\frac{d^4v}{dx^4} = w}$$

Boundary conditions



Method of integration

Given: A simply supported beam AB carries a uniform load w per unit length.
Find: (a) equation of the elastic curve using the double-integration approach, (b) derive the equation of the elastic curve using the multiple-integration approach, (c) obtain the maximum deflection and the slope.

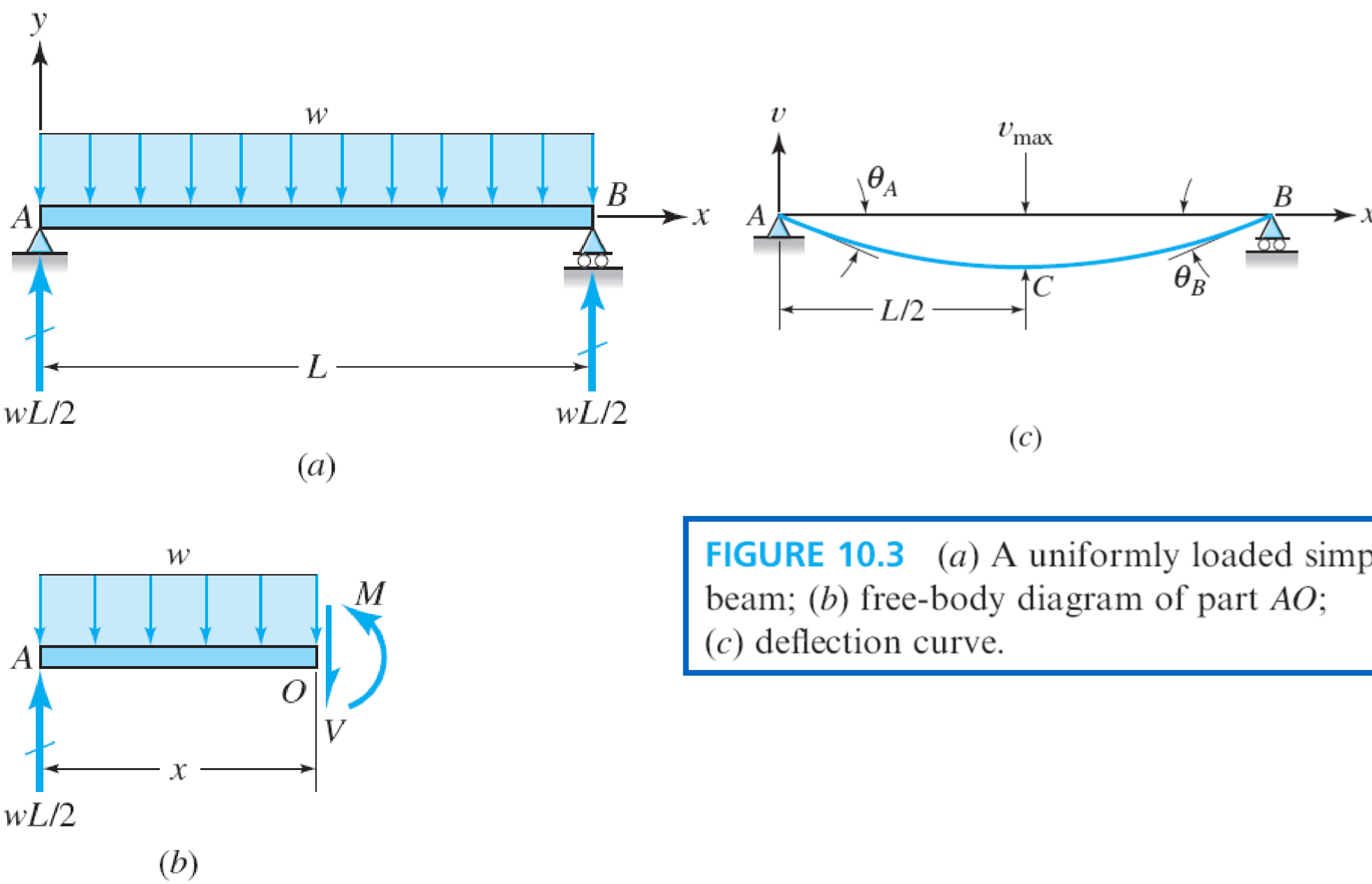


FIGURE 10.3 (*a*) A uniformly loaded simple beam; (*b*) free-body diagram of part *AO*; (*c*) deflection curve.

Bending moment at x,

$$\boxed{M = \frac{1}{2}\,wLx - \frac{1}{2}\,wx^2}$$

(1) Double integration method

$$\boxed{EI\,\frac{d^2v}{dx^2} = \frac{1}{2}\,wLx - \frac{1}{2}\,wx^2}$$

$$EI\,\frac{dv}{dx} = \frac{1}{4}\,wLx^2 - \frac{1}{6}\,wx^3 + C_1$$

$$EIv = \frac{1}{12}\,wLx^3 - \frac{1}{24}\,wx^4 + C_1x + C_2$$

With B.C.
v=0 at x=0, dv/dx=0 at x=L/2

$C_2 = 0$ and $C_1 = -\,w\,L^3/24$

Solution: slope and deflection

$$\frac{dv}{dx} = -\frac{w}{24EI}(L^3 - 6Lx^2 + 4x^3)$$

$$v = -\frac{w}{24EI}(L^3x - 2Lx^3 + x^4)$$

(2) Multiple integration method:

$$EI\,\frac{d^4v}{dx^4} = -w$$

$$EI\,\frac{d^3v}{dx^3} = -wx + C_1$$

$$EI\,\frac{d^2v}{dx^2} = -\frac{1}{2}\,wx^2 + C_1x + C_2$$

$$EI\,\frac{dv}{dx} = -\frac{1}{6}\,wx^3 + \frac{1}{2}\,C_1x^2 + C_2x + C_3$$

$$EIv = -\frac{1}{24}\,wx^4 + \frac{1}{6}\,C_1x^3 + \frac{1}{2}\,C_2x^2 + C_3x + C_4$$

B.C.s:

v=0 at x=0, L

dv/dx=0 at x=L/2

M =0 at x=0, L

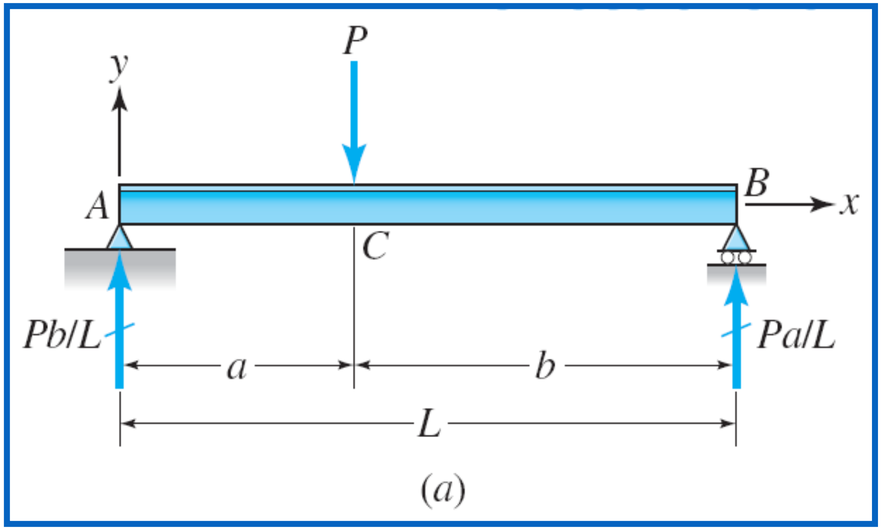
(3) Largest displacements and slopes

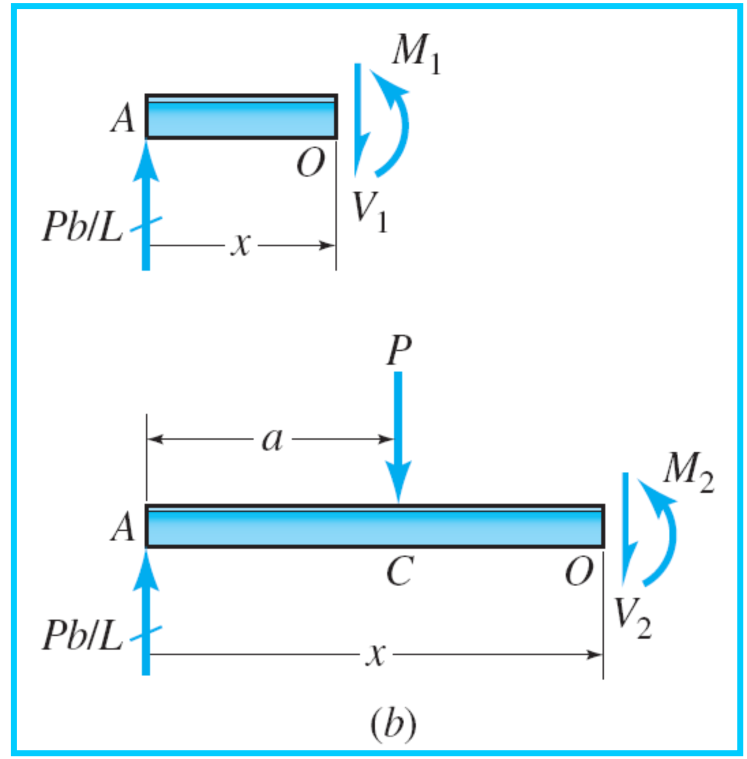
$$v_{\max} = v_C = -\frac{5wL^4}{384EI} = \frac{5wL^4}{384EI} \downarrow$$

$$\theta_A = v'(0) = -\frac{wL^3}{24EI} = \frac{wL^3}{24EI}$$

$$\theta_B = v'(L) = \frac{wL^3}{24EI}$$

Multi-interval method:





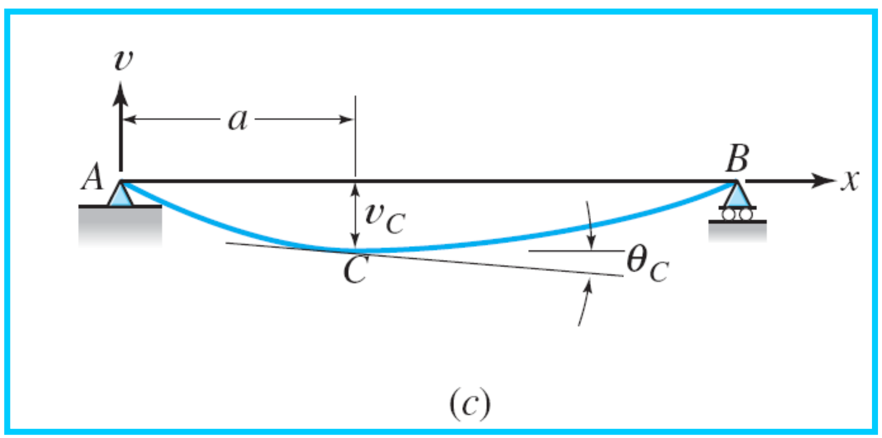
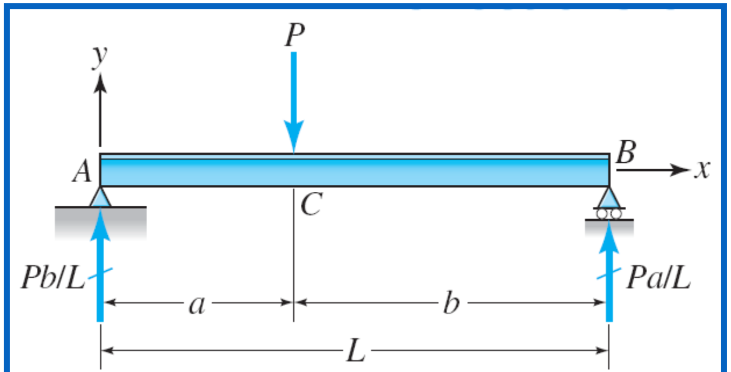


FIGURE 10.5 (*a*) Simply supported beam carries a load *P*; (*b*) free-body diagrams of two parts *AO*; (*c*) deflection curve.

a) Equation of elastic curve

$$M_1 = \frac{Pb}{L} x \qquad (0 \leq x \leq a)$$

$$M_2 = \frac{Pb}{L} x - P(x - a) \qquad (a \leq x \leq L)$$



(a)

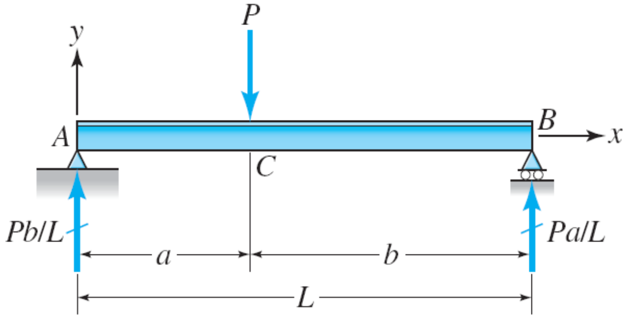
Double integration method,

For segment AC

$$EIv_1'' = \frac{Pb}{L}x$$

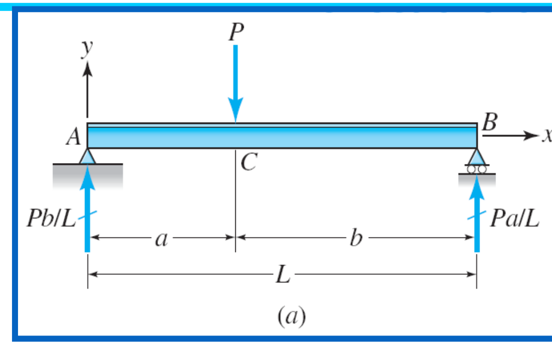
$$EIv_1' = \frac{Pb}{2L}x^2 + C_1$$

$$EIv_1 = \frac{Pb}{6L}x^3 + C_1 x + C_2$$



(a)

For segment *CB*



$$EIv_2'' = \frac{Pb}{L}x - P(x-a)$$

$$EIv_2' = \frac{Pb}{2L}x^2 - \frac{P}{2}(x-a)^2 + C_3$$

$$EIv_2 = \frac{Pb}{6L}x^3 - \frac{P}{6}(x-a)^3 + C_3x + C_4$$

Boundary and continuity of slope conditions

- Slope at both parts of the beam must be equal, i.e.,

$$\boxed{v_1'(a) = v_2'(a)}$$

- This condition yields that $C_1 = C_3$

- In a similar manner, using the other conditions yield:

$$v_1(a) = v_2(a): \quad \frac{Pa^3 b}{6L} + C_1 a + C_2 = \frac{Pa^3 b}{6L} + C_3 a + C_4;$$

$$v_1(0) = 0:$$

$$v_2(L) = 0: \quad \frac{PbL^3}{6L} - \frac{Pb^3}{6} + C_3 L = 0;$$

- This condition yields that $C_2 = C_4$; $C_2 = 0$ and $C_3 = -\ Pb(L_2 – b_2)/6L$

- The elastic curves for right and left segments:

$$v_1 = -\frac{Pbx}{6EIL}(L^2 - b^2 - x^2) \qquad (0 \leq x \leq a)$$

$$v_2 = -\frac{Pbx}{6EIL}(L^2 - b^2 - x^2) - \frac{P(x-a)^3}{6EI} \qquad (a \leq x \leq L)$$

Similarly, the *slopes* for the two parts of the beam are found:

$$v_1' = -\frac{Pb}{6EIL}(L^2 - 3x^2 - b^2) \qquad (0 \leq x \leq a)$$

$$v_2' = -\frac{Pb}{6EIL}(L^2 - 3x^2 - b^2) - \frac{P(x-a)^2}{2EI} \qquad (a \leq x \leq L)$$

- The deflection and slope at point C are:

$$\boxed{\begin{aligned} v_C &= -\frac{Pba}{6EIL}(L^2 - a^2 - b^2) \\ \theta_C &= -\frac{Pb}{6EIL}(L^2 - 3a^2 - b^2) \end{aligned}}$$

- The slope at points A and B are:

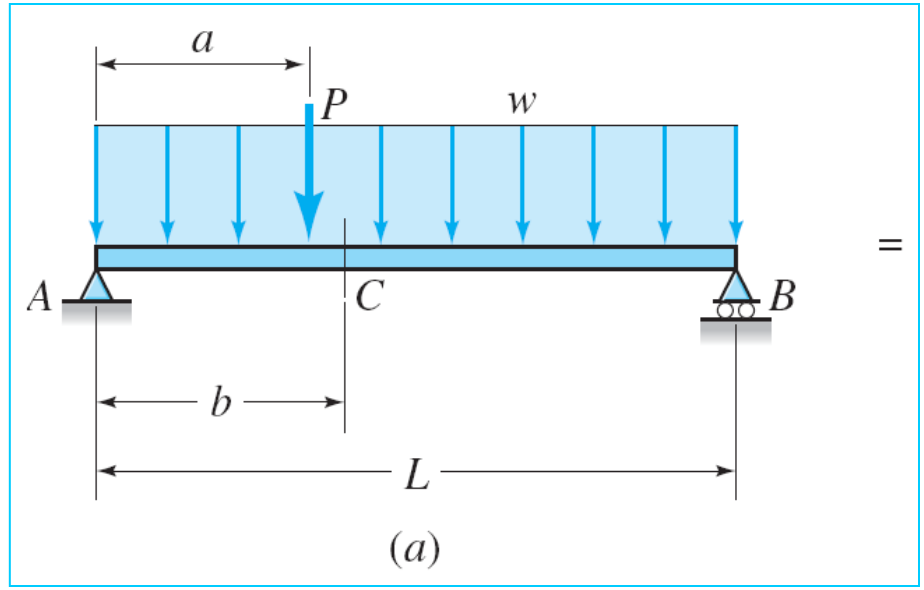
$$\boxed{\begin{aligned} \theta_A &= -\frac{Pb(L^2 - b^2)}{6EIL} = -\frac{Pab(L + b)}{6EIL} \\ \theta_B &= \frac{Pab(L + a)}{6EIL} \end{aligned}}$$

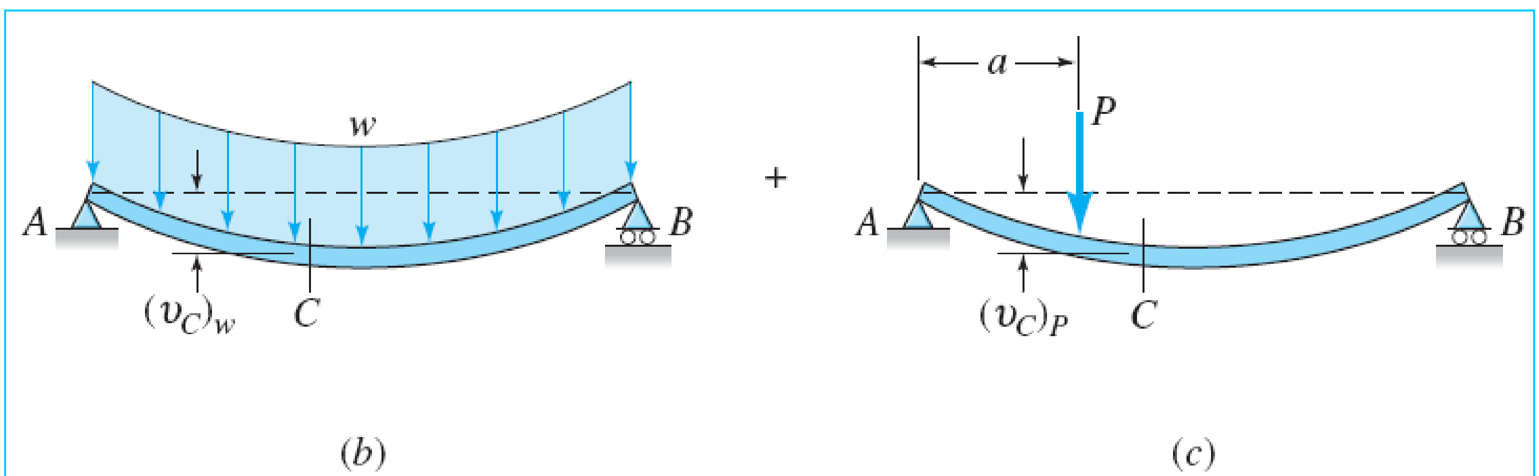
The largest deflection occurs at:

$$\boxed{x_m = \sqrt{\frac{L^2 - b^2}{3}}}$$

Method of Superposition

Valid for beams with small deflections with linear Hooke's law





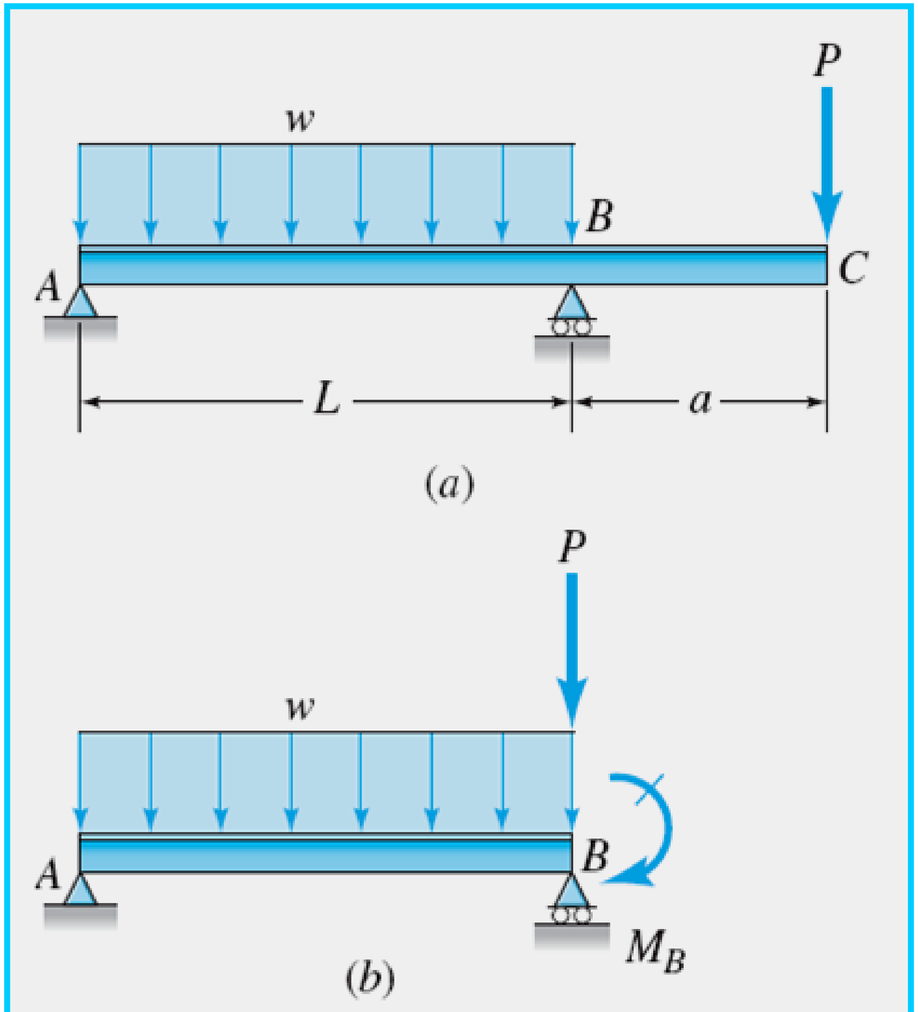
Method of Superposition

$$\boxed{v_C = (v_C)_w + (v_C)_P}$$

For the special case when *a = b = L/2:*

$$\boxed{v_C = \frac{5wL^4}{384EI} + \frac{PL^3}{48EI} \downarrow}$$

Method of Superposition



$$\theta_B = \frac{wL^3}{24EI} - \frac{PaL}{3EI}$$

$$(v_C)_M = \theta_B a = \frac{wL^3 a}{24EI} - \frac{Pa^2 L}{3EI}$$

$$v_C = \frac{wL^3 a}{24EI} - \frac{Pa^2}{3EI}(L + a)$$

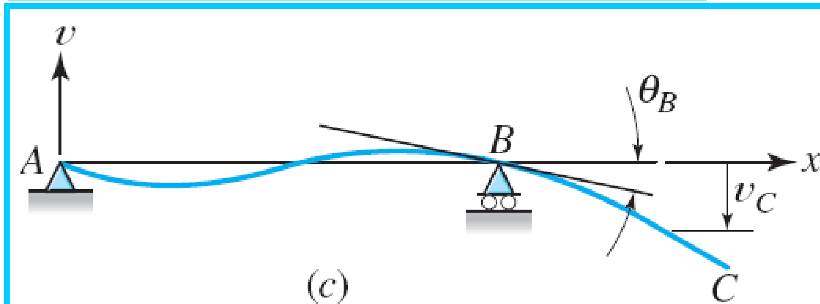


FIGURE 10.12 (*a*) Beam under two loads; (*b*) free-body diagram of part *AB*; (*c*) deflection curve.

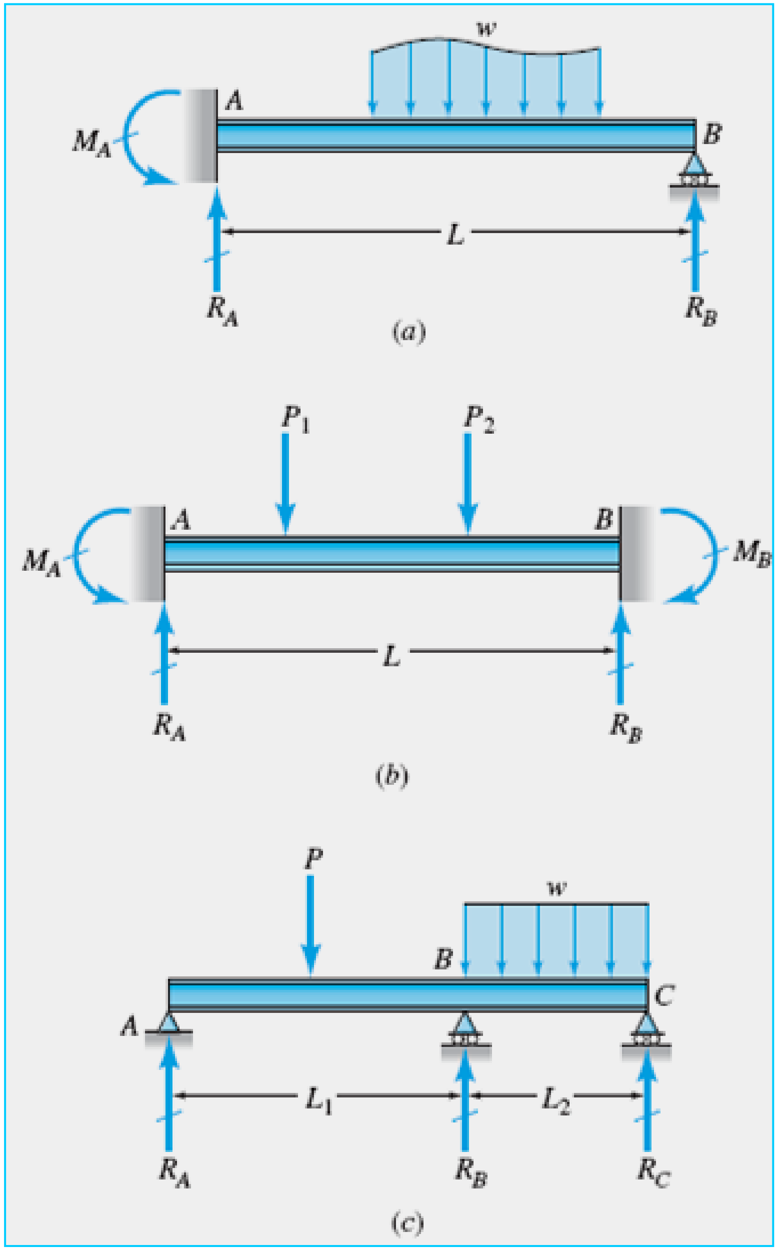
Statically Indeterminate Beams

- Beams that have more number of unknown reactions loads (forces or moments) than can be obtained by conditions of equilibrium

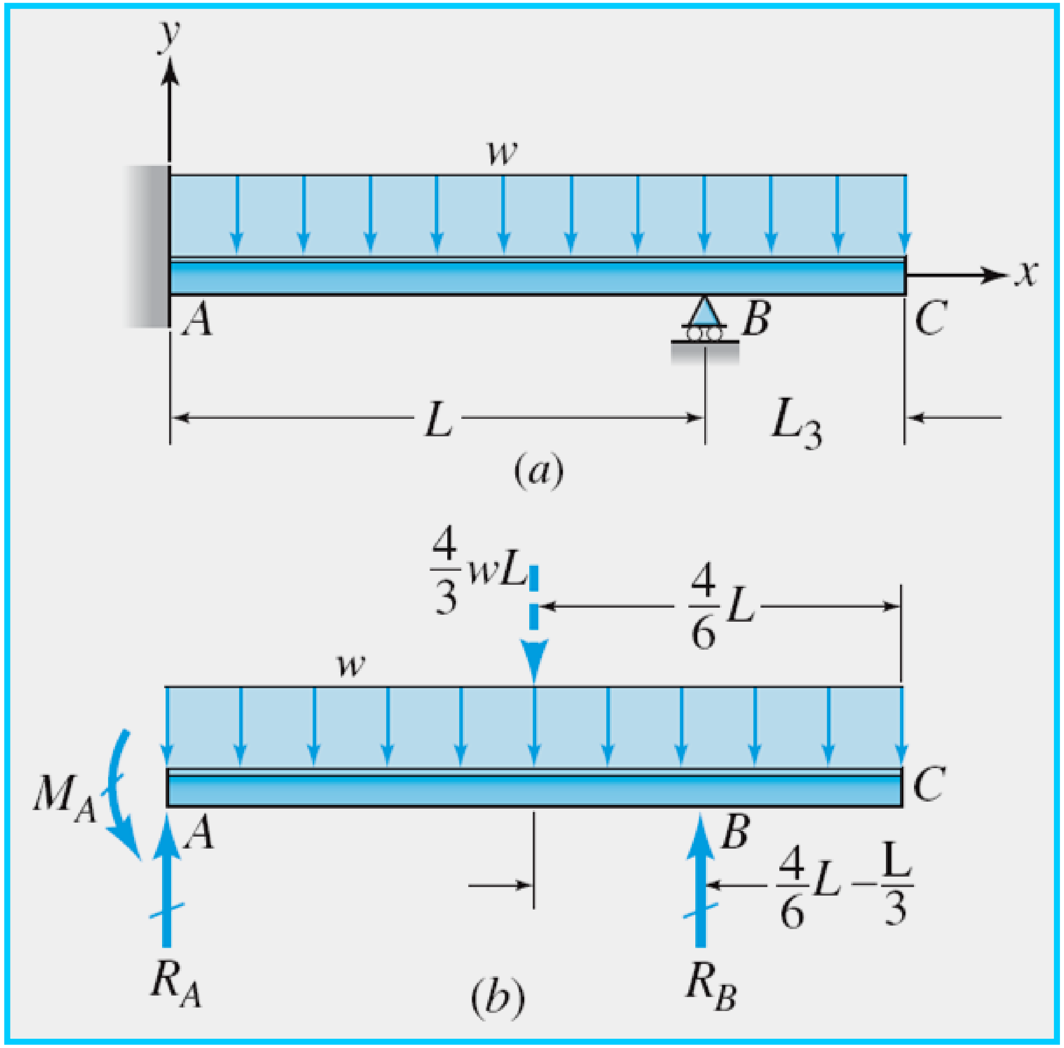
- The deflections of such beams must be taken into account and equations of compatibility obtained to supplement the static equilibrium conditions (see chapters 4 and 5)

Examples

FIGURE 10.13:
Types of statically indeterminate beams: (a) fixed-simply supported beam or propped cantilever beam; (b) fixed-end beam; (c) continuous beam.



Example: method of Integration



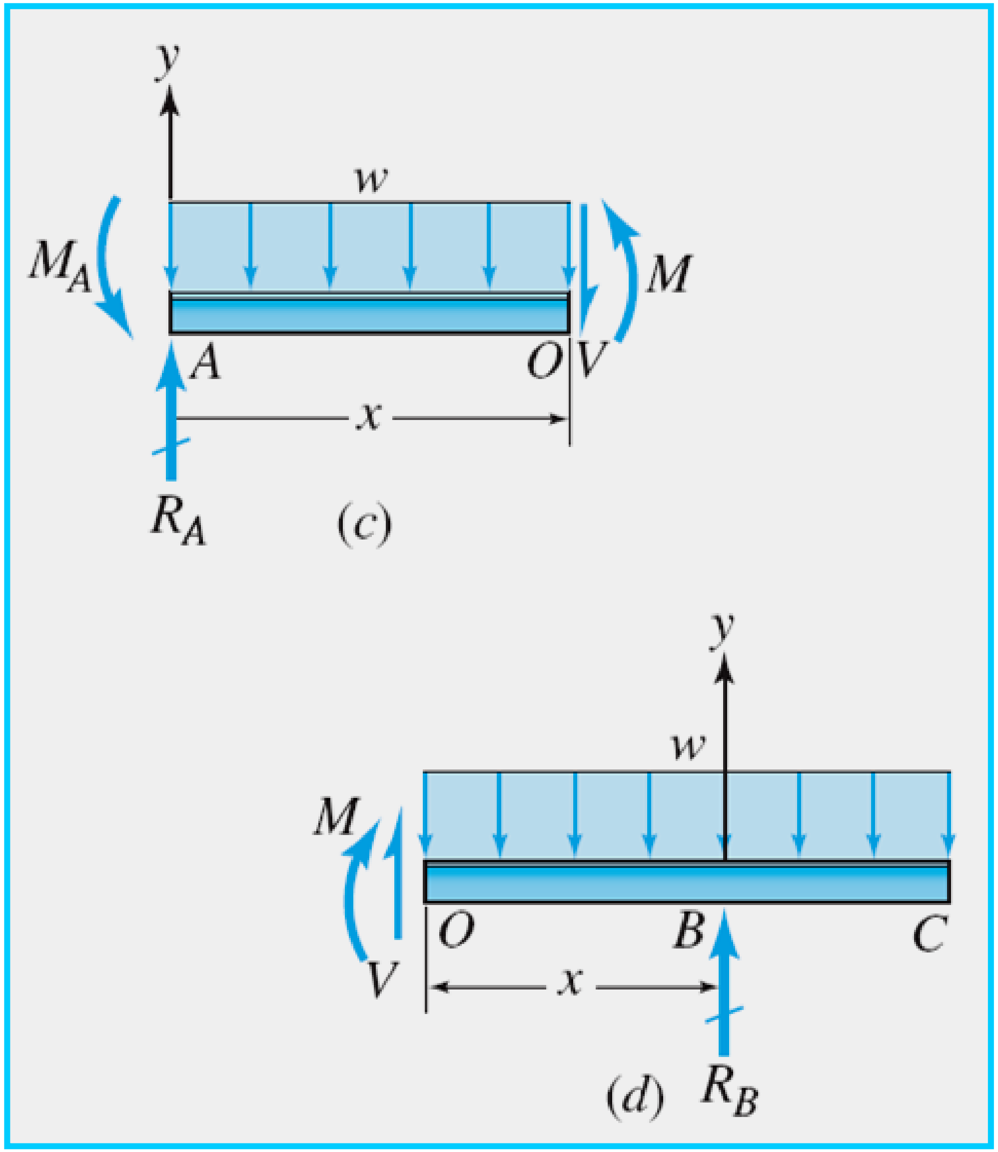


FIGURE 10.14: (a) A propped beam with an overhang under uniform load; (b) freebody diagram of the entire beam; (c, d) free-body diagrams of parts AO and OC, respectively.

(a) Double-integration method

Conditions of equilibrium give:

$$\boxed{\begin{aligned} R_A + R_B &= \frac{4}{3}wL \\ R_A L - M_A &= \left(\frac{4}{3}wL\right)\left(\frac{4L}{6} - \frac{L}{3}\right) = \frac{4}{9}wL^2 \end{aligned}}$$

$$\boxed{M = R_A x - M_A - \frac{1}{2}wx^2}$$

Conditions of deflection give:

$$\boxed{\begin{aligned} EI\,\frac{d^2v}{dx^2} &= R_A x - M_A - \frac{1}{2}\,wx^2 \\ EI\,\frac{dv}{dx} &= \frac{1}{2}\,R_A x^2 - M_A x - \frac{1}{6}\,wx^3 + C_1 \\ EIv &= \frac{1}{6}\,R_A x^3 - \frac{1}{2}\,M_A x^2 - \frac{1}{24} wx^4 + C_1 x + C_2 \end{aligned}}$$

Boundary conditions:

$C_1 = C_2 = 0$

$$4R_A L - 12M_A = wL^2$$

$$M_A = \frac{7}{72}\,wL^2 \circlearrowleft \quad R_A = \frac{13}{24}\,wL \uparrow \quad R_B = \frac{19}{24}\,wL \uparrow$$

(b) Multiple-integration method

$$EI\,\frac{d^4v}{dx^4} = -w$$

$$EI\,\frac{d^3v}{dx^3} = -wx + C_1$$

$$EI\,\frac{d^2v}{dx^2} = -\frac{1}{2}\,wx^2 + C_1x + C_2$$

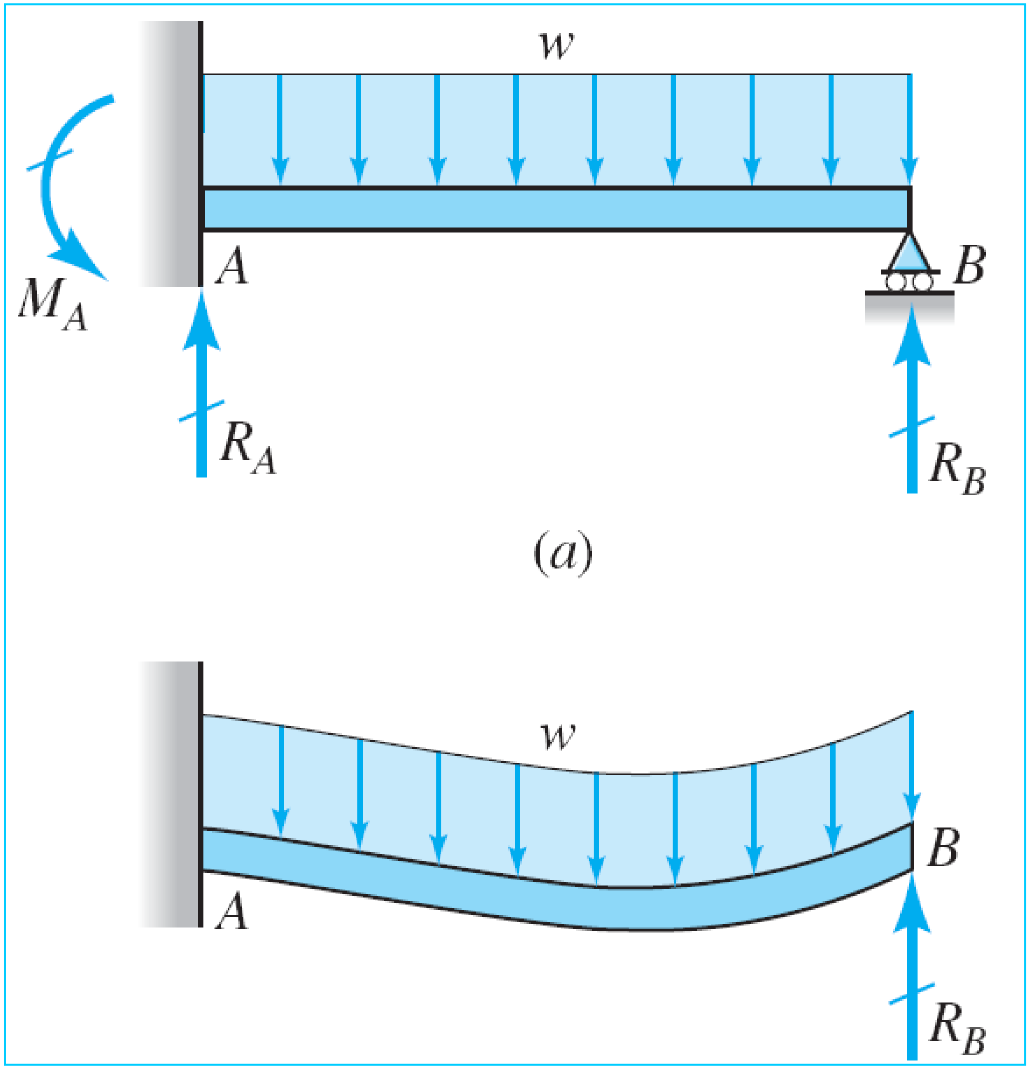
$$EI\,\frac{dv}{dx} = -\frac{1}{6}\,wx^3 + \frac{1}{2}\,C_1x^2 + C_2x + C_3$$

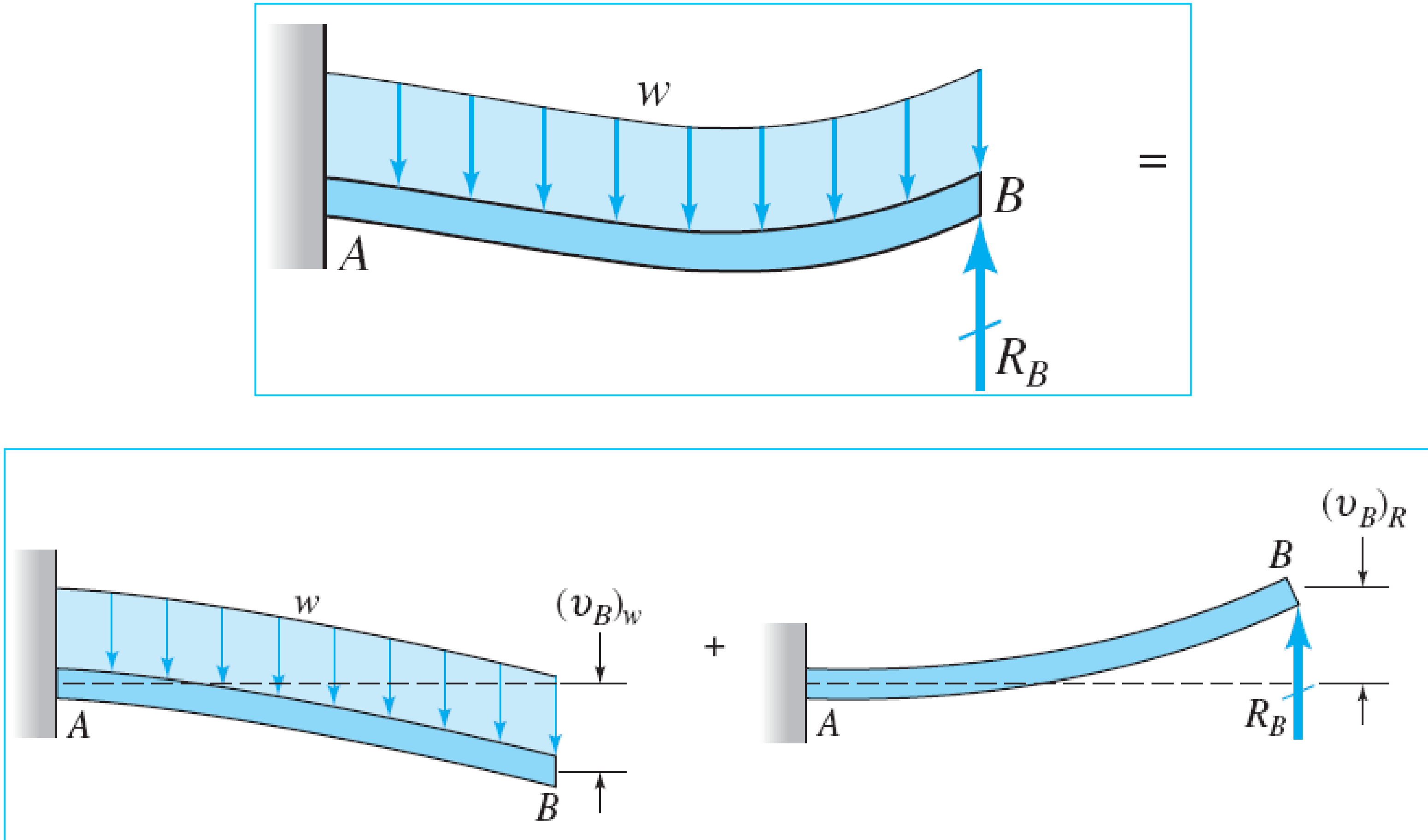
$$EIv = -\frac{1}{24}\,wx^4 + \frac{1}{6}\,C_1x^3 + \frac{1}{2}\,C_2x^2 + C_3x + C_4$$

Statically Indeterminate Beams — Method of Superposition

- The redundant reactions are considered as unknown loads and the corresponding supports are removed or modified accordingly. Then superposition is used, the deformation diagrams are drawn, and expressions are written for the displacements caused by the individual loads (both known and unknown). Next, these redundant reactions are computed such that the displacements fulfill the geometric boundary conditions. Finally, all other reactions can be readily determined from conditions of equilibrium.

Example





(c)