Junho Song
junhosong@snu.ac.kr

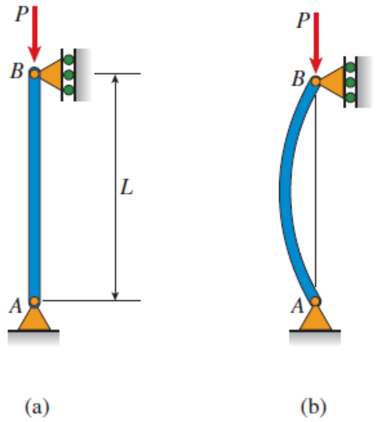
Chapter 11 Columns

11.1 Introduction

⦿ Buckling Failures of Columns

1. Failures investigated so far in this course: failures caused by excessive s________ or d___________ → **strength** and **stiffness** of members are important

2. **Buckling** failure of columns: long, slender members loaded axially in c___________ deflects l____________ → b____________ → may collapse eventually (instead of failures by direct compression of the material)



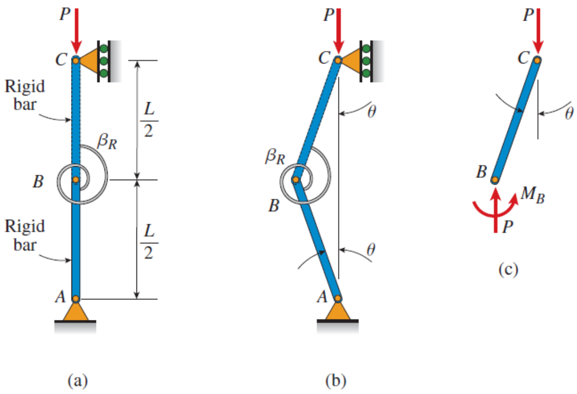
3. Example: compressing a plastic slender ruler, stepping on an aluminum can, think plate of a bridge under compression, etc.

4. Buckling is one of the major causes of failures in structures → should be considered in design process

11.2 Buckling and Stability

⦿ Idealized Structure to Investigate Buckling and Stability ("Buckling Model")

1. Rigid bars AB and BC joined by a pin connection ~ rotational spring with stiffness β_R is added at the pin → an idealized structure analogous to the column structure shown above (elasticity is concentrated vs distributed)



2. Hooke's law for the rotational spring

$$M = \beta_R \theta$$

3. If the bars are perfectly aligned, the axial load P acts through the longitudinal line → spring is uns_________, and the bars are in direct c_____________

4. Suppose point B moves a small distance laterally (by external disturbances, forces or imperfect geometry) → rigid bars rotate through small angles θ

Junho Song
junhosong@snu.ac.kr

5. Axial forces and "**restoring moment**" M_B developed in the spring show opposite effects → Axial force tends to __________ the lateral displacement, and M_B tends to ____________ it

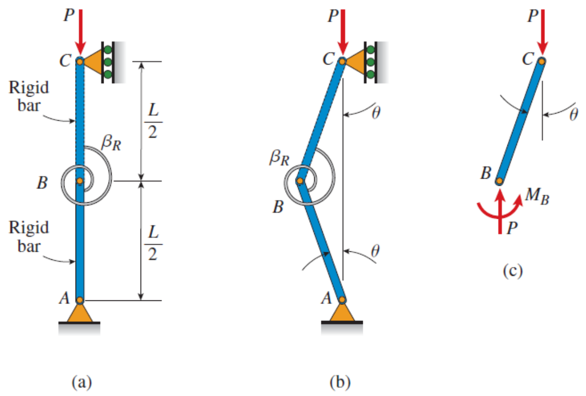
6. What happens after the disturbing force is removed?

   1) Small P → θ keeps __________ → returns to the original position: **Stable**

   2) Large P → θ keeps __________ → fails by lateral buckling: **Unstable**

   "How large P should be to make the system unstable?" → Critical load

⊙ Critical Load P_{cr}



1. Moment in the spring: $M_B = 2\beta_R\theta$

2. Under small angle θ, the lateral displacement at point B: $\theta L/2$

3. Moment equilibrium for bar BC

$$M_B - P \cdot \left(\frac{\theta L}{2}\right) = 0$$

$$\left(2\beta_R - \frac{PL}{2}\right) \cdot \theta = 0$$

4. First solution of equilibrium equation: $\theta = 0$ → trivial solution representing the equilibrium at perfectly straight alignment regardless of the magnitude of the load

5. Second solution of equilibrium equation:

$$P_{cr} = \frac{4\beta_R}{L}$$

   → The structure is in equilibrium regardless of the magnitude of the angle θ

   → Critical load is the *only load* for which the structure will be in equilibrium in the disturbed position, i.e. $\theta \neq 0$

6. What if $P \neq P_{cr}$, i.e. can't sustain the equilibrium?

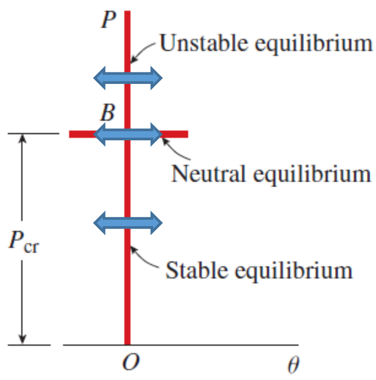
   1) If $P < P_{cr}$, restoring moment is dominant → structure is __________

   2) If $P > P_{cr}$, effect of the axial force is dominant → structure is __________

7. From the critical load derived above, it is seen that one can increase the stability by ___________ing stiffness or ___________ing length

 Junho Song
junhosong@snu.ac.kr

⊙ Summary



1. $\theta = 0$: no disturbance → equilibrium for any P

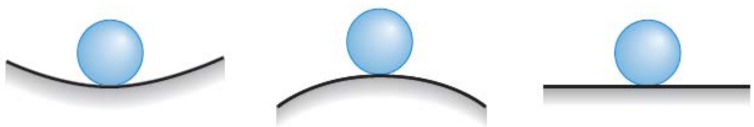
2. Disturbance introduced to cause $\theta \neq 0$ and the source of the disturbance removed

    1) $P < P_{cr}$: goes back to the original equilibrium (**stable** equilibrium)

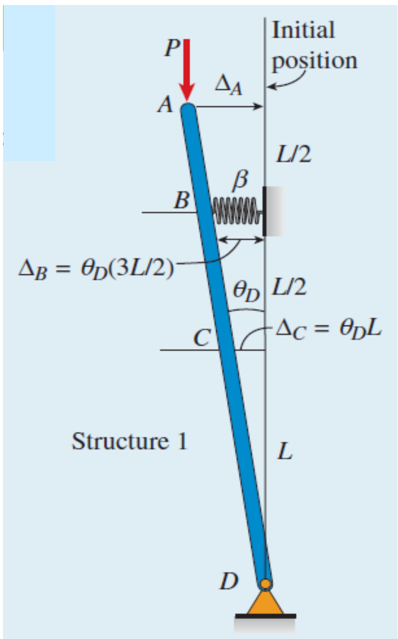
    2) $P = P_{cr}$: can sustain the equilibrium regardless of θ (**neutral** equilibrium) ~ "bifurcation" point

    3) $P > P_{cr}$: cannot sustain the equilibrium (**unstable** equilibrium)

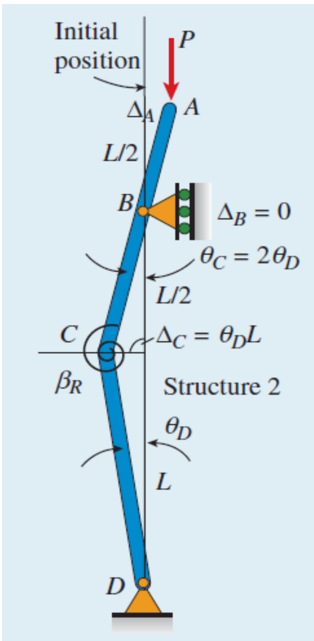
3. These are analogous to a ball placed upon a smooth surface



⊙ **Example 11-1**: Consider two idealized columns. The first one consists of a single rigid bar $ABCD$ pinned at D and laterally supported at B by a spring with translational stiffness β. The second column consists of two rigid bars ABC and CD that are joined at C by an elastic connection with rotational stiffness $\beta_R = \left(\frac{2}{5}\right)\beta L^2$. Find an expression for critical load P_{cr} for each column.

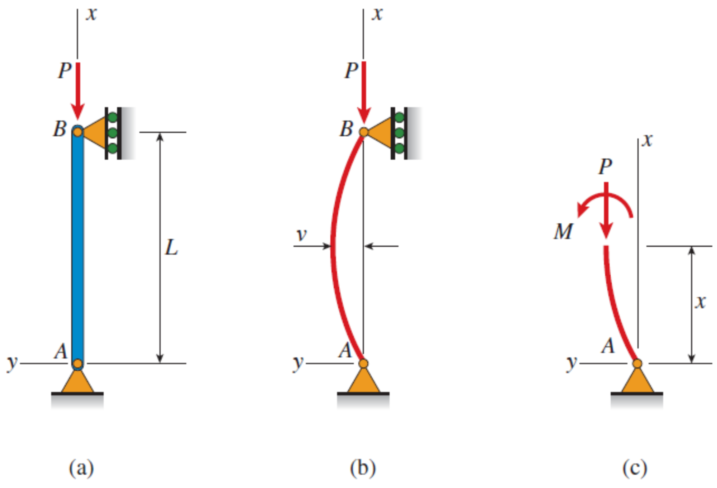


Junho Song
junhosong@snu.ac.kr



11.3 Columns with Pinned Ends

⦿ Differential Equation for Deflection of an "Ideal Column" (i.e. perfectly straight) with Pinned Ends



1. Bending-moment equation:

   $EIv'' = M$

2. Moment equilibrium equation:

   $M + Pv = 0$

3. Therefore, the deflection equation of the deflection curve is

   $EIv'' + Pv = 0$

4. Homogeneous, linear, differential equation of second order with constant coefficients

⦿ Solution of Differential Equation

1. For convenience, we introduce $k^2 = P/EI$
2. Rewrite the differential equation:

   $v'' + k^2 v = 0$

3. From mathematics, the general solution of the equation is

   $v = C_1 \sin kx + C_2 \cos kx$

4. Boundary conditions to determine C_1 and C_2:

   $v(0) =$ $\qquad$ $v(L) =$

5. From the first condition, $C_2 =$
6. Thus the deflection of the column is $v(x) = C_1 \sin kx$
7. From the second condition, $C_1 \sin kL =$
8. **Case 1**: $C_1 =$ → $v(x) =$, i.e. the column remains straight (for any kL)
9. **Case 2**: $\sin kL =$ → "Buckling equation"

Junho Song
junhosong@snu.ac.kr

The column sustains equilibrium if $kL = n\pi,\ n = 1,2,3,\ldots$

The corresponding axial (critical) loads are

$$P = \frac{n^2\pi^2 EI}{L^2}$$

10. Deflection curves at neutral equilibrium at critical loads are

$$v(x) = C_1 \sin kx = C_1 \sin \frac{n\pi x}{L}$$

⦿ Critical Loads

1. The lowest critical load for a column with pinned ends:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

2. The corresponding buckled shape (mode shape):

$$v(x) = C_1 \sin \frac{\pi x}{L}$$

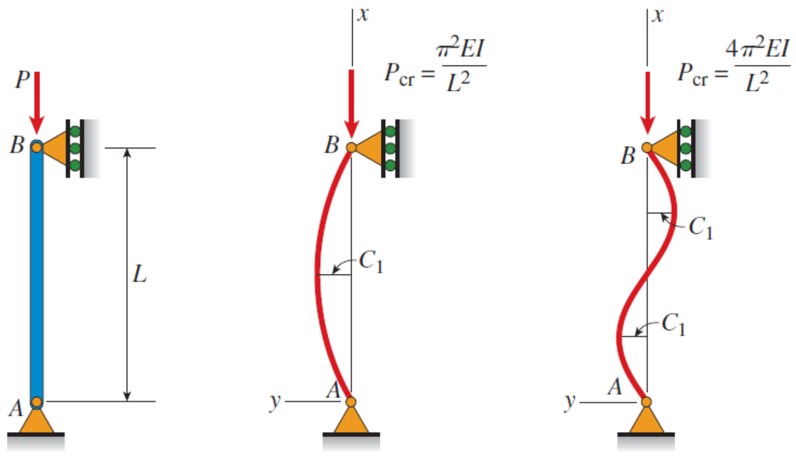
3. Note: the amplitude C_1 of the buckled shape is un____________ (but small)

4. $n = 1$: **"Fundamental"** buckling mode

5. As n increases, "higher modes" appear

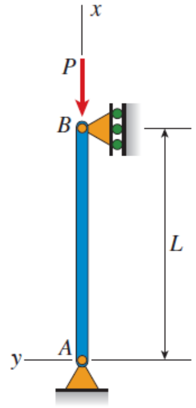
→ No practical interest because the fundamental load is reached first

→ To make higher modes occur, lateral supports should be provided at intermediate points



(a) (b) (c)

11.1 Columns with Pinned Ends (continued)



⦿ Critical Stress

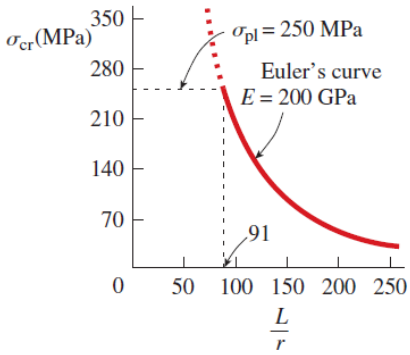
1. **Critical stress**: the stress in the column when $P =$

$$\sigma_{\text{cr}} = \frac{P_{\text{cr}}}{A} = \frac{\pi^2 EI}{AL^2}$$

2. Using **the radius of gyration** $r = \sqrt{I/A}$

$$\sigma_{\text{cr}} = \frac{\pi^2 E}{(L/r)^2}$$

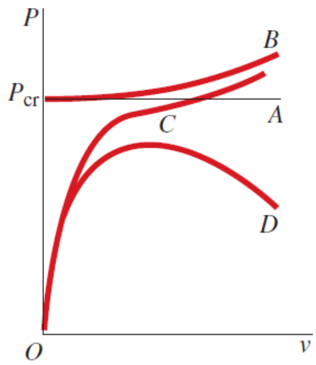
where L/r is called "slenderness ratio"



3. **Euler's curve**: critical stress versus the slenderness ratio

- Long and slender columns: buckle at ________ stress
- Short and stubby columns: buckle at __________ stress
- The curve is valid only for $\sigma < \sigma_{\text{pl}}$ because we use ____________'s law

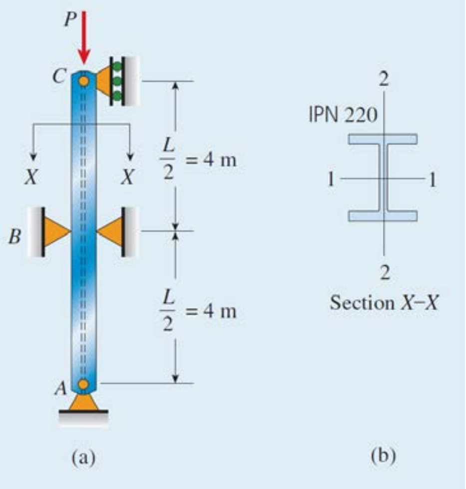
⦿ Effects of Large Deflections, Imperfections, and Inelastic Behavior



1. Ideal elastic column with **small defections** (Curve *A*): No deflection or undetermined deflection at $P = P_{\text{cr}}$
2. Ideal elastic column with **large deflection** (Curve *B*): Should use exact (nonlinear) expression for the curvature, i.e. instead of v'' → Once the column begins buckling, an increasing load is required to cause an increase in the deflections
3. Elastic column with **imperfections** (Curve *C*): imperfections such as initial curvature → imperfections produce deflections from the onset of loading; the larger the imperfections, the further curve *C* moves to the right
4. **Inelastic** column with **imperfections** (Curve D): As the material reaches the proportional limit, it becomes easier to increase deflections

Junho Song
junhosong@snu.ac.kr

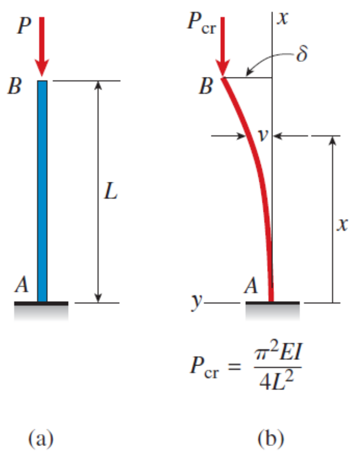
⦿ **Example 11-2**: A long, slender column *ABC* is pin supported at the ends and compressed by an axial load P. Lateral support is provided at the midpoint *B* (only in the direction within the plane). The column is constructed of a standard steel shape (IPN 220; Table E-2) having $E = 200\ \mathrm{GPa}$ and proportional limit $\sigma_{\mathrm{pl}} = 300\ \mathrm{MPa}$. The total length $L = 8\ \mathrm{m}$. Determine the allowable load P_{allow} using a factor of safety $n = 2.5$ with respect to Euler buckling of the column.



Junho Song
junhosong@snu.ac.kr

11.4 Columns with Other Support Conditions

⦿ Column Fixed at the Base and Free at the Top



1. Bending moment at distance x from the base is

   $M = P(\delta - v)$

2. Bending moment equation: $EIv'' = M =$

3. Using the notation $k^2 = P/EI$ again, the equation becomes

   $v'' + k^2 v = k^2 \delta$

4. Homogeneous solution (the same as the pinned-pinned case):

   $v_H = C_1 \sin kx + C_2 \cos kx$

5. Particular solution:

   $v_P =$

6. Consequently, the general solution is

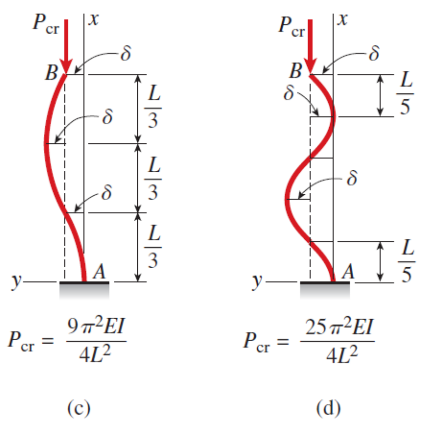
   $v(x) = v_H + v_P = C_1 \sin kx + C_2 \cos kx +$

7. Boundary conditions: $v(0) =$, $v'(0) =$ and $v(L) =$

8. From the first condition, $C_2 =$

9. From the second boundary condition, $C_1 =$

10. Finally, the solution is $v(x) = \delta(1 - \cos kx)$ → shape is identified but the amplitude is und___________



11. From the third boundary condition, $\delta \cos kL =$

12. The nontrivial solution (i.e. buckling equation) is $\cos kL = 0$

13. Therefore, $kL = \frac{n\pi}{2}$, $n = 1,3,5, \ldots$

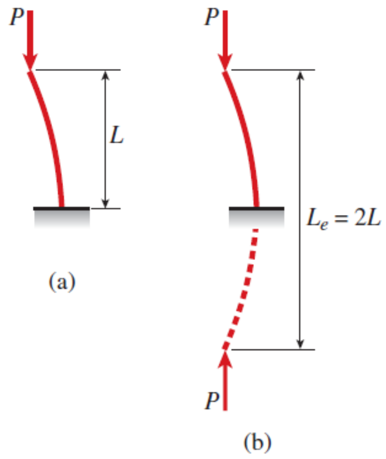
14. The critical loads are

$$P_{cr} = \frac{n^2 \pi^2 EI}{4L^2}, \quad n = 1,3,5, \ldots \text{ and for } n = 1, P_{cr} = \frac{\pi^2 EI}{4L^2}$$

Junho Song
junhosong@snu.ac.kr

15. Buckled mode shapes are $v(x) = \delta\left(1 - \cos\frac{n\pi x}{2L}\right)$

⦿ Effective Lengths of Columns



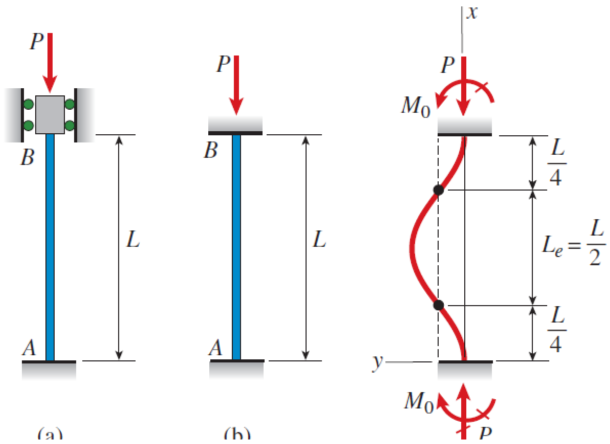
1. **Effective length** of a column: the length of the equivalent pinned-end column having a deflection curve matching the deflection of the given column
2. As seen in the figure, the effective length of a column fixed at the base and free at the top is

   $L_e =$

3. From the critical loads of the two column cases, we can derive a general formula for the critical load,

$$P_{\rm cr} = \frac{\pi^2 EI}{L_e^2}$$

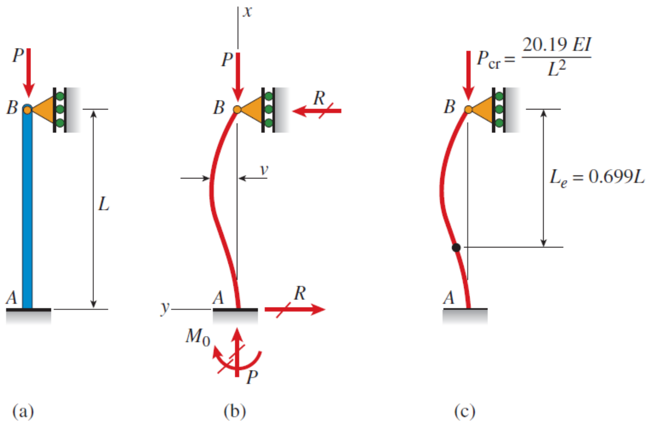
⦿ Column with Both Ends Fixed against Rotation



1. According to the deflected shape sketched based on the boundary conditions, it is noted that $L_e =$
2. Therefore, the critical load is

$$P_{\rm cr} = \frac{4\pi^2 EI}{L^2}$$

⦿ Column Fixed at the Base and Pinned at the Top



1. By solving the differential equation (details in the textbook), we find the buckling equation

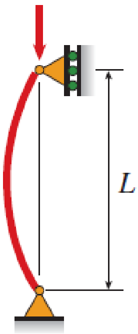
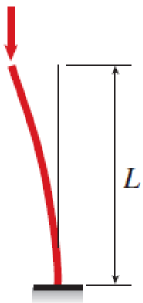
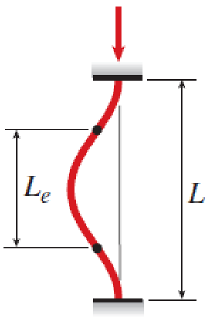
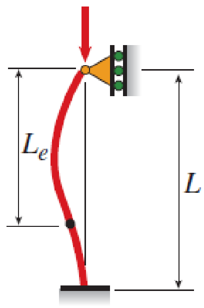
   $kL = \tan kL$

2. Solving this equation numerically, $kL = 4.4934$
3. The corresponding critical load is $\mathrm{P_{cr}} = \frac{20.19EI}{L^2} = \frac{2.046\pi^2 EI}{L^2}$

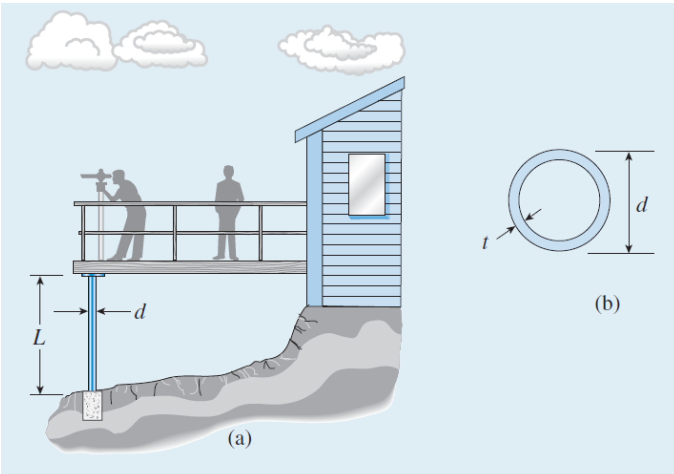
Junho Song
junhosong@snu.ac.kr

4. The effective length is $L_e = 0.699L \approx 0.7L$

⦿ Summary of Results

(a) Pinned-pinned column	(b) Fixed-free column	(c) Fixed-fixed column	(d) Fixed-pinned column
$P_{cr} = \frac{\pi^2 EI}{L^2}$	$P_{cr} = \frac{\pi^2 EI}{4L^2}$	$P_{cr} = \frac{4\pi^2 EI}{L^2}$	$P_{cr} = \frac{2.046\, \pi^2 EI}{L^2}$
			
$L_e = L$	$L_e = 2L$	$L_e = 0.5L$	$L_e = 0.699L$
$K = 1$	$K = 2$	$K = 0.5$	$K = 0.699$

⦿ **Example 11-3**: A viewing platform is supported by a row of aluminum pipe columns having length $L = 3.25\ \mathrm{m}$ and outer diameter $d = 100\ \mathrm{mm}$. Because of the manner in which the columns are constructed, we model each column as a fixed-pinned column. The columns are being designed to support compressive loads $P = 200\ \mathrm{kN}$. Determine the minimum required thickness t of the columns if a factor of safety $n = 3$ is required with respect to Euler buckling. The modulus of Elasticity of the aluminum is $E = 72\ \mathrm{GPa}$ and the proportional stress limit is $480\ \mathrm{MPa}$.



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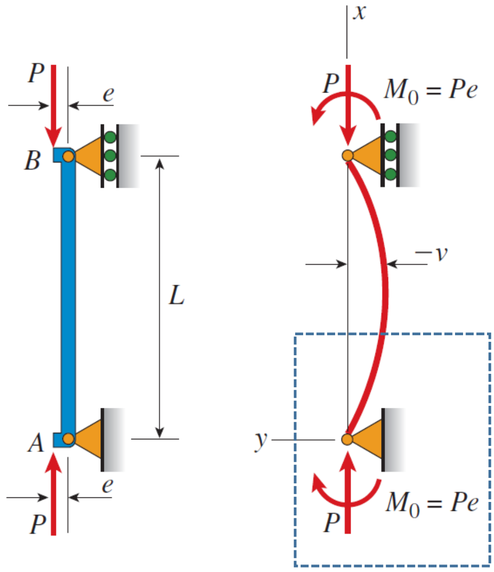
Junho Song
junhosong@snu.ac.kr

11.5 Columns with Eccentric Axial Loads

⊙ Differential Equation of Columns with Eccentricity



(a) (b)

1. Consider a column with a small eccentricity e under axial load P
2. This is equivalent to a column under *centric* load P but with additional couples

   $M_0 =$
3. Bending moment in the column is obtained from a free-body-diagram from the moment equilibrium (around A)

   $M = M_0 + P(-v) = Pe - Pv$
4. Differential equation

   $EIv'' = M = Pe - Pv$

   $v'' + k^2 v =$
5. The general solution: $v = C_1 \sin kx + C_2 \cos kx + e$
6. Boundary conditions: $v(0) =$ $\qquad v(L) =$
7. These conditions yield

   $C_2 =$

   $$C_1 = -\frac{e(1-\cos kL)}{\sin kL} = -e \ \tan\frac{kL}{2}$$
8. The equation of the deflection curve is

   $$v(x) = -e\left(\tan\frac{kL}{2}\sin kx + \cos kx - 1\right)$$

   Note: the deflection for the centric load was $v(x) = C_1 \sin kx = C_1 \sin\frac{n\pi x}{L}$

   Undetermined (centric load) vs determined (eccentric load)
9. Critical load (both pinned ends)

   $$P_{\rm cr} = \frac{\pi^2 EI}{L^2}$$

⦿ Maximum Deflection

1. Maximum deflection δ occurs at the midpoint, thus

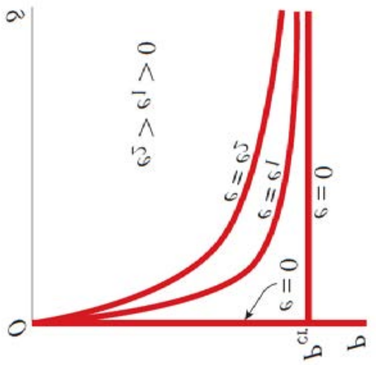
$$\begin{aligned}\delta &= -v\left(\frac{L}{2}\right)\\ &= e\left(\tan\frac{kL}{2}\sin\frac{kL}{2} + \cos\frac{kL}{2} - 1\right)\\ &= e\left(\sec\frac{kL}{2} - 1\right)\end{aligned}$$

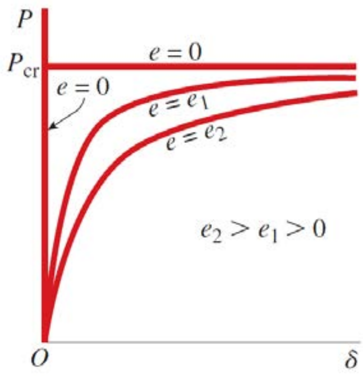
2. Consider an alternative expression for k

$$k = \sqrt{\frac{P}{EI}} = \sqrt{\frac{P\pi^2}{P_{\mathrm{cr}}L^2}} = \frac{\pi}{L}\sqrt{\frac{P}{P_{\mathrm{cr}}}}$$

3. Using this, the maximum deflection is described in terms of the ratio P/P_{cr}

$$\delta = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{\mathrm{cr}}}}\right) - 1\right]$$



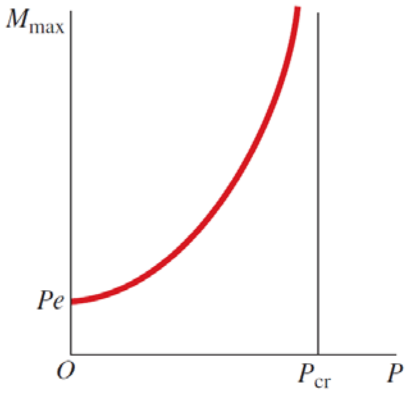


4. **Load-deflection diagram** (↗)

- The deflection increases as the load P increases, but nonlinear even if linear elastic material is used → s___________ rule does not work
- When the imperfection is increased from e_1 to e_2: the maximum deflection increases by
- As the load P approaches the critical load P_{cr} the deflection increases without limit
- An ideal column with a centrally applied load ($e = 0$) is the limiting case of a column with an eccentric load ($e > 0$)

Junho Song
junhosong@snu.ac.kr

⦿ Maximum Bending Moment



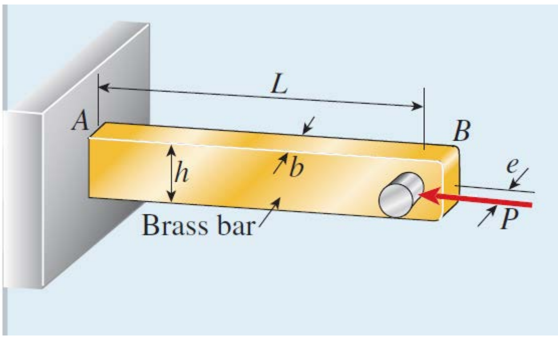
1. Maximum bending moment occurs when $v =$

   $M_{max} = P(e+ \quad)$

2. Thus the maximum bending moment is

$$M_{max} = Pe \sec\frac{kL}{2} = Pe \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$$

⦿ **Example 11-4**: A brass bar AB projecting from the side of a large machine is loaded at end B by a force $P = 7\ \mathrm{kN}$ with an eccentricity $e = 11\ \mathrm{mm}$. The bar has a rectangular cross section with height $h = 30\ \mathrm{mm}$ and width $b = 15\ \mathrm{mm}$. What is the longest permissible length L_{max} of the bar if the deflection at the end is limited to $3\ \mathrm{mm}$? For the brass, use $E = 100\ \mathrm{GPa}$.



Junho Song
junhosong@snu.ac.kr

11.6 Secant Formula for Columns

⦿ Maximum Stress in a Column under an Eccentric Load

1. Maximum stress occurs at the (concave/convex) side of the column

$$\sigma_{\max} = \frac{P}{A} + \frac{M_{\max} c}{I}$$

2. Maximum moment

$$M_{\max} = Pe \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{\mathrm{cr}}}}\right)$$

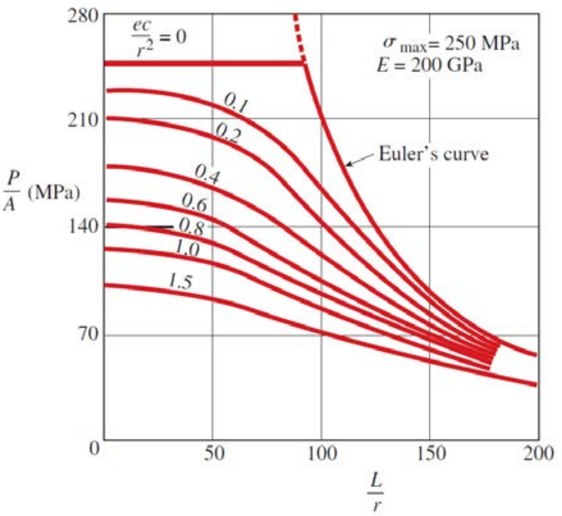
3. From $P_{\mathrm{cr}} = \pi^2 EI/L^2$ and $I = Ar^2$ where r is the radius of gyration, the maximum moment is described as

$$M_{\max} = Pe \sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)$$

4. Substituting this into the maximum stress formula above,

$$\sigma_{\max} = \frac{P}{A} + \frac{Pec}{I}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)$$

$$= \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right]$$



5. This so-called "secant formula" describes the maximum compressive stress in a column under eccentric load in terms of $E, P/A$, L/r (s__________ ratio) and ec/r^2 (ecc____________ ratio)

6. For given $\sigma_{\max}$ and E, one can find the possible pairs of P/A and L/r for each eccentricity level (ec/r^2) and plot a graph of secant formula (↑)

7. For centric load, i.e. $ec/r^2 =$, the critical stress is

$$\sigma_{\mathrm{cr}} = \frac{P_{\mathrm{cr}}}{A} = \frac{\pi^2 EI}{AL^2} = \frac{\pi^2 E}{(L/r)^2}$$

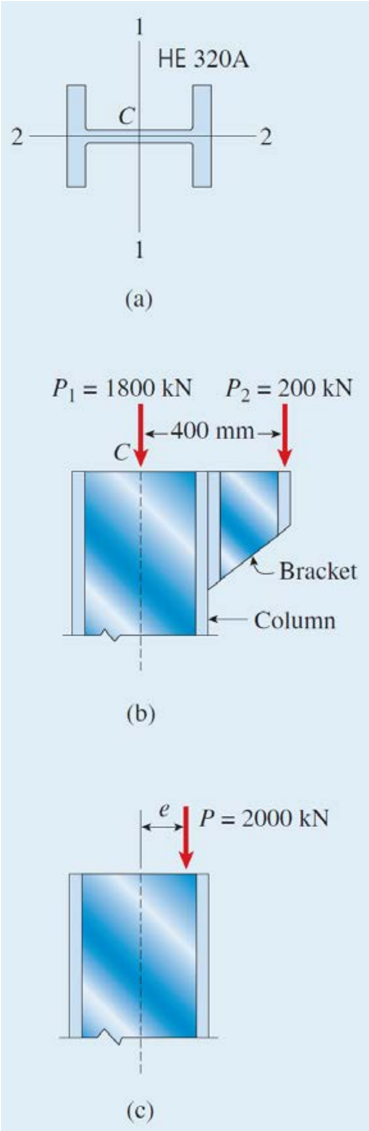
8. Secant formula and graph let us know the load-carrying capacity of a column in terms of slenderness and eccentricity (Trend?)

Junho Song
junhosong@snu.ac.kr

⦿ **Example 11-5**: A steel wide-flange column of HE 320A shape is pin-supported at the ends and has a length of 7.5 m. The column supports a centrally applied load $P_1 = 1800\ \mathrm{kN}$ and an eccentrically applied load $P_2 = 200\ \mathrm{kN}$. Bending takes place about axis 1-1 of the cross section, and the eccentric load acts on axis 2-2 at a distance of 400 mm from the centroid *C*.

(a) Using the secant formula, and assuming $E = 210\ \mathrm{GPa}$, calculate the maximum compressive stress in the column.

(b) If the yield stress for the steel is $\sigma_Y = 300\ \mathrm{MPa}$, what is the factor of safety with respect to yielding?



Junho Song
junhosong@snu.ac.kr

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Many thanks for your hard work in this semester to learn Mechanics of Materials. I wish you the very best on your remaining course work and future career and life.

Cheers,
Junho