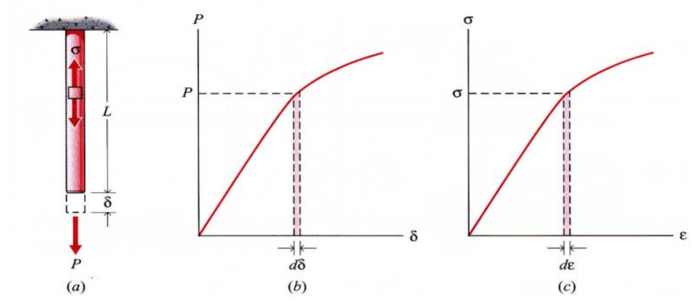
10. Strain Energy Methods and Failure Theories with Applications

10.1 Introduction

- Part A discusses the concept of strain energy and its application to stress and deformation determinations in members subjected to impact loading

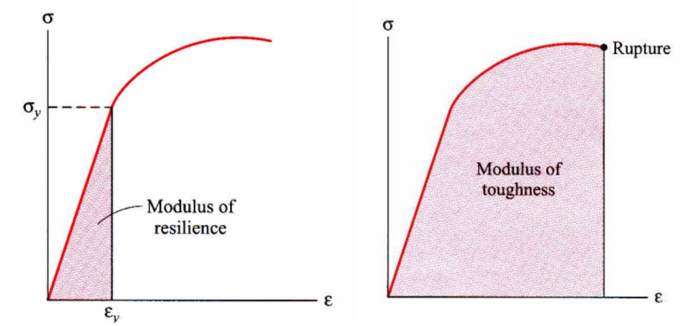
10.2 Strain Energy

- Strain energy by axial loading



Work done by external force: $W_k = \int_0^{\delta} P d\delta$

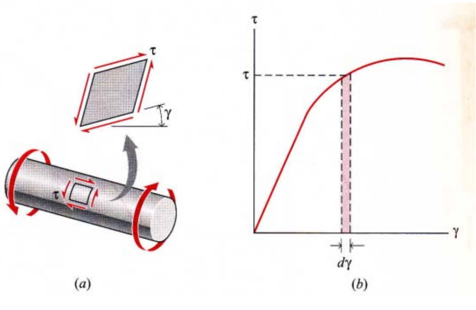
Work stored in the bar: $W_k = U = \int_0^{\varepsilon} \sigma A L d\varepsilon = AL \int_0^{\varepsilon} \sigma d\varepsilon = \frac{AL}{E} \int_0^{\sigma} \sigma d\sigma = AL\left(\frac{\sigma^2}{2E}\right)$



Strain energy intensity: $u = \frac{\sigma^2}{2E}$

Modulus of resilience: $u_R = \frac{\sigma_y^2}{2E}$

Modulus of toughness: $u_T = \int_0^{\varepsilon_R} \sigma d\varepsilon$



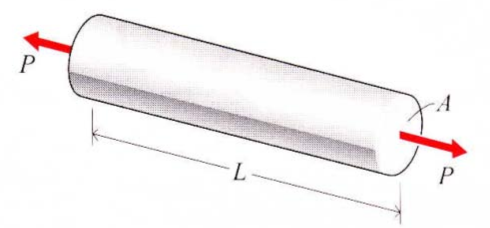
For shear loading: $u = \int_0^{\gamma} \tau d\gamma = \frac{\tau^2}{2G}$

Strain energy: $U = \int_V \frac{\sigma^2}{2E} dV, \quad U = \int_V \frac{\tau^2}{2G} dV$

10.3 Elastic strain energy for various loads

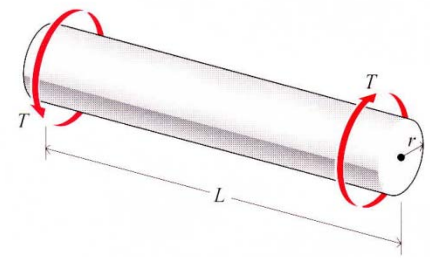
1) Strain energy for axial loading

$$U=\int_V \frac{\sigma^2}{2E}dV=\int_0^L \frac{(P/A)^2}{2E}A\,dx=\int_0^L \frac{P^2}{2AE}dx\left(=\frac{P^2L}{2AE}\right)$$



2) Strain energy for torsional loading

$$U=\int_V \frac{\tau^2}{2G}dV=\int_V \frac{(T\rho/J)^2}{2G}dV=\int_0^L \frac{T^2}{2GJ^2}\left(\int_A \rho^2 dA\right)dx$$
$$=\int_0^L \frac{T^2}{2GJ}dx\left(=\frac{T^2L}{2GJ}\right)$$

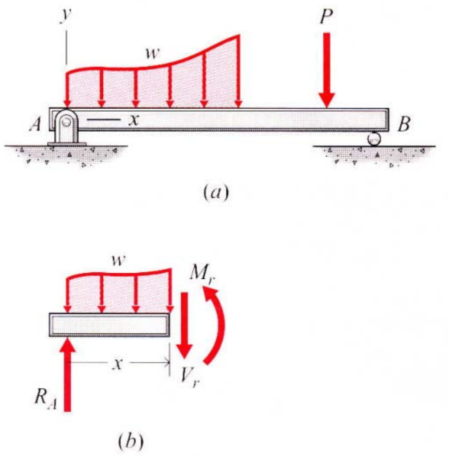


10.3 Elastic strain energy for various loads

3) Strain energy for bending loads

$$U = \int_V \frac{\sigma^2}{2E} dV = \int_V \frac{(-M_r y / I)^2}{2E} dV = \int_0^L \frac{M_r^2}{2EI^2}\left(\int_A y^2 dA\right) dx$$

$$= \int_0^L \frac{M_r^2}{2EI} dx$$



4) Strain energy for transverse shear

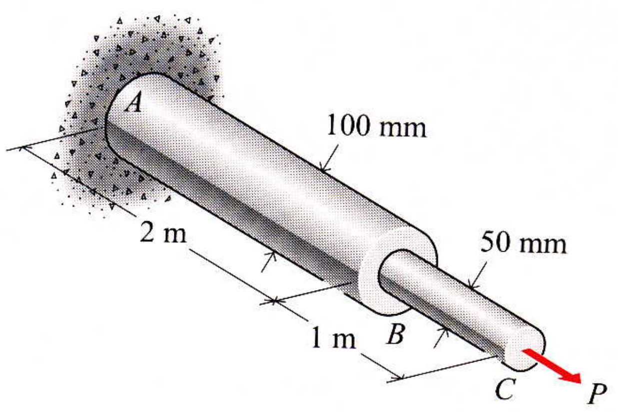
$$U = \int_V \frac{\tau^2}{2G} dV = \int_V \frac{(V_r Q / It)^2}{2G} dV = \int_0^L \frac{V_r^2}{2GI^2}\left(\int_A \frac{Q^2}{t^2} dA\right) dx$$

10.3 Elastic strain energy for various loads

- Example Problem 10-1

The modulus of elasticity E for both bars is 200 GPa. If the axial load P is 50 kN,

- Strain energy for the assembly

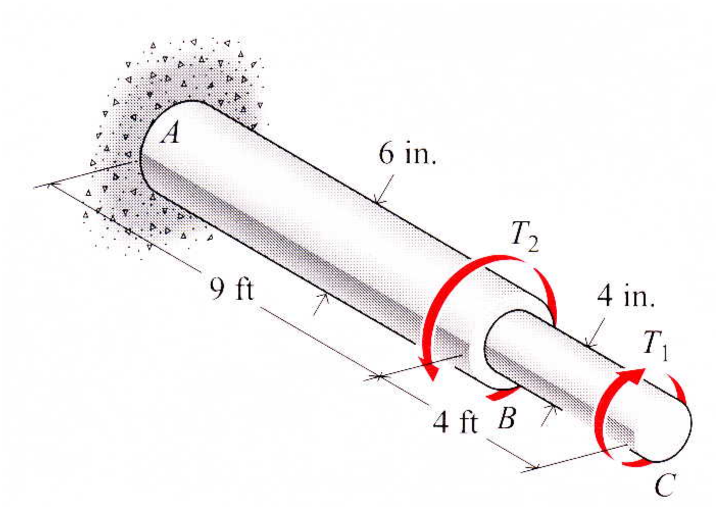


10.3 Elastic strain energy for various loads

• Example Problem 10-2

Segment *AB* and *BC* are of the same material (G = 11,000 ksi). If T_1 = 50 kip·in. and T_2 = 70 kip ·in.

- Strain energy for the assembly



10.3 Elastic strain energy for various loads

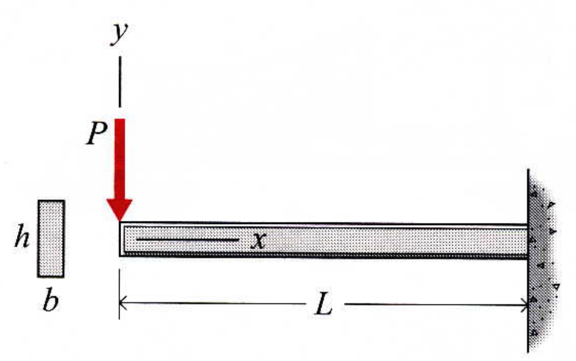
- Example Problem 10-3

The cantilever beam has a constant cross section and is subjected to a concentrated force *P* at its free end.

- Strain energy in the beam due to bending

- Strain energy in the beam due to transverse shear

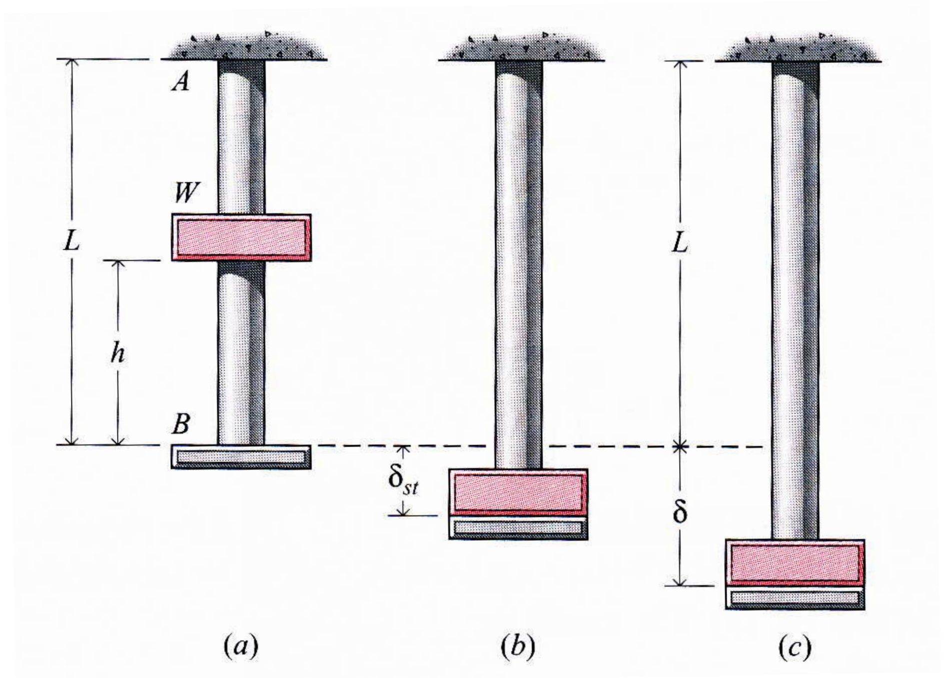
- Compare results of above



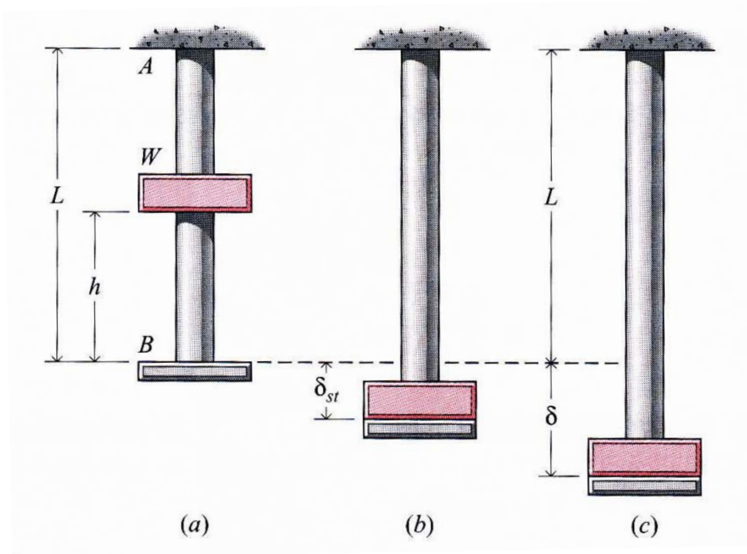
10.4 Impact loading

- Dynamic force (loading): the force necessary to produce a change in acceleration of a body. Ex) pressure on the airplane wings, collision of two automobiles, and a man jumping on a diving board

- Impact load: a suddenly applied load.



10.4 Impact loading



1) Strain Energy Method

$$W\left(h+\delta\right)=\frac{\sigma^2 AL}{2E} \quad \leftarrow \quad \delta=\frac{\sigma L}{E}$$

$$AL\sigma^2-2WL\sigma-2WhE=0$$

$$\sigma=\frac{W}{A}+\sqrt{\left(\frac{W}{A}\right)^2+\frac{2WhE}{AL}} \quad \leftarrow \quad \delta=\frac{\sigma L}{E}$$

$$\delta=\frac{WL}{AE}+\sqrt{\left(\frac{WL}{AE}\right)^2+\frac{2WhL}{AE}}, \quad \delta_{st}=\frac{WL}{AE}$$

$$\rightarrow \delta=\delta_{st}\left[1+\sqrt{1+\frac{2h}{\delta_{st}}}\right]$$

$$\delta=\frac{\sigma L}{E}, \quad \delta_{st}=\frac{WL}{AE}=\frac{\sigma_{st}L}{E}$$

$$\rightarrow \sigma=\sigma_{st}\left[1+\sqrt{1+\frac{2h}{\delta_{st}}}\right]$$

$$When \;\; h=0$$

$$\delta=2\delta_{st}$$

$$\sigma=2\sigma_{st}$$

$$When \;\; h\gg\delta_{st}$$

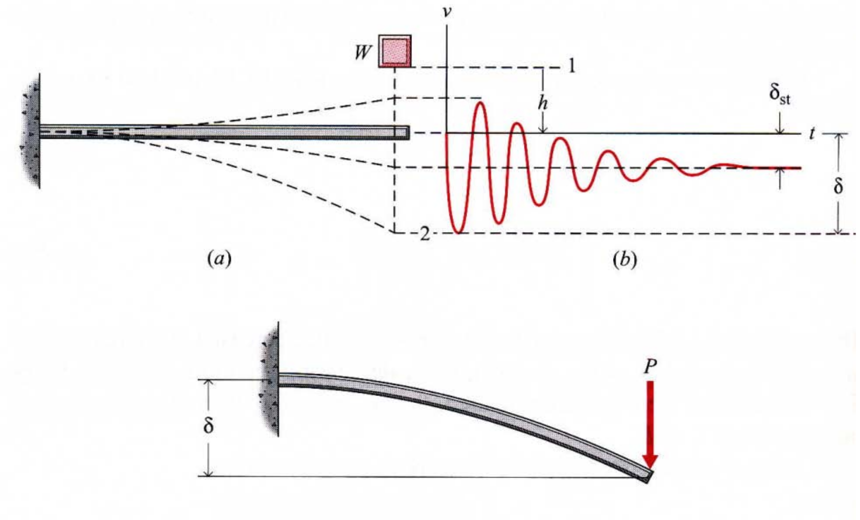
$$\delta=\delta_{st}+\sqrt{\delta_{st}^2+2\delta_{st}h}\approx\sqrt{2\delta_{st}h}$$

$$\sigma=\frac{E}{L}\delta\approx\frac{E}{L}\sqrt{2\left(\frac{WL}{AE}\right)h}=\sqrt{2\frac{E}{AL}Wh}$$

$$\sigma=\sqrt{2\frac{E}{AL}\left(\frac{mV^2}{2}\right)}=\sqrt{\frac{mEV^2}{AL}}$$

10.4 Impact loading

2) Work-kinetic Energy Method



$$\frac{P}{\delta}=\frac{W}{\delta_{st}}$$

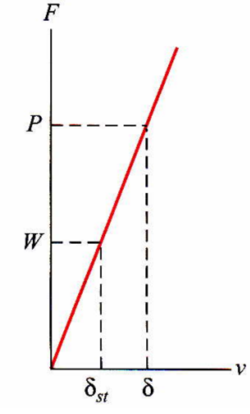
$$U_{1\to 2}=T_2-T_1$$

$$W\left(h+\delta\right)-\frac{1}{2}P\delta=0$$

$$P^2-2WP-\frac{2W^2h}{\delta_{st}}=0$$

$$P=W\left[1+\sqrt{1+\frac{2h}{\delta_{st}}}\right]$$

$$\delta=\delta_{st}\left[1+\sqrt{1+\frac{2h}{\delta_{st}}}\right]\quad\left(\because\frac{P}{\delta}=\frac{W}{\delta_{st}}\right)$$



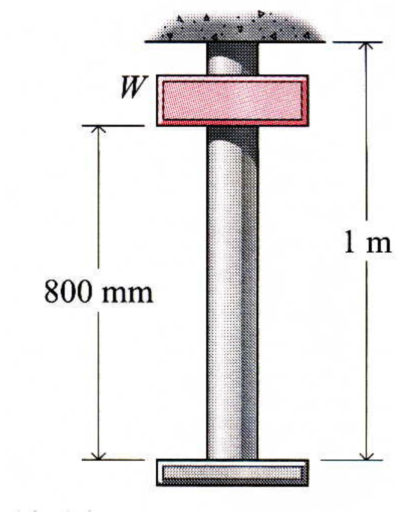
$\rightarrow$ same as the result of the previous case by the strain-energy method

10.4 Impact loading

- Example Problem 10-4

30-mm-diameter aluminum alloy (E = 70 GPa) rod 1 m long is fitted with a flange at the bottom.

- The max. mass that the collar may have if the yield strength (σ_y = 270 MPa) of the rod must not be exceeded.



10.4 Impact loading

- Example Problem 10-8

The beam is of 75 mm wide and 25 mm deep. Each of the supporting coil spring has a modulus of 9 kN/m.

- The max. stress in the beam when the block of weight $W = 45$ N is dropped from a height $h = 15$ mm. The modulus of elasticity of the beam is 200 Gpa.