Ch.15 Kinematics of Rigid Bodies

Translation

Rotation About a Fixed Axis

-Rotation About a Representative Slab

Equations Defining the Rotation of a Rigid Body About a Fixed Axis

General Plane Motion

Absolute and Relative Velocity in Plane Motion

Instantaneous Center of Rotation in Plane Motion

Absolute and Relative Acceleration in Plane Motion

Analysis of Plane Motion in Terms of a Parameter

Rate of Change With Respect to a Rotating Frame

Coriolis Acceleration

Motion About a Fixed Point

Applications

The linkage between train wheels is an example of curvilinear translation – the link stays horizontal as it swings through its motion.

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How can we determine the velocity of the tip of a turbine blade?

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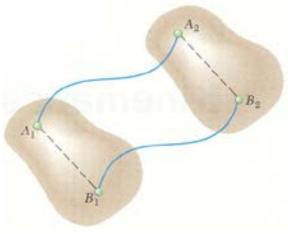
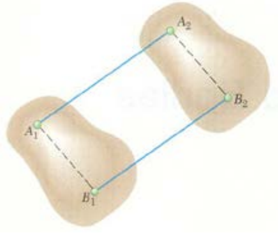
Introduction





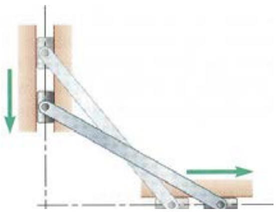
Kinematics of rigid bodies: relations between time and the positions, velocities, and accelerations of the particles forming a rigid body.

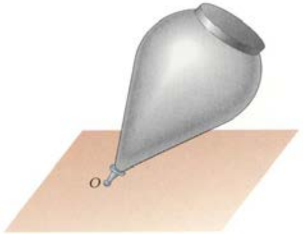
Classification of rigid body motions:

Translation:

rectilinear translation

curvilinear translation





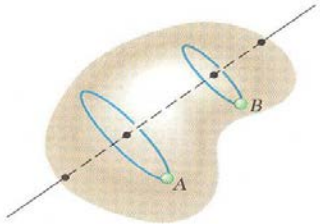
Rotation about a fixed axis

– General plane motion

– Motion about a fixed point

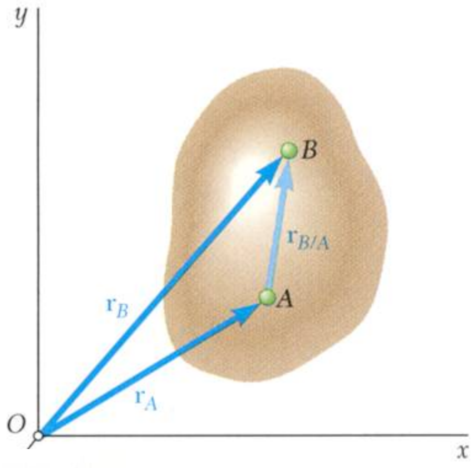
–

– General motion

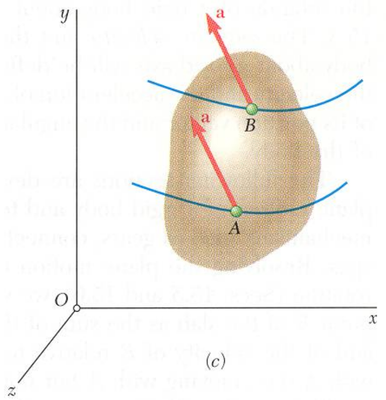


15.1 Translation and Fixed Axis Rotation

15.1A Translation



(a)



(c)

- <u>Consider rigid body in translation</u>:
  - direction of any straight line inside the body is constant,
  - all particles forming the body move in parallel lines.
- For any two particles in the body,

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

- Differentiating with respect to time,

$$\dot{\vec{r}}_B = \dot{\vec{r}}_A + \dot{\vec{r}}_{B/A} = \dot{\vec{r}}_A$$

$$\vec{v}_B = \vec{v}_A$$

All particles have the same velocity.

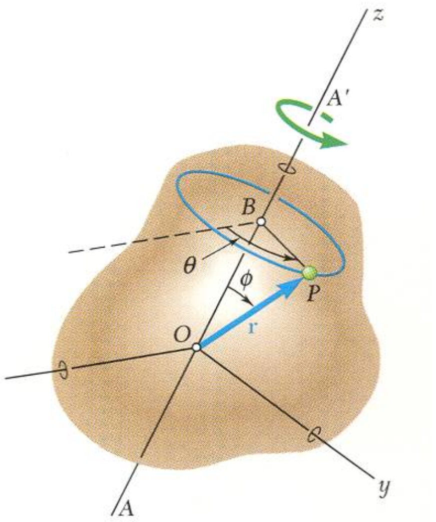
- Differentiating with respect to time again,

$$\ddot{\vec{r}}_B = \ddot{\vec{r}}_A + \ddot{\vec{r}}_{B/A} = \ddot{\vec{r}}_A$$

$$\vec{a}_B = \vec{a}_A$$

All particles have the same acceleration.

15.1B Rotation About a Fixed Axis.

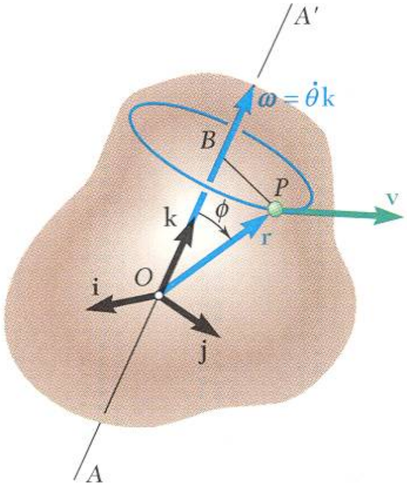


- **Consider rotation of rigid body about a fixed axis** ***AA′***

- **Velocity vector** $\vec{v} = d\vec{r}/dt$ **of the particle** ***P*** **is tangent to the path with magnitude** $v = ds/dt$

$$\Delta s = (BP)\Delta\theta = (r\sin\phi)\Delta\theta$$

$$v = \frac{ds}{dt} = \lim_{\Delta t \to 0}(r\sin\phi)\frac{\Delta\theta}{\Delta t} = r\dot{\theta}\sin\phi \qquad \textbf{(15.4)}$$



- The same result is obtained from(15.5,15.6)

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

$$\vec{\omega} = \omega \vec{k} = \dot{\theta} \vec{k} = angular\ velocity$$

Concept Quiz

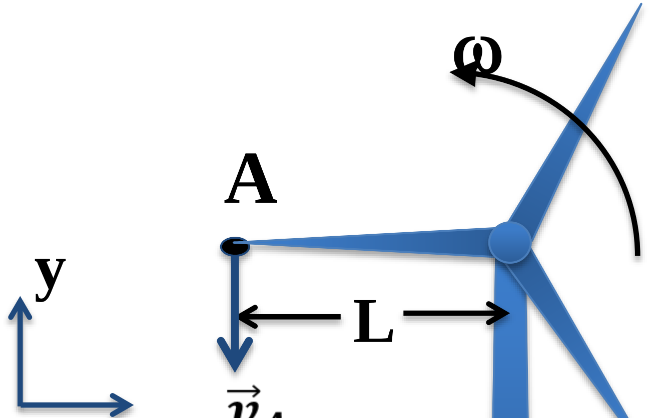
What is the direction of the velocity of point A on the turbine blade?

a) →

b) ←

c) ↑

d) ↓



Sol)

$$\vec{v}_A = \vec{\omega} \times \vec{r} \qquad \vec{v}_A = \omega \hat{k} \times -L\hat{i} \qquad \vec{v}_A = -L\omega \hat{j}$$

answer ; d)

Acceleration

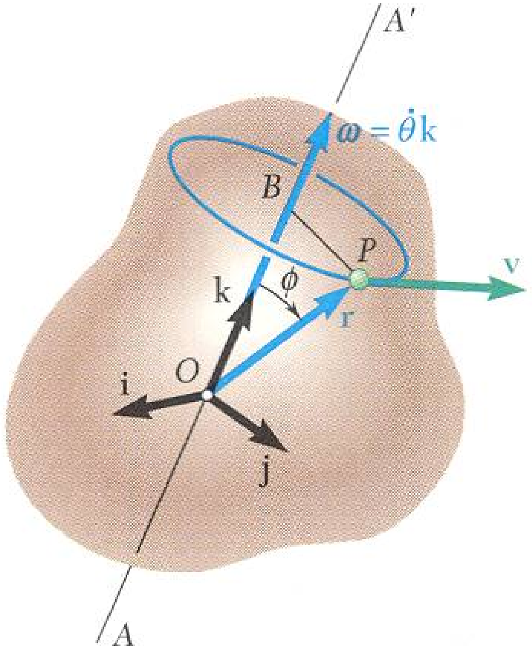


Fig.15.9

- **Differentiating to determine the acceleration,**

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}$$

and

$$\frac{d\vec{\omega}}{dt} = \vec{\alpha} = angular\ acceleration$$

$$= \alpha\vec{k} = \dot{\omega}\vec{k} = \ddot{\theta}\vec{k}$$

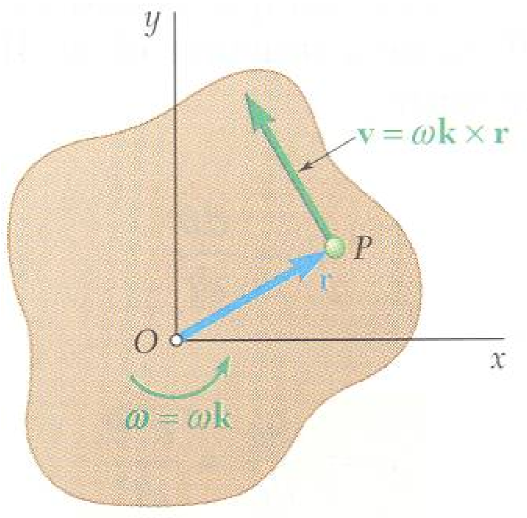
- **Acceleration of *P* is combination of two vectors,**

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\omega} \times \vec{r}$$

$$\vec{\alpha} \times \vec{r} = \text{tangential acceleration component}$$

$$\vec{\omega} \times \vec{\omega} \times \vec{r} = \text{radial acceleration component}$$

Representative Slab



- **Consider the motion of a representative slab in a plane perpendicular to the axis of rotation.**
- **Velocity of any point *P* of the slab,**

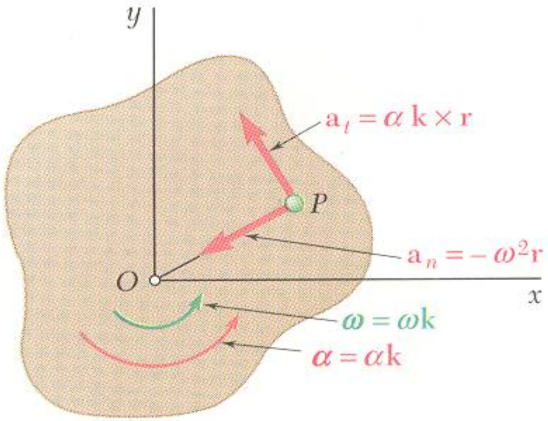
$$\vec{v} = \vec{\omega} \times \vec{r} = \omega \vec{k} \times \vec{r}$$

$$v = r\omega$$

- **Acceleration of any point *P* of the slab,**

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\omega} \times \vec{r}$$

$$= \alpha \vec{k} \times \vec{r} - \omega^2 \vec{r}$$



Resolving the acceleration into tangential and normal components,

$$\vec{a}_t = \alpha \vec{k} \times \vec{r} \qquad a_t = r\alpha$$

$$\vec{a}_n = -\omega^2 \vec{r} \qquad a_n = r\omega^2$$

Concept Quiz

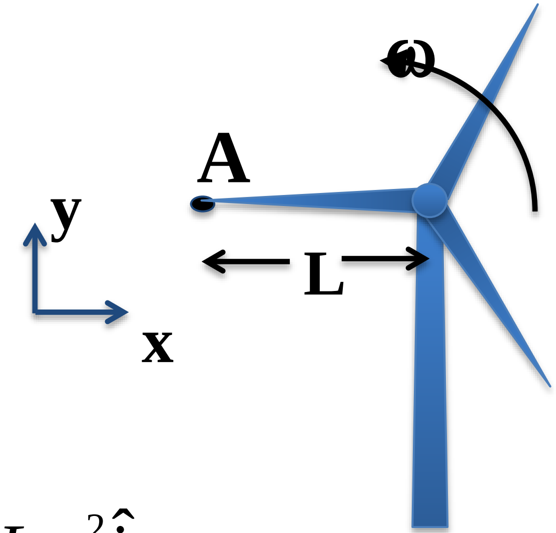
What is the direction of the normal acceleration of point A on the turbine blade?

a) $\rightarrow$

b) $\leftarrow$

c) $\uparrow$

d) $\downarrow$



Sol)

$$\vec{a}_n = -\omega^2 \vec{r} \qquad \vec{a}_n = -\omega^2(-L\hat{i}) \qquad \vec{a}_n = L\omega^2\hat{i}$$

15.1C Equations Defining the Rotation of a Rigid Body About a Fixed Axis

- **Motion of a rigid body rotating around a fixed axis is often specified by the type of angular acceleration.**
- **Recall**

$$\omega = \frac{d\theta}{dt} \qquad \text{or} \qquad dt = \frac{d\theta}{\omega}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}$$

- *Uniform Rotation (angular acceleration=0)*

$$\theta = \theta_0 + \omega t$$

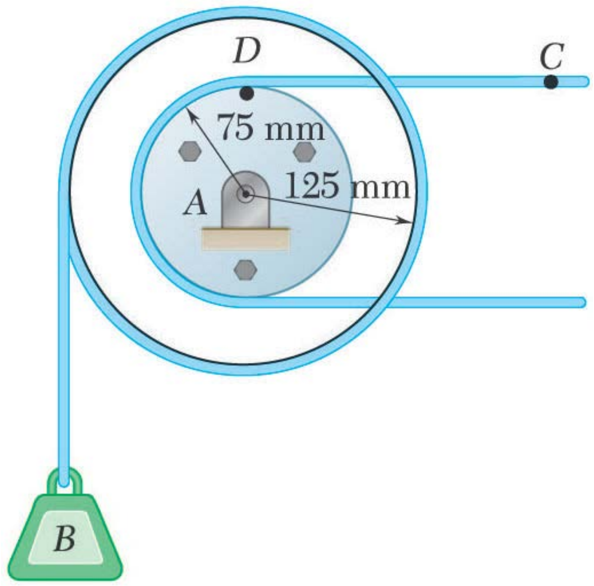
- *Uniformly Accelerated Rotation(angular acceleration =* **constant):**

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Sample Problem 15.3



Cable *C* has a constant acceleration of 225 mm/s^2 and an initial velocity of

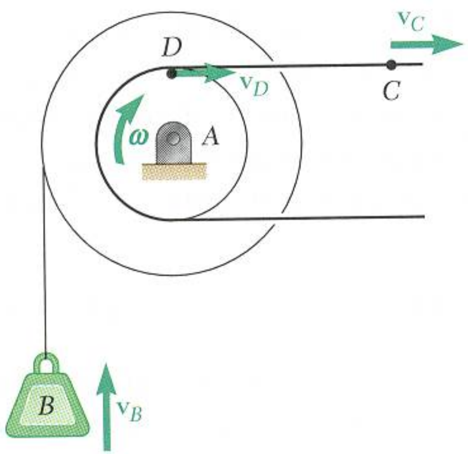
300 mm/s, both directed to the right.

Determine *(a)* the number of revolutions of the pulley in 2 s, *(b)* the velocity and change in position of the load *B* after 2 s, and *(c)* the acceleration of the point *D* on the rim of the inner pulley at t = 0.

STRATEGY:

- Due to the action of the cable, the tangential velocity and acceleration of *D* are equal to the velocity and acceleration of *C*. Calculate the initial angular velocity and acceleration.
- Apply the relations for uniformly accelerated rotation to determine the velocity and angular position of the pulley after 2 s.

- **Evaluate the initial tangential and normal acceleration components of *D*.**



- **MODELING and ANALYSIS:**

The tangential velocity and acceleration of *D* are equal to the velocity and acceleration of *C*.

$$(\vec{v}_D)_0 = (\vec{v}_C)_0 = 300\,\mathrm{mm/s} \rightarrow \qquad (\vec{a}_D)_t = \vec{a}_C = 225\,\mathrm{mm/s^2} \rightarrow$$

$$(v_D)_0 = r\omega_0 \qquad (a_D)_t = r\alpha$$

$$\omega_0 = \frac{(v_D)_0}{r} = \frac{300}{75} = 4\,\mathrm{rad/s} \circlearrowright \qquad \alpha = \frac{(a_D)_t}{r} = \frac{225}{75} = 3\,\mathrm{rad/s^2} \circlearrowright$$

- **Apply the relations for uniformly accelerated rotation to determine velocity and angular position of pulley after 2 s.**

$$\omega = \omega_0 + \alpha t = 4\,\mathrm{rad/s} + \left(3\,\mathrm{rad/s^2}\right)(2\ \mathrm{s}) = 10\,\mathrm{rad/s} \circlearrowright$$

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 = (4\,\mathrm{rad/s})(2\ \mathrm{s}) + \tfrac{1}{2}\left(3\,\mathrm{rad/s^2}\right)(2\ \mathrm{s})^2$$

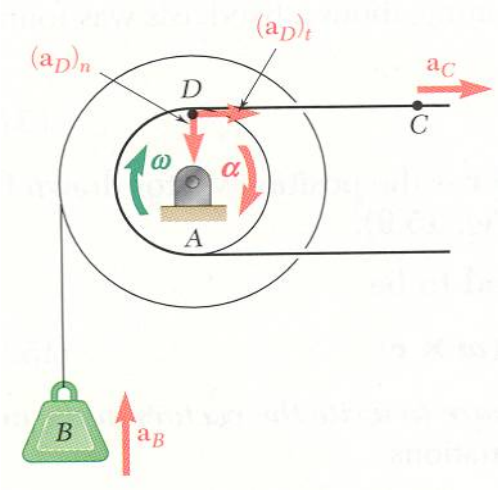
$$= 14\ \mathrm{rad}$$

$$N = (14\ \mathrm{rad})\left(\frac{1\ \mathrm{rev}}{2\pi\ \mathrm{rad}}\right) = \text{number of revs} \qquad \boxed{N = 2.23\,\mathrm{rev}}$$

$$v_B = r\omega = (125\ \text{mm})(10\,\text{rad/s}) = 1250\ \text{mm/s}$$

$$\Delta y_B = r\theta = (125\ \text{mm})(14\ \text{rad}) = 1750\ \text{mm}$$

$$\boxed{\begin{aligned} \vec{v}_B &= 1.25\,\text{m/s} \uparrow \\ \Delta y_B &= 1.75\ \text{m} \uparrow \end{aligned}}$$

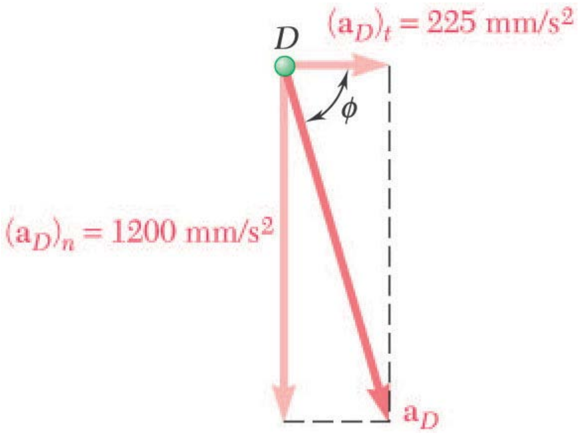


- **Evaluate the initial tangential and normal acceleration components of *D*.**

$$(\vec{a}_D)_t = \vec{a}_C = 225\,\text{mm/s}^2 \rightarrow$$

$$(a_D)_n = r_D\omega_0^2 = (75\ \text{mm})(4\,\text{rad/s})^2 = 1200\,\text{mm/s}^2$$

$$\boxed{(\vec{a}_D)_t = 225\,\text{mm/s}^2 \rightarrow \qquad (\vec{a}_D)_n = 1200\,\text{mm/s}^2 \downarrow}$$



Magnitude and direction of the total acceleration,

$$a_D = \sqrt{(a_D)_t^2 + (a_D)_n^2}$$

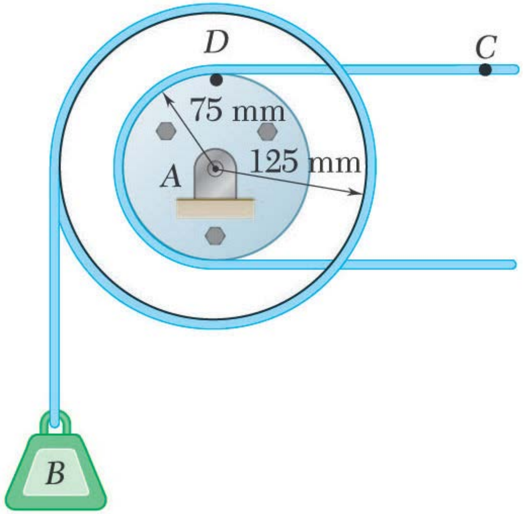
$$= \sqrt{(225)^2 + (1200)^2} = 1221\,\text{mm/s}^2$$

$$\boxed{a_D = 1.221\,\text{m/s}^2}$$

$$\tan\phi = \frac{(a_D)_n}{(a_D)_t}$$

$$= \frac{1200}{225}$$

$$\boxed{\phi = 79.4°}$$

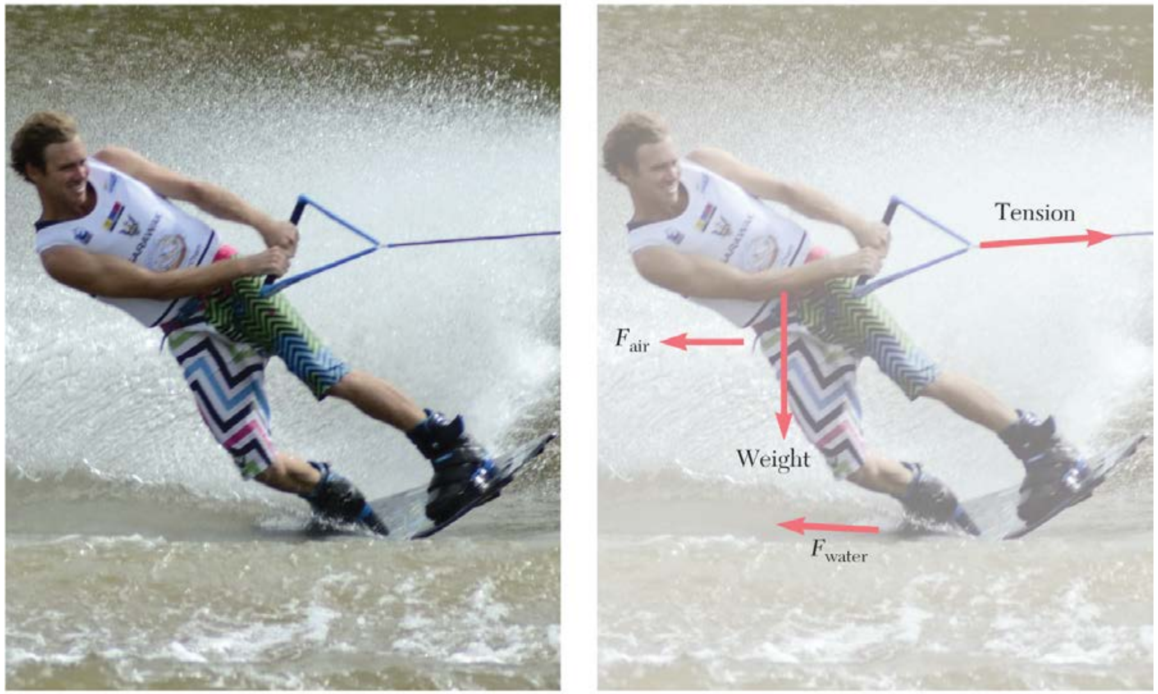


REFLECT and THINK:

- **A double pulley acts similarly to a system of gears; for every 75 mm that point C moves to the right, point B moves 125 mm upward. This is also similar to the rear tire of your bicycle. As you apply tension to the chain, the rear sprocket rotates a small amount, causing the rear wheel to rotate through a much larger angle.**

15.2 General Plane Motion : Velocity

As the man approaches to release the bowling ball, his arm has linear velocity and acceleration from both translation (the man moving forward) as well as rotation (the arm rotating about the shoulder).



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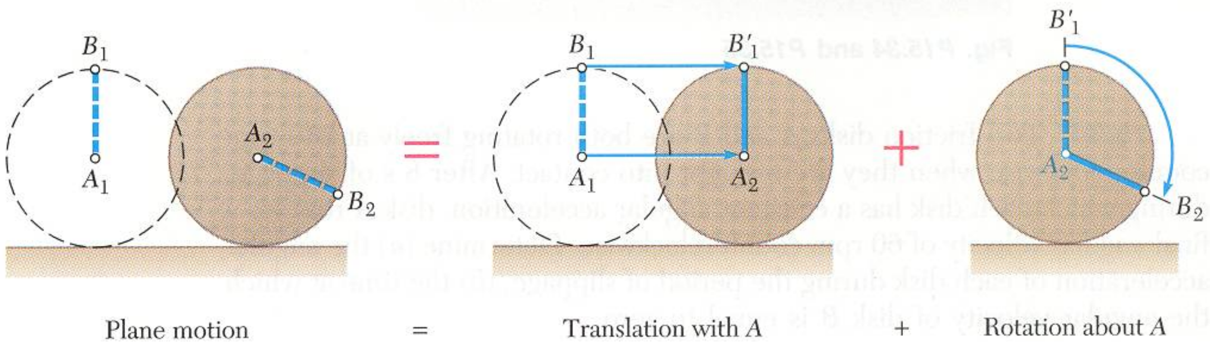
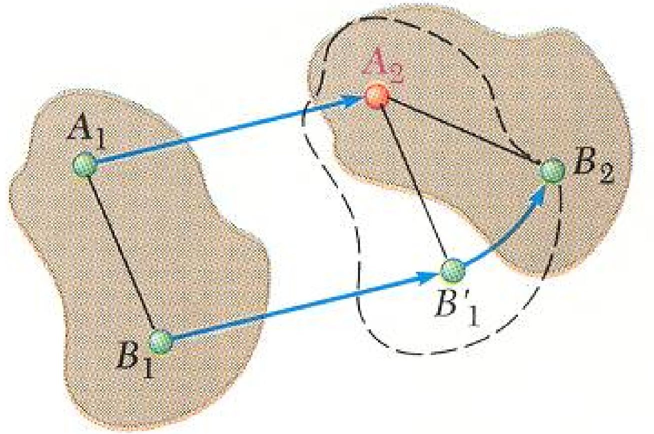
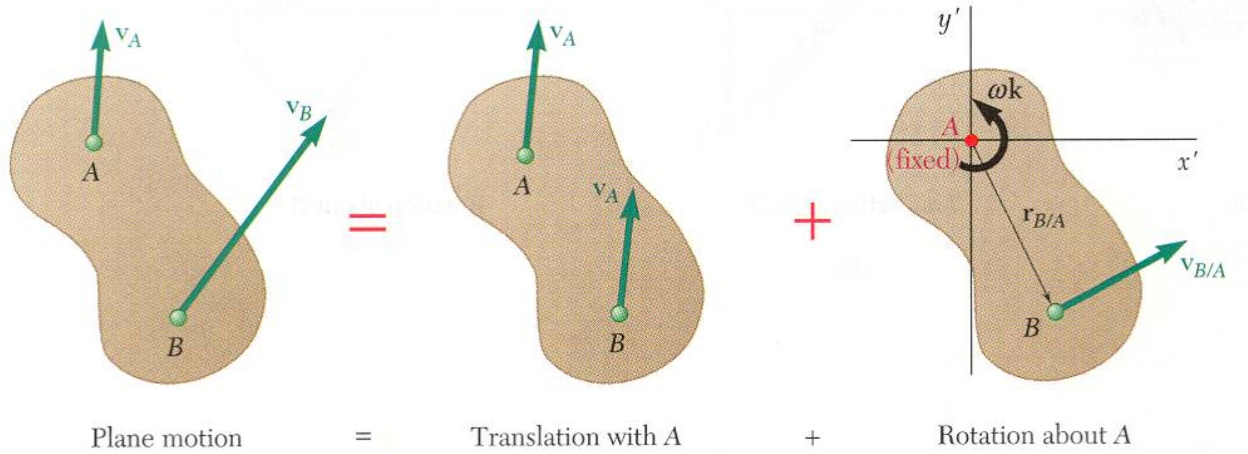
15.2A Analyzing General Plane Motion

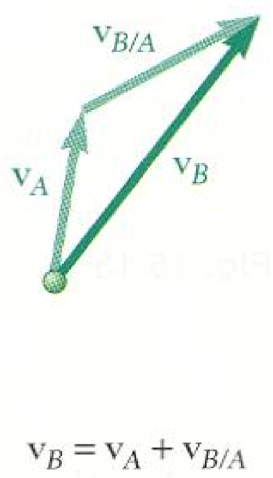


- *General plane motion* is neither a translation nor a rotation.
- General plane motion can be considered as the *sum* of a translation and rotation.
- Displacement of particles A and B to A_2 and B_2 can be divided into two parts:
    - translation to A_2 and B_1'
    - rotation of B_1' about A_2 to B_2



15.2B Absolute and Relative Velocity in Plane Motion



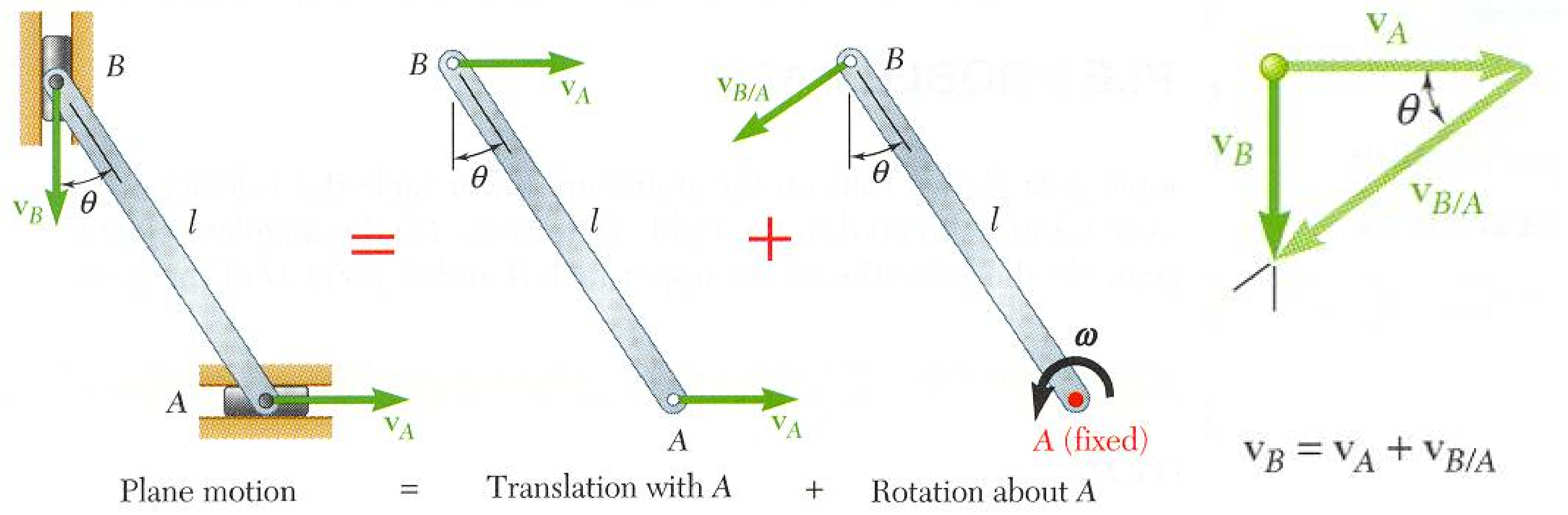


- **Any plane motion can be replaced by a translation of an arbitrary reference point *A* and a simultaneous rotation about *A*.**

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

$$\vec{v}_{B/A} = \omega\vec{k} \times \vec{r}_{B/A} \qquad v_{B/A} = r\omega$$

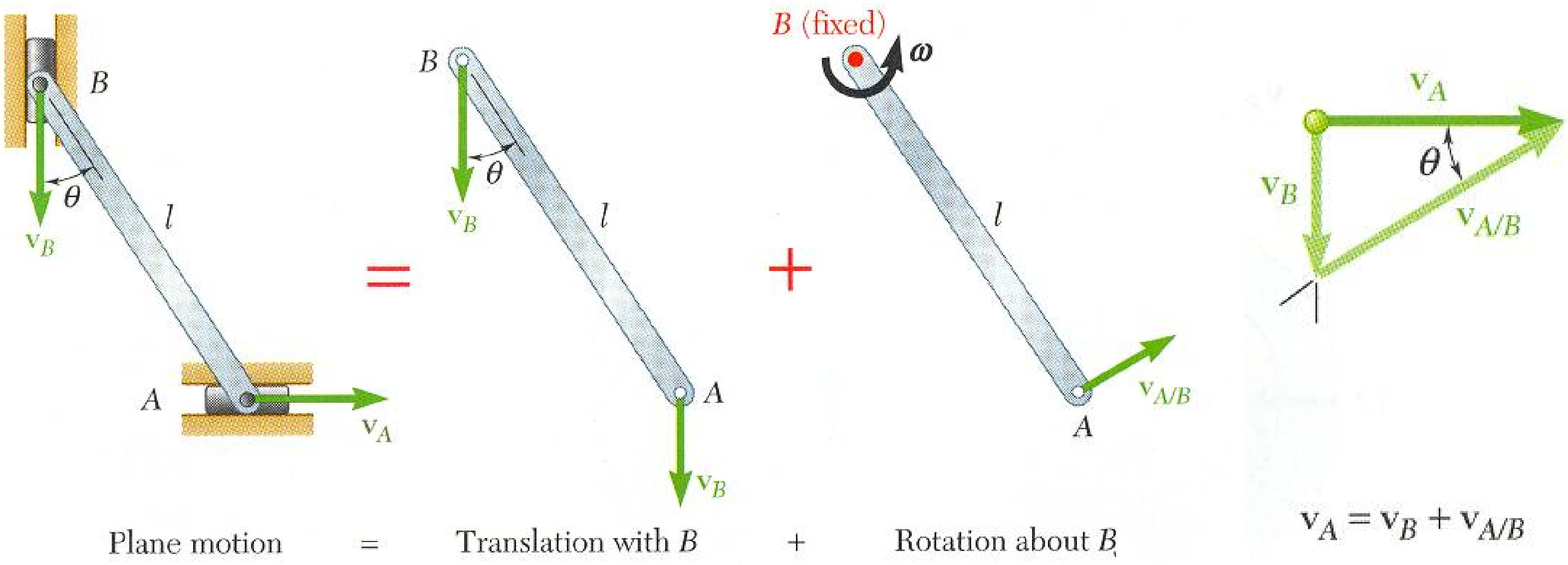
$$\vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$



- **Assuming that the velocity v_A of end A is known, wish to determine the velocity v_B of end B and the angular velocity w in terms of v_A, l, and q.**
- **The direction of v_B and $v_{B/A}$ are known. Complete the velocity diagram.**

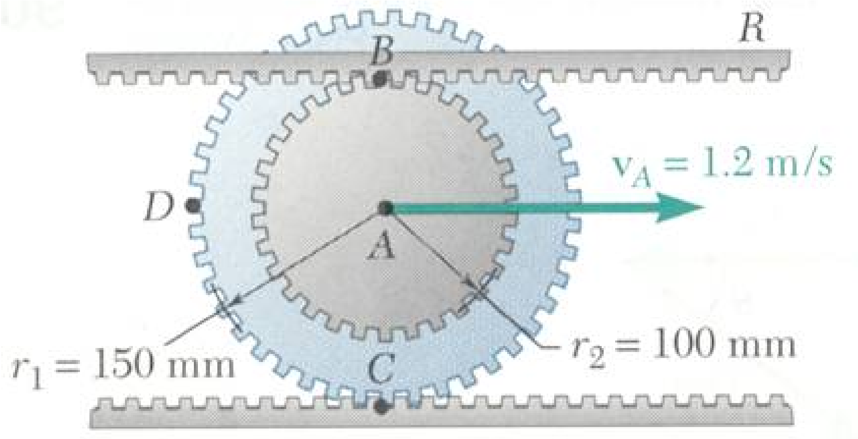
$$\frac{v_B}{v_A} = \tan\theta \qquad \frac{v_A}{v_{B/A}} = \frac{v_A}{l\omega} = \cos\theta$$

$$v_B = v_A \tan\theta \qquad \omega = \frac{v_A}{l\cos\theta}$$



- **Selecting point *B* as the reference point and solving for the velocity v_A of end *A* and the angular velocity *w* leads to an equivalent velocity triangle.**
- **$v_{A/B}$ has the same magnitude but opposite sense of $v_{B/A}$. The sense of the relative velocity is dependent on the choice of reference point.**
- **Angular velocity *w* of the rod in its rotation about *B* is the same as its rotation about *A*. Angular velocity is not dependent on the choice of reference point.**

Sample Problem 15.6



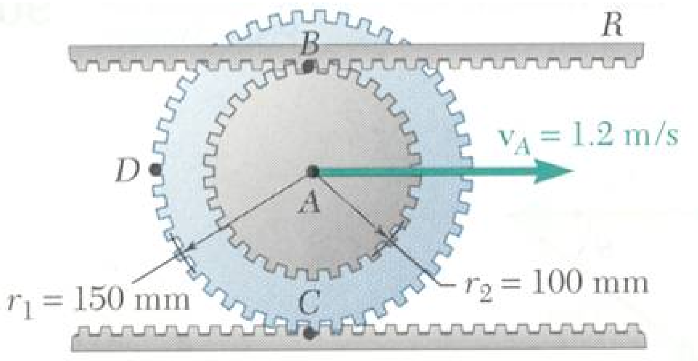
The double gear rolls on the stationary lower rack: the velocity of its center is 1.2 m/s. Determine *(a)* the angular velocity of the gear, and *(b)* the velocities of the upper rack *R* and point *D* of the gear.

STRATEGY:

- **The displacement of the gear center in one revolution is equal to the outer circumference. Relate the translational and angular displacements. Differentiate to relate the translational and angular velocities.**
- **The velocity for any point *P* on the gear may be written as**

  $$\vec{v}_P = \vec{v}_A + \vec{v}_{P/A} = \vec{v}_A + \omega\vec{k} \times \vec{r}_{P/A}$$

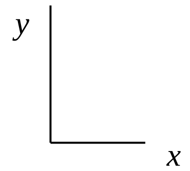
  Evaluate the velocities of points *B* and *D*.



MODELING and ANALYSIS

- The displacement of the gear center in one revolution is equal to the outer circumference.

For $x_A > 0$ (moves to right), $w < 0$ (rotates clockwise).



$$\frac{x_A}{2\pi r} = -\frac{\theta}{2\pi} \qquad x_A = -r_1\theta$$

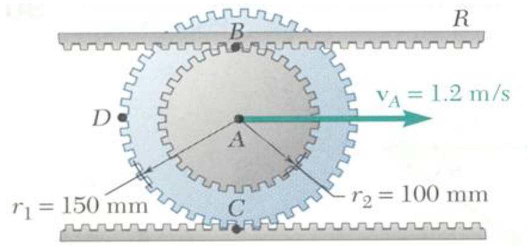
Differentiate to relate the translational and angular velocities.

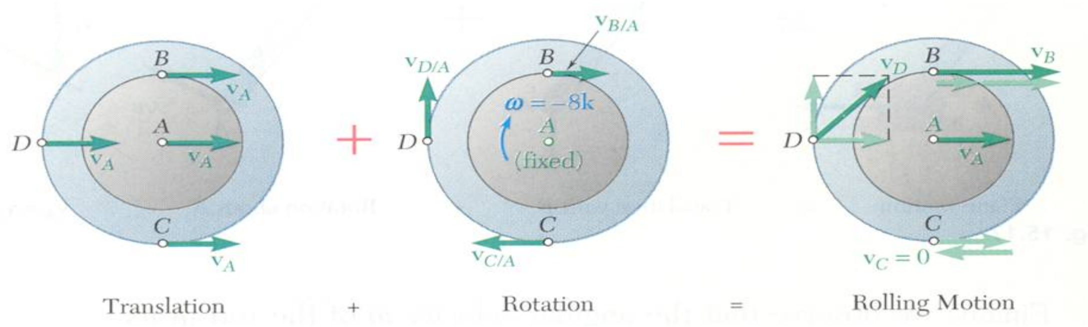
$$v_A = -r_1\omega$$

$$\omega = -\frac{v_A}{r_1} = -\frac{1.2\,\mathrm{m/s}}{0.150\,\mathrm{m}}$$

$$\boxed{\vec{\omega} = \omega\vec{k} = -(8\,\mathrm{rad/s})\vec{k}}$$

- **For any point *P* on the gear,** $\vec{v}_P = \vec{v}_A + \vec{v}_{P/A} = \vec{v}_A + \omega\vec{k} \times \vec{r}_{P/A}$





Velocity of the upper rack is equal to velocity of point *B*:

$$\vec{v}_R = \vec{v}_B = \vec{v}_A + \omega\vec{k} \times \vec{r}_{B/A}$$

$$= (1.2\,\mathrm{m/s})\vec{i} + (8\,\mathrm{rad/s})\vec{k} \times (0.10\,\mathrm{m})\vec{j}$$

$$= (1.2\,\mathrm{m/s})\vec{i} + (0.8\,\mathrm{m/s})\vec{i}$$

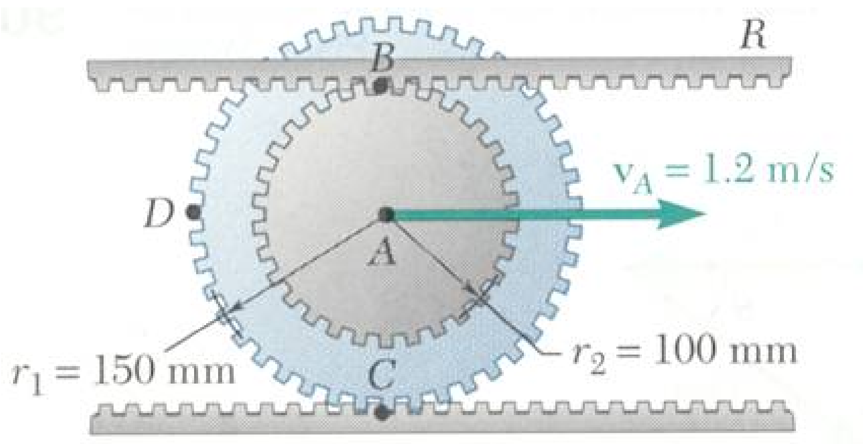
$$\boxed{\vec{v}_R = (2\,\mathrm{m/s})\vec{i}}$$

Velocity of the point *D*:

$$\vec{v}_D = \vec{v}_A + \omega\vec{k} \times \vec{r}_{D/A}$$

$$= (1.2\,\mathrm{m/s})\vec{i} + (8\,\mathrm{rad/s})\vec{k} \times (-0.150\,\mathrm{m})\vec{i}$$

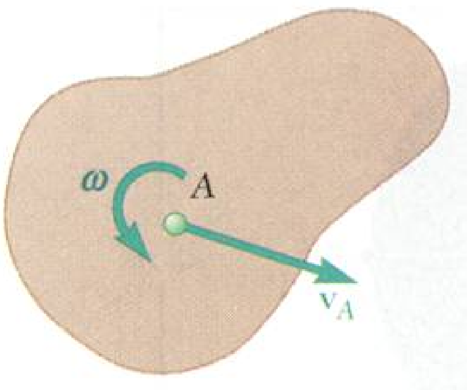
$$\boxed{\begin{aligned} &\vec{v}_D = (1.2\,\mathrm{m/s})\vec{i} + (1.2\,\mathrm{m/s})\vec{j} \\ &v_D = 1.697\,\mathrm{m/s} \end{aligned}}$$



REFLECT and THINK:

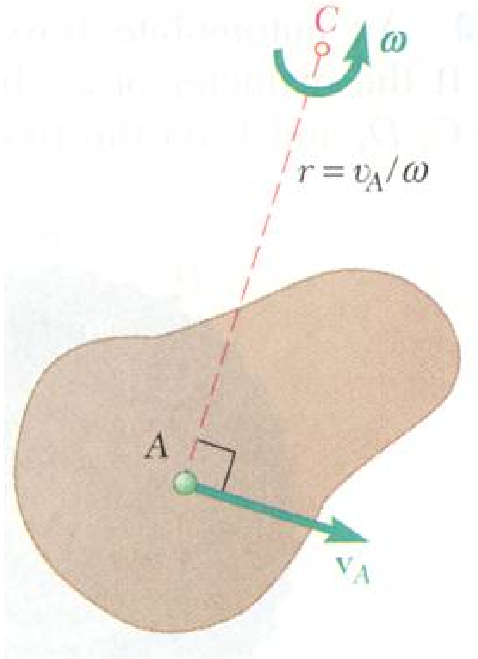
- **Note that point A was free to translate, and Point C, since it is in contact with the fixed lower rack, has a velocity of zero. Every point along diameter CAB has a velocity vector directed to the right and the magnitude of the velocity increases linearly as the distance from point C increases.**

Instantaneous Center of Rotation



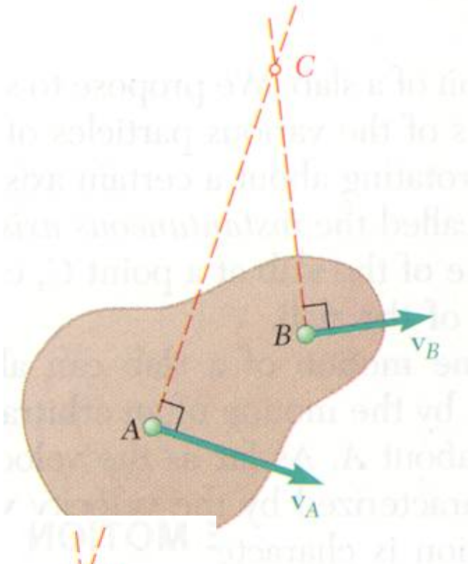
*<u>Plane motion of all particles</u> in a slab can always be replaced by the translation of an arbitrary point *A* and a rotation about *A* with an angular velocity that is independent of the choice of *A*.

*The same translational and rotational velocities at *A* are obtained by allowing the slab to rotate with the same angular velocity about the point *C* on a perpendicular to the velocity at *A*.



*The velocity of all other particles in the slab are the same as originally defined since the angular velocity and translational velocity at *A* are equivalent.

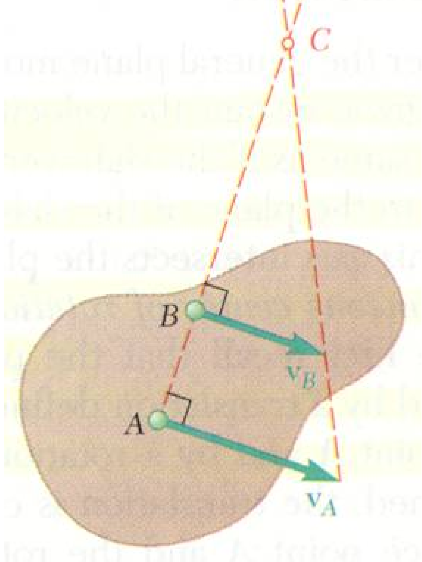
*As far as the velocities are concerned, the slab seems to rotate about the *instantaneous center of rotation C*.



*If the velocity at two points A and B are known, the instantaneous center of rotation lies at the intersection of the perpendiculars to the velocity vectors through A and B .

*If the velocity vectors are parallel, the instantaneous center of rotation is at infinity and the angular velocity is zero.

*If the velocity vectors at A and B are perpendicular to the line AB, the instantaneous center of rotation lies at the intersection of the line AB with the line joining the extremities of the velocity vectors at A and B.



*If the velocity magnitudes are equal, the instantaneous center of rotation is at infinity and the angular velocity is zero.

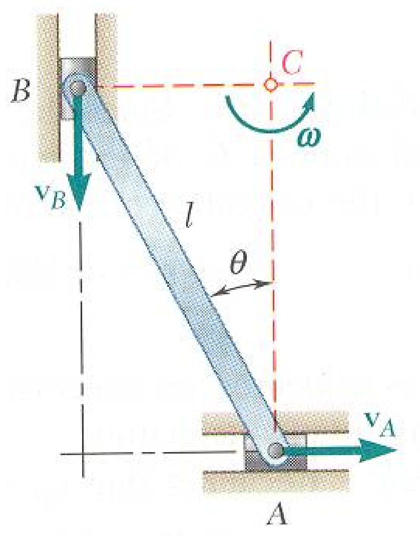


Fig. 15.20

*The instantaneous center of rotation lies at the intersection of the perpendiculars to the velocity vectors through *A* and *B* .with

$$\omega = \frac{v_A}{AC} = \frac{v_A}{l\cos\theta}$$

Then,

$$v_B = \left(BC\right)\omega = \left(l\sin\theta\right)\frac{v_A}{l\cos\theta}$$

$$= v_A \tan\theta$$

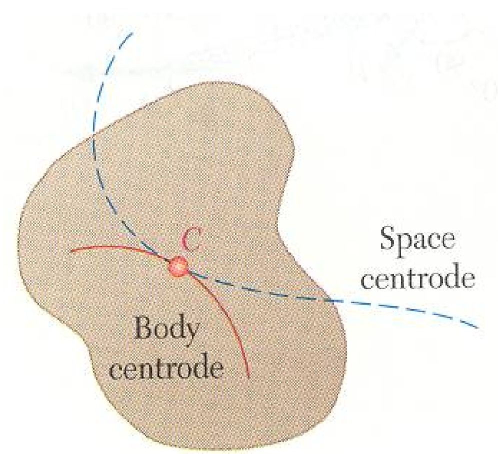
*The velocities of all particles on the rod are as if they were rotated about *C.*

*The particle at the center of rotation has zero velocity.

*The particle coinciding with the center of rotation changes with time and the acceleration of the particle at the instantaneous center of rotation is not zero.

- The acceleration of the particles in the slab cannot be determined as if the slab were simply rotating about *C.*

- The trace of the locus of the center of rotation on the body is the body centrode and in space is the space centrode.



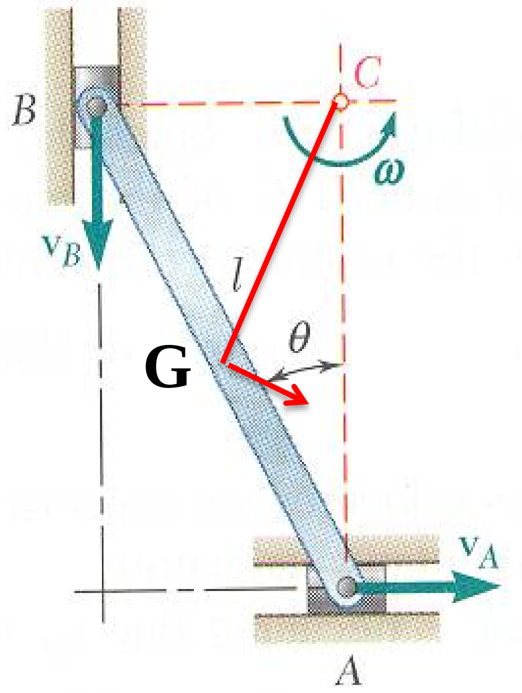
***At the instant shown, what is the approximate direction of the velocity of point G, the center of bar AB?**

a) $\rightarrow$

b) $\nearrow$

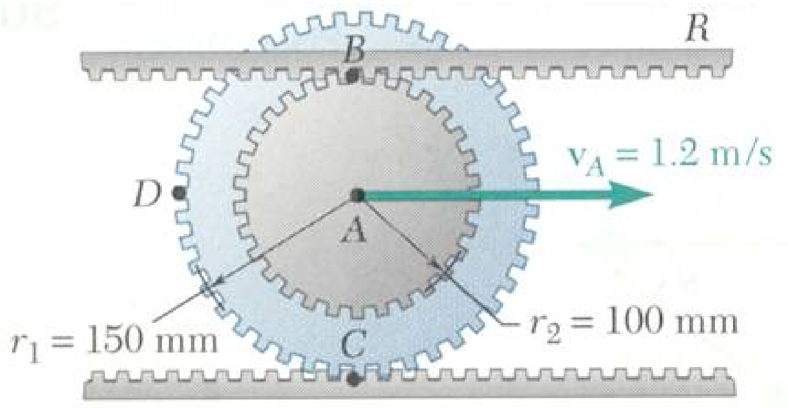
c) $\searrow$

d) $\downarrow$



answer ; c)

Sample Problem 15.9

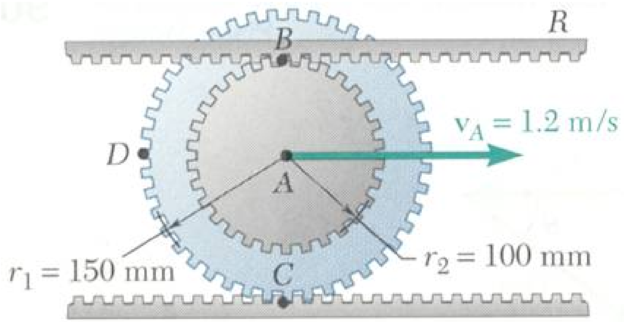


The double gear rolls on the stationary lower rack: the velocity of its center is 1.2 m/s.

Determine *(a)* the angular velocity of the gear, and *(b)* the velocities of the upper rack *R* and point *D* of the gear.

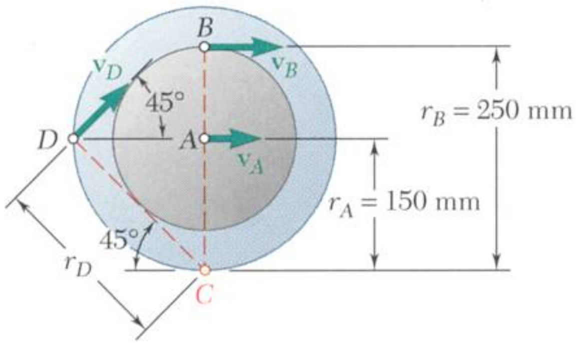
STRATEGY:

- **The point *C* is in contact with the stationary lower rack and, instantaneously, has zero velocity. It must be the location of the instantaneous center of rotation.**
- **Determine the angular velocity about *C* based on the given velocity at *A*.**
- **Evaluate the velocities at *B* and *D* based on their rotation about *C*.**



MODELING and ANALYSIS:

- The point *C* is in contact with the stationary lower rack and, instantaneously, has zero velocity. It must be the location of the instantaneous center of rotation.
- Determine the angular velocity about *C* based on the given velocity at *A*.



$$v_A = r_A \omega \qquad \omega = \frac{v_A}{r_A} = \frac{1.2\,\mathrm{m/s}}{0.15\,\mathrm{m}} = 8\,\mathrm{rad/s}$$

- Evaluate the velocities at *B* and *D* based on their rotation about *C*.

$$v_R = v_B = r_B \omega = (0.25\,\mathrm{m})(8\,\mathrm{rad/s}) \qquad \boxed{\vec{v}_R = (2\,\mathrm{m/s})\vec{i}}$$

$$r_D = (0.15\text{ m})\sqrt{2} = 0.2121\text{ m}$$

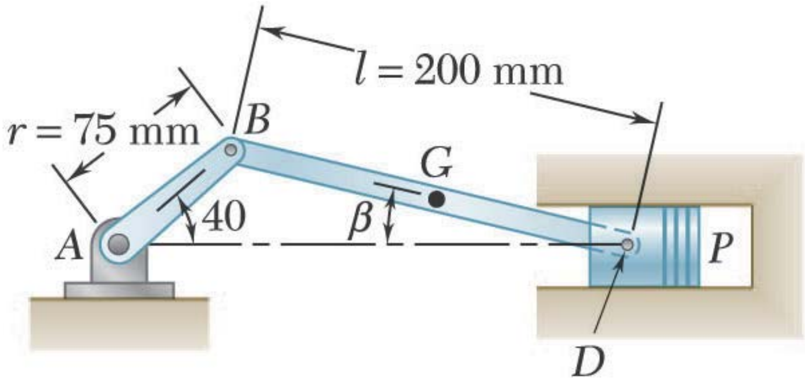
$$v_D = r_D \omega = (0.2121\text{ m})(8\,\mathrm{rad/s})$$

$$\boxed{\begin{aligned} v_D &= 1.697\,\mathrm{m/s} \\ \vec{v}_D &= \left(1.2\vec{i} + 1.2\vec{j}\right)(\mathrm{m/s}) \end{aligned}}$$

REFLECT and THINK:

The results are the same as in Sample Prob. 15.6, as you would expect, but it took much less computation to get them.

Sample Problem 15.10



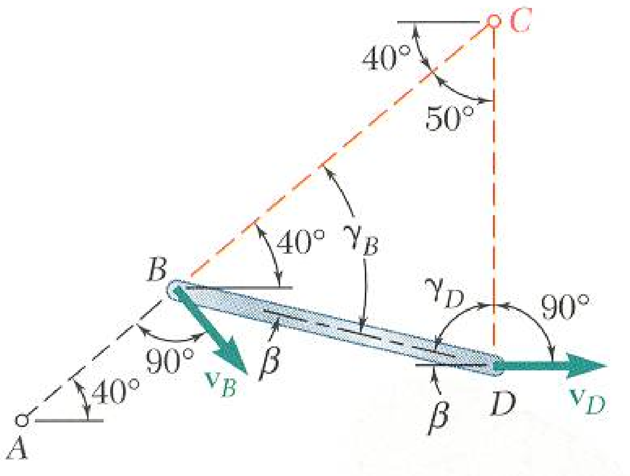
The crank *AB* has a constant clockwise angular velocity of 2000 rpm.

For the crank position indicated, determine *(a)* the angular velocity of the connecting rod *BD*, and *(b)* the velocity of the piston *P*.

Use method of instantaneous center of rotation

STRATEGY:

- Determine the velocity at *B* from the given crank rotation data.
- The direction of the velocity vectors at *B* and *D* are known. The instantaneous center of rotation is at the intersection of the perpendiculars to the velocities through *B* and *D*.
- Determine the angular velocity about the center of rotation based on the velocity at *B*.
- Calculate the velocity at *D* based on its rotation about the instantaneous center of rotation.



MODELING and ANALYSIS:

- **From Sample Problem 15.3,**

$$v_B = 15{,}705\,\text{mm/s}$$

$$\beta = 13.95^\circ$$

- **The instantaneous center of rotation is at the intersection of the perpendiculars to the velocities through *B* and *D*.**

$$\gamma_B = 40^\circ + \beta = 53.95^\circ$$

$$\gamma_D = 90^\circ - \beta = 76.05^\circ$$

$$\frac{BC}{\sin 76.05^\circ} = \frac{CD}{\sin 53.95^\circ} = \frac{200\text{ mm}}{\sin 50^\circ}$$

$$BC = 253.4\text{ mm} \quad CD = 211.1\text{ mm}$$

Determine the angular velocity about the center of rotation based on the velocity at *B*.

$$v_B = (BC)\omega_{BD}$$

$$\omega_{BD} = \frac{v_B}{BC} = \frac{15{,}705\,\text{mm/s}}{253.4\text{ mm}}$$

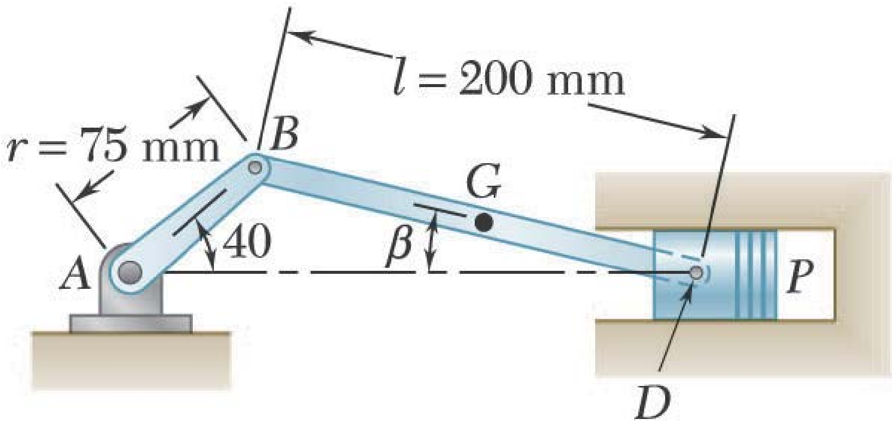
$$\boxed{\omega_{BD} = 62.0\,\text{rad/s}}$$

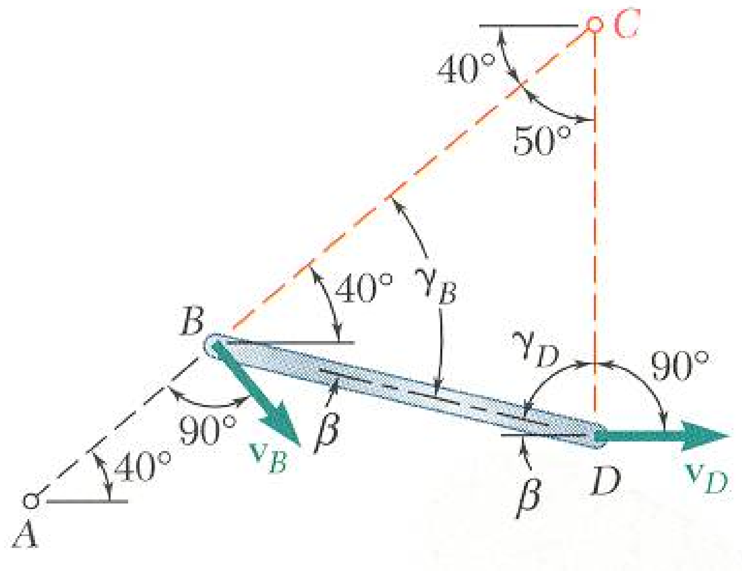
Calculate the velocity at *D* based on its rotation about the instantaneous center of rotation.

$$v_D = (CD)\omega_{BD} = (211.1\text{ mm})(62.0\,\text{rad/s})$$

$$\boxed{v_P = v_D = 13{,}080\,\text{mm/s} = 13.08\,\text{m/s}}$$

Instantaneous Center of Zero Velocity





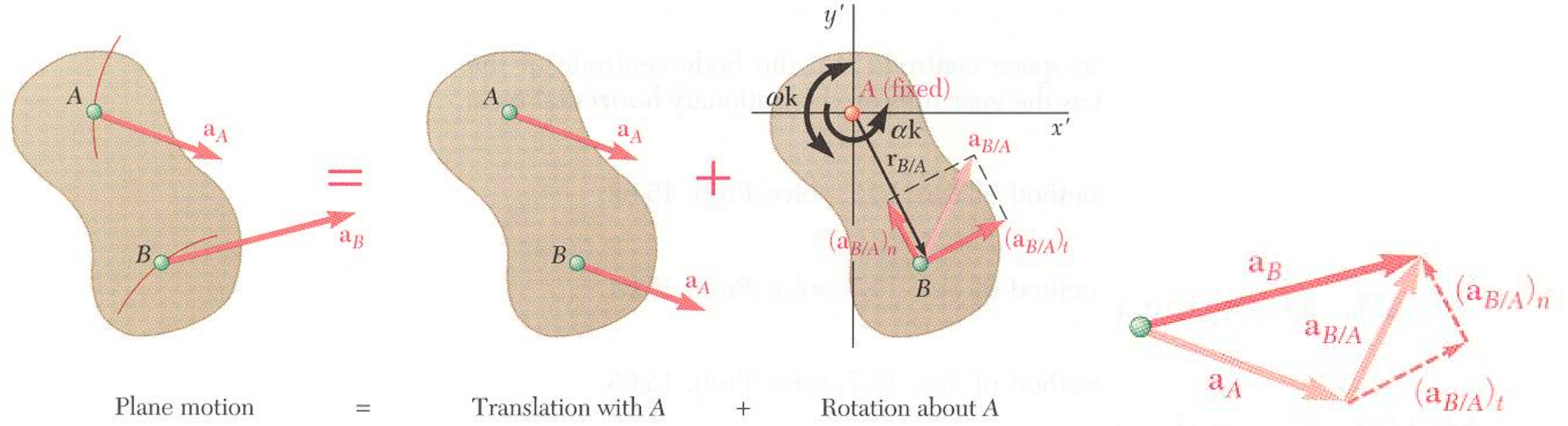
REFLECT and THINK:

What happens to the location of the instantaneous center of velocity if the crankshaft angular velocity increases from 2000 rpm in the previous problem to 3000 rpm?

What happens to the location of the instantaneous center of velocity if the angle b is 0?

15.4 GENERAL PLANE MOTION : ACCELERATION

15.4 A Absolute and Relative Acceleration in Plane Motion



- **Absolute acceleration of a particle of the slab,**
-

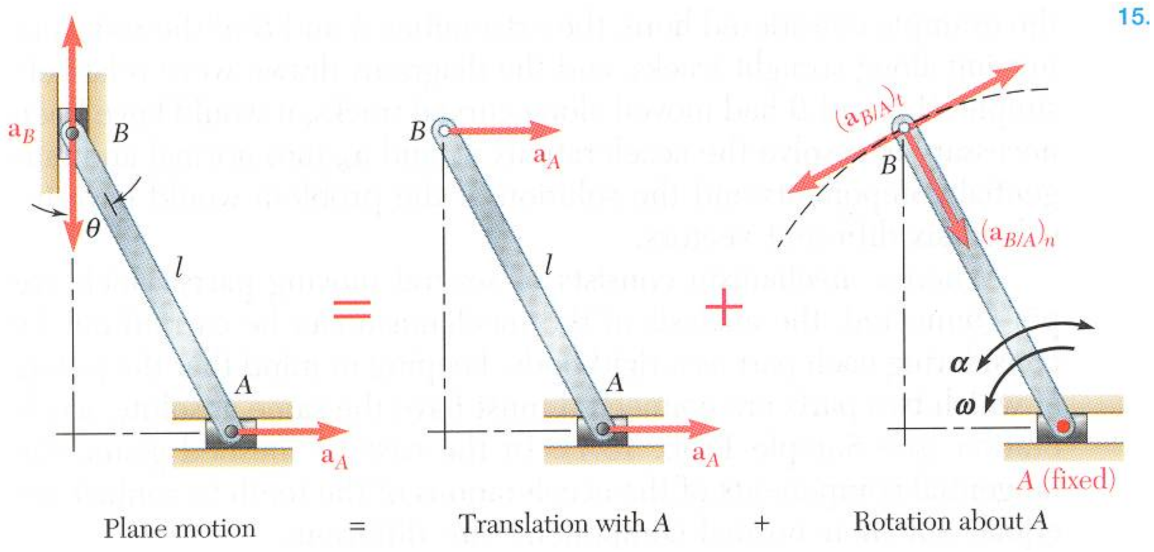
$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \qquad \textbf{(15.21)}$$

- **Relative acceleration** $\vec{a}_{B/A}$ **associated with rotation about *A* includes tangential and**

normal components,

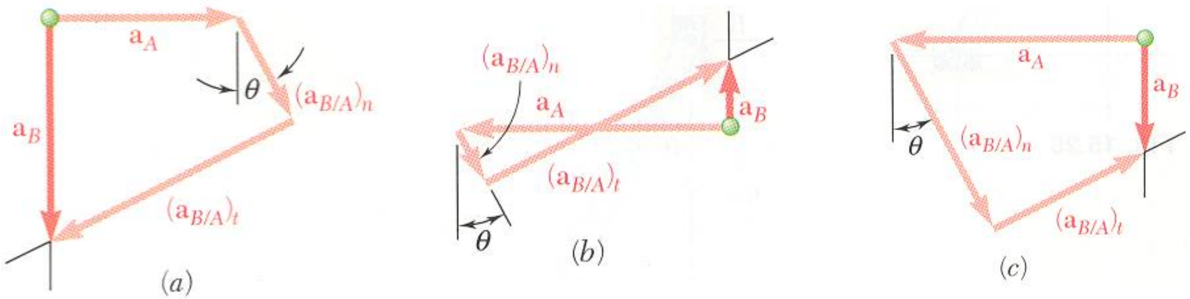
$$\left(\vec{a}_{B/A}\right)_t = \alpha\vec{k} \times \vec{r}_{B/A} \qquad \left(a_{B/A}\right)_t = r\alpha$$

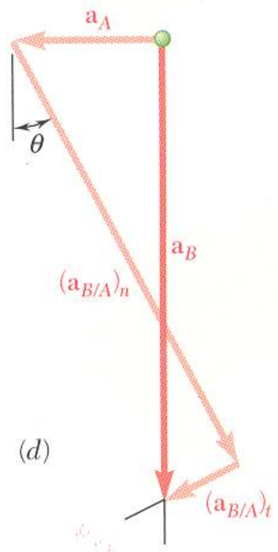
$$\left(\vec{a}_{B/A}\right)_n = -\omega^2 \vec{r}_{B/A} \qquad \left(a_{B/A}\right)_n = r\omega^2 \qquad \textbf{(15.22)}$$



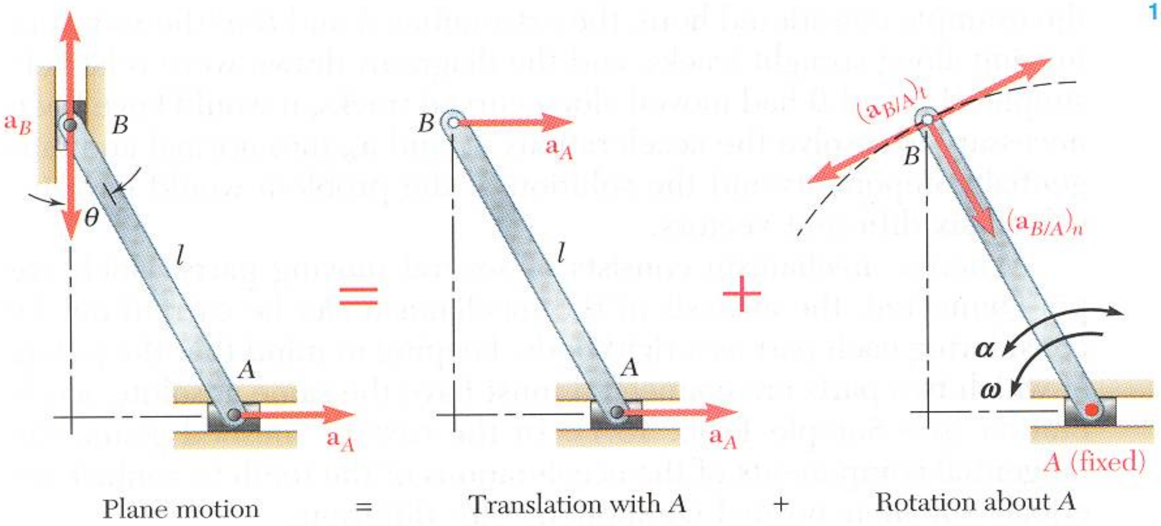
- **Given** $\vec{a}_A$ and $\vec{v}_A$, **determine** $\vec{a}_B$ and $\vec{\alpha}$.

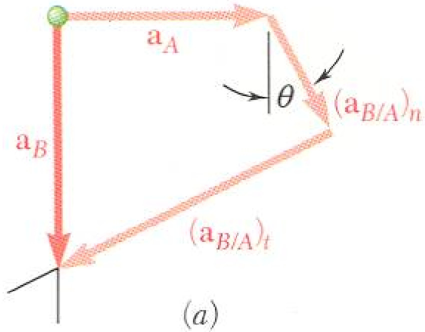
$$\begin{aligned}\vec{a}_B &= \vec{a}_A + \vec{a}_{B/A} \\ &= \vec{a}_A + \left(\vec{a}_{B/A}\right)_n + \left(\vec{a}_{B/A}\right)_t\end{aligned}$$





- **Vector result depends on sense of $\vec{a}_A$ and the relative magnitudes of a_A and $\left(a_{B/A}\right)_n$**
- **Must also know angular velocity *w*.**





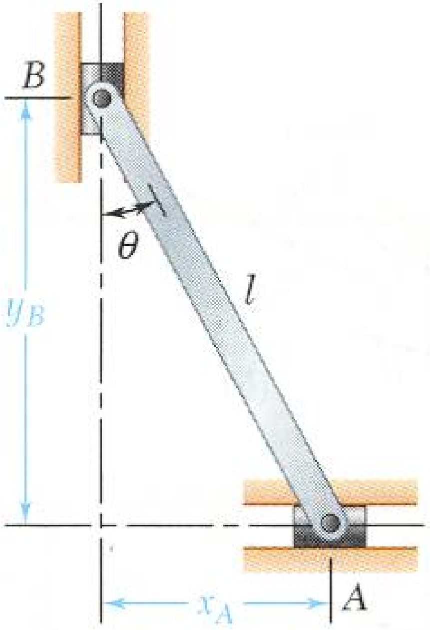
- **Write** $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ **in terms of the two component equations,**

$+\rightarrow$ ***x*** **components:** $0 = a_A + l\omega^2 \sin\theta - l\alpha\cos\theta$

$+\uparrow$ ***y*** **components:** $-a_B = -l\omega^2 \cos\theta - l\alpha\sin\theta$

- **Solve for** a_B **and** a**.**

15.4B Analysis of Plane Motion in Terms of a Parameter



- **In some cases, it is advantageous to determine the absolute velocity and acceleration of a mechanism directly.**

$$x_A = l\sin\theta$$

$$\begin{aligned} v_A &= \dot{x}_A \\ &= l\dot{\theta}\cos\theta \\ &= l\omega\cos\theta \end{aligned}$$

$$y_B = l\cos\theta$$

$$\begin{aligned} v_B &= \dot{y}_B \\ &= -l\dot{\theta}\sin\theta \\ &= -l\omega\sin\theta \end{aligned}$$

$$\begin{aligned} a_A &= \ddot{x}_A \\ &= -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta \\ &= -l\omega^2\sin\theta + l\alpha\cos\theta \end{aligned}$$

$$\begin{aligned} a_B &= \ddot{y}_B \\ &= -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta \\ &= -l\omega^2\cos\theta - l\alpha\sin\theta \end{aligned}$$

15.5 Analyzing Motion with Respect to a Rotating Frame

Rotating coordinate systems are often used to analyze mechanisms (such as amusement park rides) as well as weather patterns.

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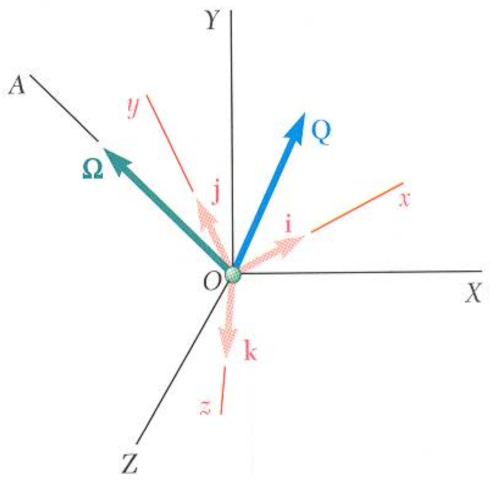
15.5A Rate of Change With Respect to a Rotating Frame



- Frame *OXYZ* is fixed.
- Frame *Oxyz* rotates about fixed axis *OA* with angular velocity $\vec{\Omega}$
- Vector function $\vec{Q}(t)$ varies in direction and magnitude.

- **With respect to the rotating *Oxyz* frame,**

$$\vec{Q} = Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}$$

$$\left(\dot{\vec{Q}}\right)_{Oxyz} = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k}$$

- **With respect to the fixed *OXYZ* frame,**

$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k} + Q_x\dot{\vec{i}} + Q_y\dot{\vec{j}} + Q_z\dot{\vec{k}}$$

$$* \ \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k} = \left(\dot{\vec{Q}}\right)_{Oxyz} =$$

Rate of change with respect to rotating frame.

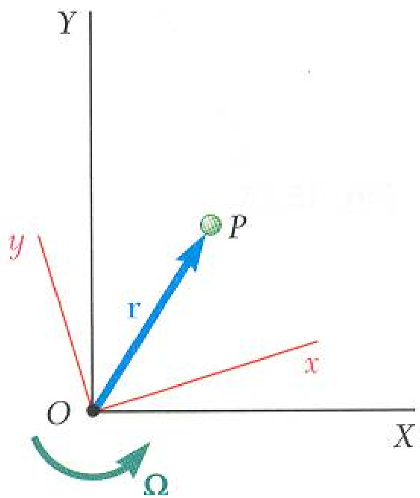
- **If $\vec{Q}$ were fixed within *Oxyz* then $\left(\dot{\vec{Q}}\right)_{OXYZ}$ is equivalent to velocity of a point in a rigid body attached to *Oxyz* and**

$$Q_x\dot{\vec{i}} + Q_y\dot{\vec{j}} + Q_z\dot{\vec{k}} = \vec{\Omega}\times\vec{Q}$$

- **With respect to the fixed *OXYZ* frame,**

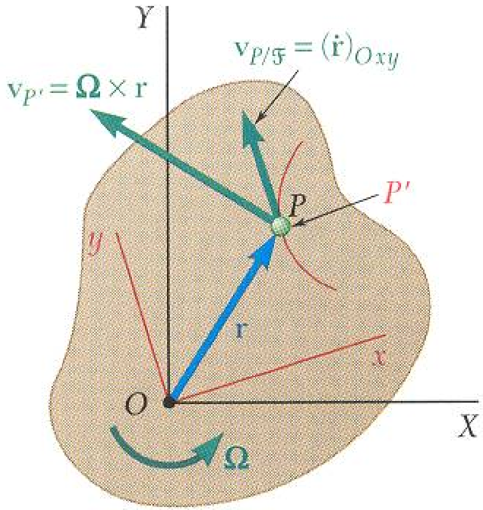
$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \left(\dot{\vec{Q}}\right)_{Oxyz} + \vec{\Omega}\times\vec{Q}$$

15.5B Plane Motion Relative to a Rotating Frame



- **Frame *OXY* is fixed and frame *Oxy* rotates with angular velocity** $\vec{\Omega}$.
- **Position vector** $\vec{r}_P$ **for the <u>particle *P* is the same in both frames</u> but the rate of change depends on the choice of frame.**
- **The absolute velocity of the particle *P* is**

$$\vec{v}_P = (\dot{\vec{r}})_{OXY} = \vec{\Omega} \times \vec{r} + (\dot{r})_{Oxy}$$



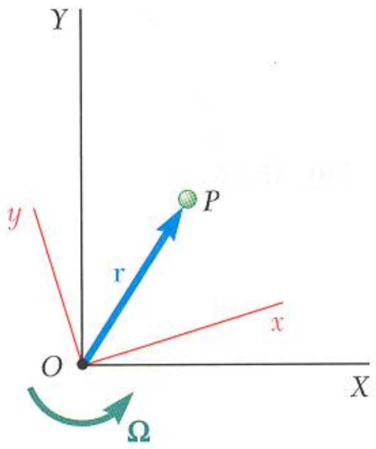
- **Imagine a rigid slab attached to the rotating frame *Oxy* or F for short. Let *P′* be a point on the slab which corresponds instantaneously to position of particle *P*.**

$\vec{v}_{P/\mathrm{F}} = (\dot{\vec{r}})_{Oxy} =$ **velocity of *P* along its path on the slab**

$\vec{v}_{P'} =$ **absolute velocity of point *P′* on the slab**

- **Absolute velocity for the particle P may be written as**

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathrm{F}}$$



- **Absolute acceleration for the particle *P* is**

$$\vec{a}_P = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\dot{\vec{r}}\right)_{OXY} + \frac{d}{dt}\left[\left(\dot{\vec{r}}\right)_{Oxy}\right]$$

but,

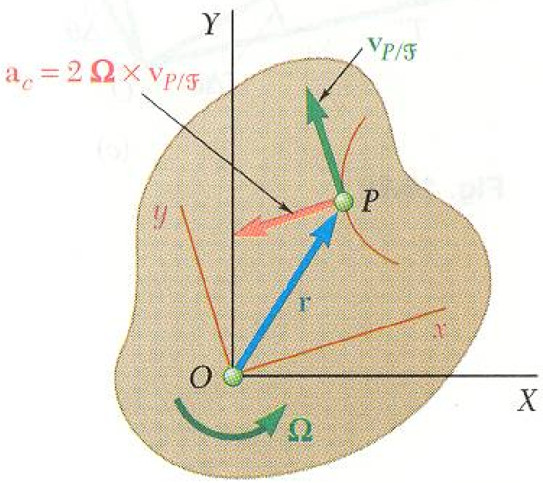
$$\left(\dot{\vec{r}}\right)_{OXY} = \vec{\Omega} \times \vec{r} + \left(\dot{\vec{r}}\right)_{Oxy}$$

$$\vec{v}_P = \vec{\Omega} \times \vec{r} + \left(\dot{r}\right)_{Oxy}$$
$$= \vec{v}_{P'} + \vec{v}_{P/\mathbf{F}}$$

$$\frac{d}{dt}\left[\left(\dot{\vec{r}}\right)_{Oxy}\right] = \left(\ddot{\vec{r}}\right)_{Oxy} + \vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxy}$$

$$\vec{a}_P = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) + 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxy} + \left(\ddot{\vec{r}}\right)_{Oxy}$$

- **Utilizing the conceptual point *P′* on the slab,**



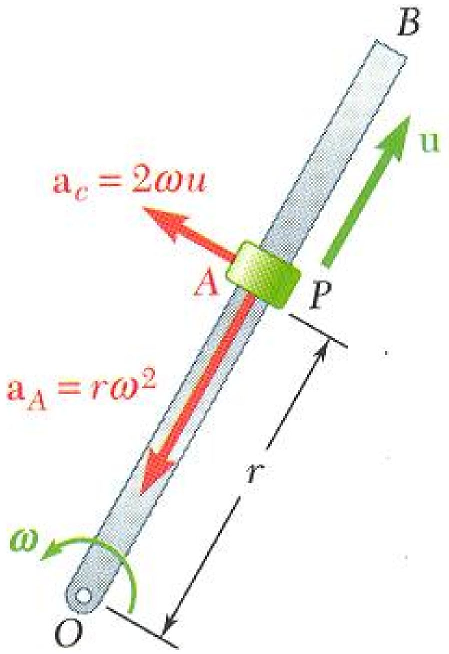
$$\vec{a}_{P'} = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$$

$$\vec{a}_{P/\mathsf{F}} = \left(\ddot{\vec{r}}\right)_{Oxy}$$

- **Absolute acceleration for the particle *P* becomes**

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathsf{F}} + 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxy}$$
$$= \vec{a}_{P'} + \vec{a}_{P/\mathsf{F}} + \vec{a}_c$$

$$\vec{a}_c = 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxy} = 2\vec{\Omega} \times \vec{v}_{P/\mathsf{F}} =$$ **Coriolis acceleration**



- Consider a collar *P* which is made to slide at constant relative velocity *u* along rod *OB.* The rod is rotating at a constant angular velocity *w.* The point *A* on the rod corresponds to the instantaneous position of *P.*
- Absolute acceleration of the collar is

$$\vec{a}_P = \vec{a}_A + \vec{a}_{P/\mathsf{F}} + \vec{a}_c$$

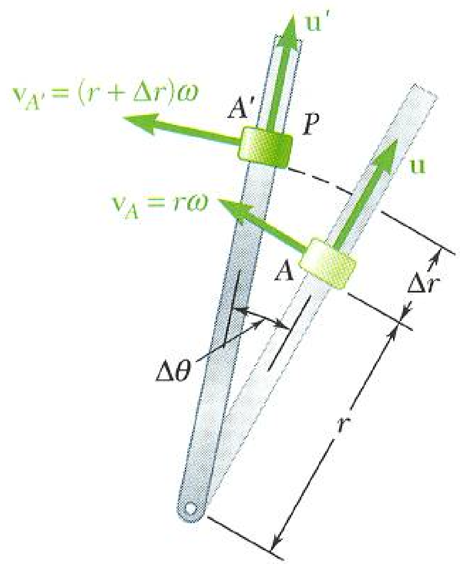
where

$$\vec{a}_A = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) \qquad a_A = r\omega^2$$

$$\vec{a}_{P/\mathbf{F}} = \left(\ddot{\vec{r}}\right)_{Oxy} = 0$$

$$\vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathbf{F}} \qquad a_c = 2\omega u$$

- The absolute acceleration consists of the radial and tangential vectors shown



at t, $\vec{v} = \vec{v}_A + \vec{u}$

at $t + \Delta t$, $\vec{v}' = \vec{v}_{A'} + \vec{u}'$

- **Change in velocity over** dt **is represented by the sum of three vectors**

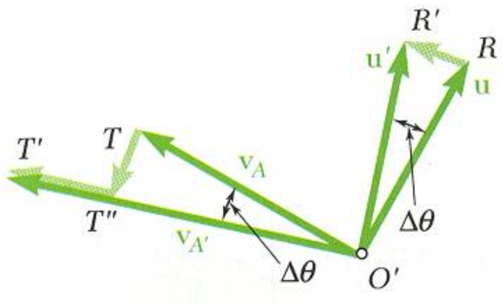
$$\Delta\vec{v} = \overrightarrow{RR'} + \overrightarrow{TT''} + \overrightarrow{T''T'}$$

- $\overrightarrow{TT''}$ **is due to change in direction of the velocity of point *A* on the rod,**

$$\lim_{\Delta t \to 0} \frac{\overrightarrow{TT''}}{\Delta t} = \lim_{\Delta t \to 0} v_A \frac{\Delta\theta}{\Delta t} = r\omega\omega = r\omega^2 = a_A$$

recall, $\vec{a}_A = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$ $\quad a_A = r\omega^2$

- $\overrightarrow{RR'}$ and $\overrightarrow{T''T'}$ **result from combined effects of relative motion of *P* and rotation of the rod**

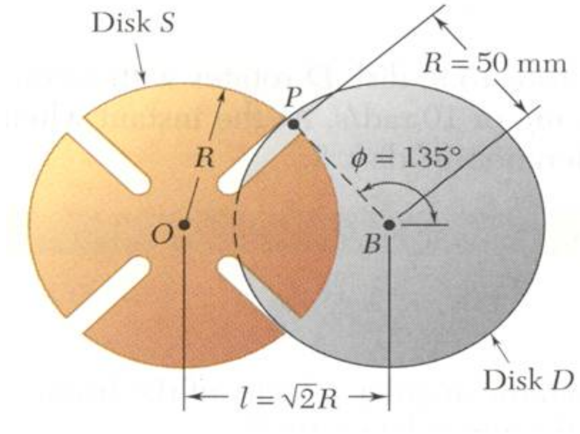


$$\lim_{\Delta t \to 0}\left(\frac{\overrightarrow{RR'}}{\Delta t} + \frac{\overrightarrow{T''T'}}{\Delta t}\right) = \lim_{\Delta t \to 0}\left(u\frac{\Delta\theta}{\Delta t} + \omega\frac{\Delta r}{\Delta t}\right)$$

$$= u\omega + \omega u = 2\omega u$$

recall, $\vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathrm{F}}$ $\quad a_c = 2\omega u$

Sample Problem 15.19

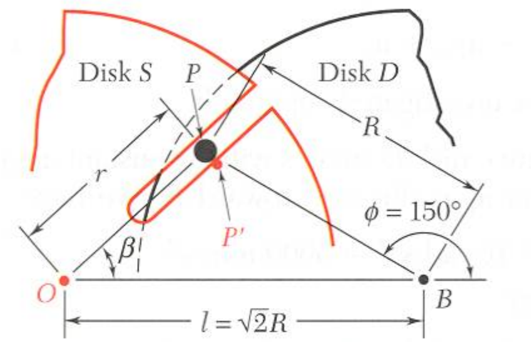


Disk D of the Geneva mechanism rotates with constant counterclockwise angular velocity *w* w_D **= 10 rad/s.**

At the instant when *f* = 150º, determine (*a*) the angular velocity of disk *S*, and (*b*) the velocity of pin *P* relative to disk *S*.

STRATEGY:

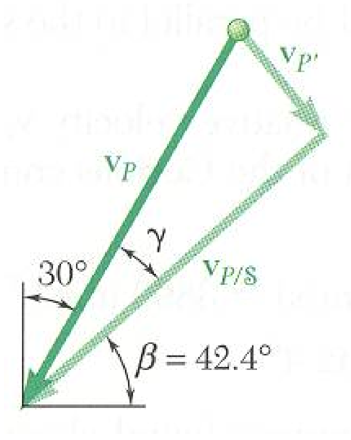
- The absolute velocity of the point P may be written as $\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/s}$
- Magnitude and direction of velocity $\vec{v}_P$ of pin P are calculated from the radius and angular velocity of disk D.
- Direction of velocity $\vec{v}_{P'}$ of point P' on S coinciding with P is perpendicular to radius OP.
- Direction of velocity $\vec{v}_{P/s}$ of P with respect to S is parallel to the slot.
- Solve the vector triangle for the angular velocity of S and relative velocity of P.



MODELING and ANALYSIS:

- The absolute velocity of the point P may be written as
  $\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/s}$
- Magnitude and direction of absolute velocity of pin P are calculated from radius and angular velocity of disk D.

$$v_P = R\omega_D = (50\text{ mm})(10\text{ rad/s}) = 500\text{ mm/s}$$



- **Direction of velocity of *P* with respect to *S* is parallel to slot.**

From the law of cosines,

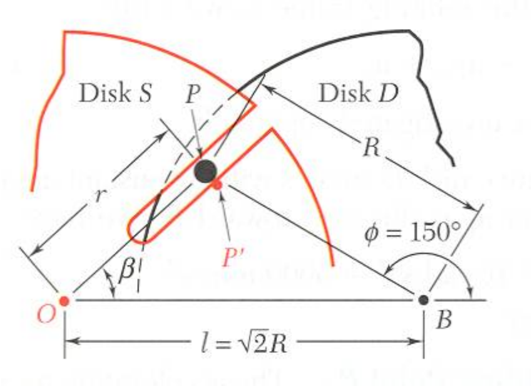
$$r^2 = R^2 + l^2 - 2Rl\cos 30° = 0.551R^2 \qquad r = 37.1\text{ mm}$$

From the law of cosines,

$$\frac{\sin\beta}{\mathrm{R}} = \frac{\sin 30°}{r} \qquad \sin\beta = \frac{\sin 30°}{0.742} \qquad \beta = 42.4°$$

The interior angle of the vector triangle is

$$\gamma = 90° - 42.4° - 30° = 17.6°$$

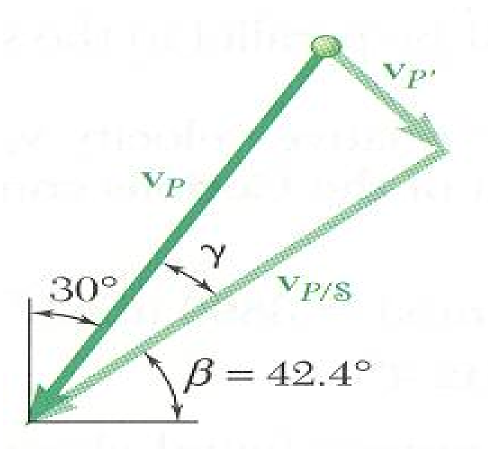


- **Direction of velocity of point *P'* on *S* coinciding with *P* is perpendicular to radius *OP*. From the velocity triangle,**

$$v_{P'} = v_P \sin\gamma = (500\text{ mm/s})\sin 17.6° = 151.2\text{ mm/s}$$

$$= r\omega_s \qquad \omega_s = \frac{151.2\text{ mm/s}}{37.1\text{ mm}}$$

$$\boxed{\vec{\omega}_s = (-4.08\text{ rad/s})\vec{k}}$$



$$v_P = 500 \text{ mm/s}$$

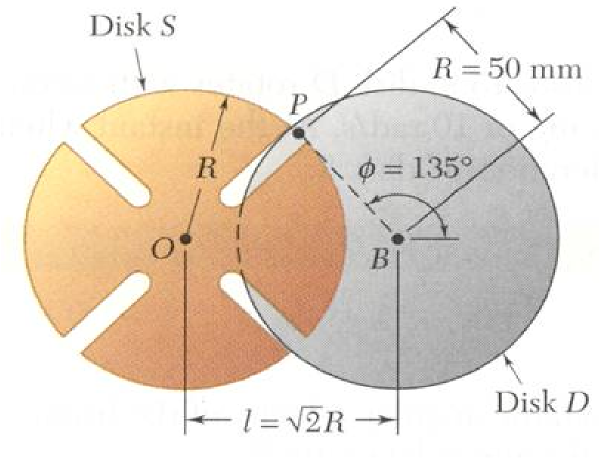
$$v_{P/s} = v_P \cos\gamma = (500 \text{ m/s})\cos 17.6°$$

$$\boxed{\vec{v}_{P/s} = (477 \text{ m/s})\left(-\cos 42.4°\vec{i} - \sin 42.4°\vec{j}\right)}$$

REFLECT and THINK:

The result of the Geneva mechanism is that disk S rotates ¼ turn each time pin P engages, then it remains motionless while pin P rotates around before entering the next slot. Disk D rotates continuously, but disk S rotates intermittently. An alternative approach to drawing the vector triangle is to use vector algebra.

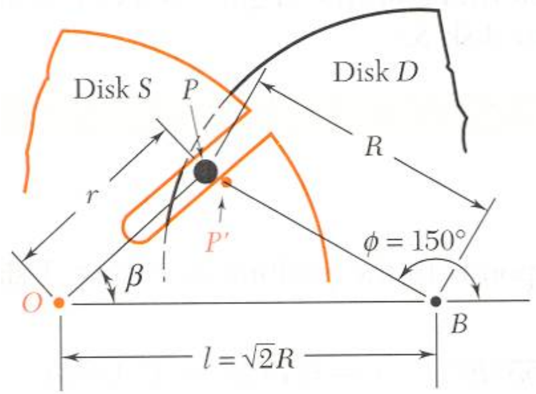
Sample Problem 15.20



In the Geneva mechanism, disk *D* rotates with a constant counter-clockwise angular velocity of 10 rad/s. At the instant when j = 150º, determine angular acceleration of disk *S*.

STRATEGY:

- The absolute acceleration of the pin *P* may be expressed as $\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/S} + \vec{a}_c$
- The instantaneous angular velocity of Disk *S* is determined as in Sample Problem 15.9.
- The only unknown involved in the acceleration equation is the instantaneous angular acceleration of Disk *S*.
- Resolve each acceleration term into the component parallel to the slot. Solve for the angular acceleration of Disk *S*.



MODELING and ANALYSIS:

- **Absolute acceleration of the pin *P* may be expressed as**

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/s} + \vec{a}_c$$

- **From Sample Problem 15.9.**

$$\beta = 42.4^\circ \qquad \vec{\omega}_S = (-4.08\,\text{rad/s})\vec{k}$$

$$\vec{v}_{P/s} = (477\,\text{mm/s})\left(-\cos 42.4^\circ \vec{i} - \sin 42.4^\circ \vec{j}\right)$$

- **Considering each term in the acceleration equation,**

$$a_P = R\omega_D^2 = (500\text{mm})(10\,\text{rad/s})^2 = 5000\,\text{mm/s}^2$$

$$\vec{a}_P = \left(5000\,\text{mm/s}^2\right)\left(\cos 30^\circ \vec{i} - \sin 30^\circ \vec{j}\right)$$

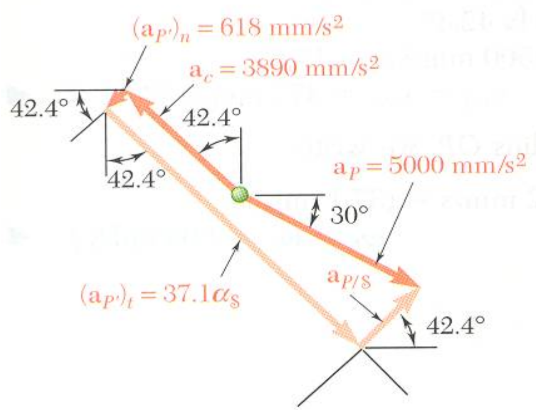
$$\vec{a}_{P'} = \left(\vec{a}_{P'}\right)_n + \left(\vec{a}_{P'}\right)_t$$

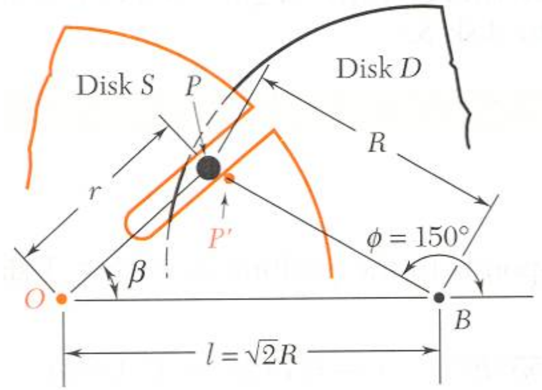
$$\left(\vec{a}_{P'}\right)_n = \left(r\omega_S^2\right)\left(-\cos 42.4^\circ \vec{i} - \sin 42.4^\circ \vec{j}\right)$$

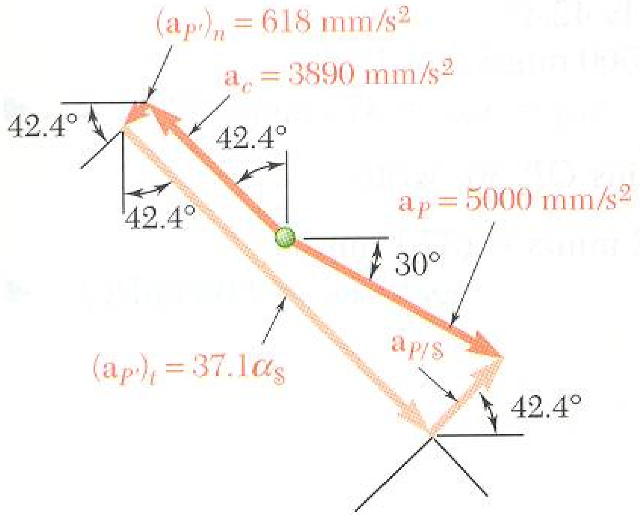
$$\left(\vec{a}_{P'}\right)_t = \left(r\alpha_S\right)\left(-\sin 42.4^\circ \vec{i} + \cos 42.4^\circ \vec{j}\right)$$

$$\left(\vec{a}_{P'}\right)_t = \left(\alpha_S\right)(37.1\text{mm})\left(-\sin 42.4^\circ \vec{i} + \cos 42.4^\circ \vec{j}\right)$$

note: a_S may be positive or negative







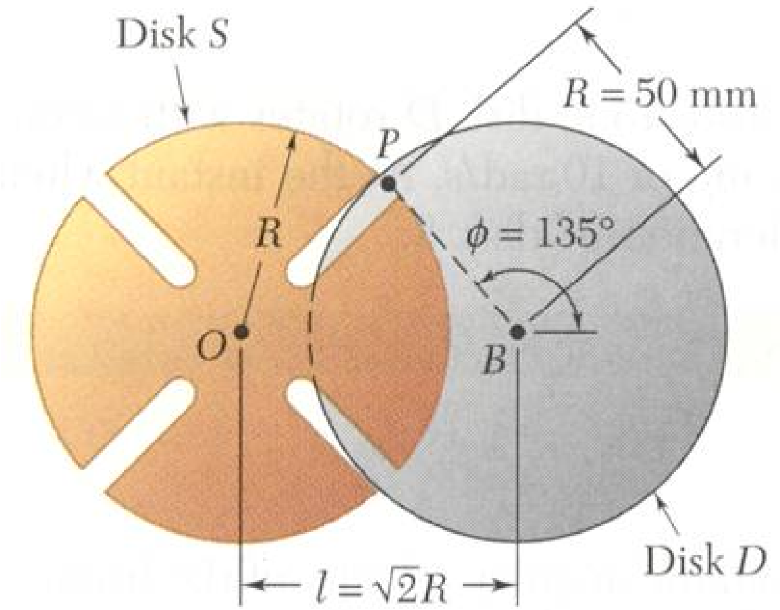
- The direction of the Coriolis acceleration is obtained by rotating the direction of the relative velocity $\vec{v}_{P/s}$ by 90° in the sense of w_S.

$$\vec{a}_c = \left(2\omega_S v_{P/s}\right)\left(-\sin 42.4°\vec{i} + \cos 42.4\vec{j}\right)$$
$$= 2(4.08\,\text{rad/s})(477\,\text{mm/s})\left(-\sin 42.4°\vec{i} + \cos 42.4\vec{j}\right)$$
$$= \left(3890\,\text{mm/s}^2\right)\left(-\sin 42.4°\vec{i} + \cos 42.4\vec{j}\right)$$

- The relative acceleration $\vec{a}_{P/s}$ must be parallel to the slot.
- Equating components of the acceleration terms perpendicular to the slot,

$$37.1\alpha_S + 3890 - 5000\cos 17.7° = 0$$
$$\alpha_S = -233\,\text{rad/s}$$

$$\boxed{\vec{\alpha}_S = (-233\,\text{rad/s})\vec{k}}$$

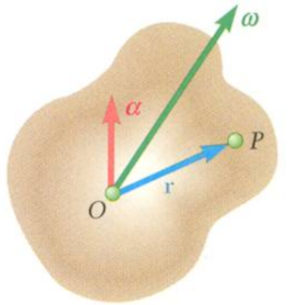


REFLECT and THINK:

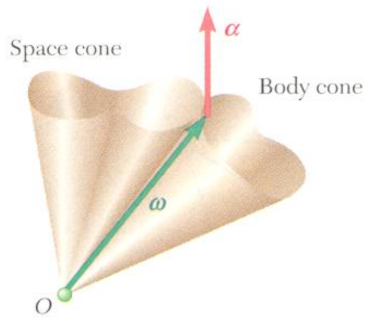
- **It seems reasonable that, since disk S starts and stops over the very short time intervals when pin P is engaged in the slots, the disk must have a very large angular acceleration. An alternative approach would have been to use the vector algebra approach.**

15.6 Motion of a Rigid Body in Space

15.6A Motion About a Fixed Point



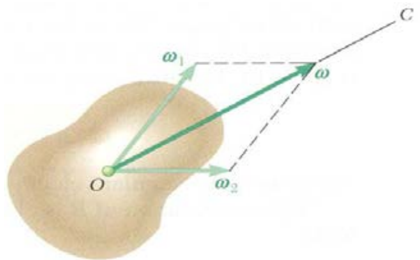
- The most general displacement of a rigid body with a fixed point *O* is equivalent to a rotation of the body about an axis through *O*.
- With the instantaneous axis of rotation and angular velocity the velocity of a particle *P* of the body is $\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$



and the acceleration of the particle *P* is

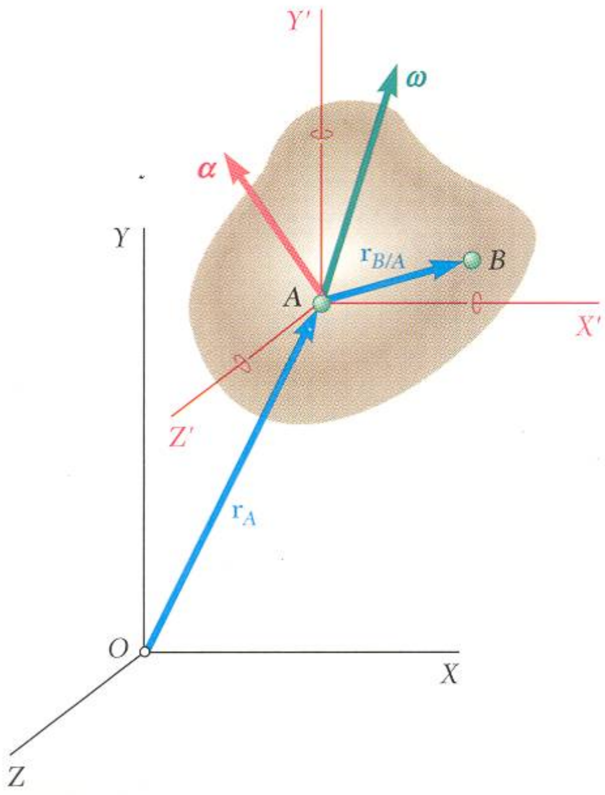
$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) \qquad \vec{\alpha} = \frac{d\vec{\omega}}{dt}.$$

- The angular acceleration $\vec{\alpha}$ represents the velocity of the tip of $\vec{\omega}$.
- As the vector $\vec{\omega}$ moves within the body and in space, it generates a body cone and space cone which are tangent along the instantaneous axis of rotation.



- Angular velocities have magnitude and direction and obey parallelogram law of addition. They are vectors.

15.6B General Motion



- For particles *A* and *B* of a rigid body,

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

- Particle *A* is fixed within the body and motion of the body relative to *AX'Y'Z'* is the motion of a body with a fixed point

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

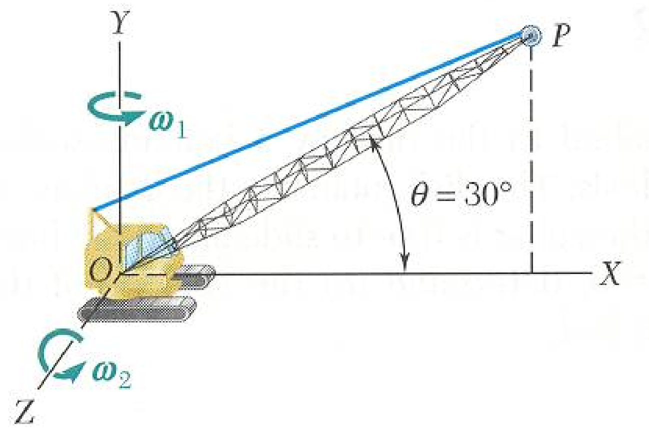
- Similarly, the acceleration of the particle *P* is

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$$= \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A} \right)$$

- Most general motion of a rigid body is equivalent to:
- a translation in which all particles have the same velocity and
- acceleration of a reference particle *A,* and of a motion in which particle *A* is assumed fixed.

Sample Problem 15.21



The crane rotates with a constant angular velocity w_1 = 0.30 rad/s and the boom is being raised with a constant angular velocity w_2 = 0.50 rad/s. The length of the boom is l = 12 m.

Determine:

- angular velocity of the boom,
- angular acceleration of the boom,
- velocity of the boom tip, and
- acceleration of the boom tip.

STRATEGY:

With $\vec{\omega}_1 = 0.30\vec{j} \quad \vec{\omega}_2 = 0.50\vec{k}$

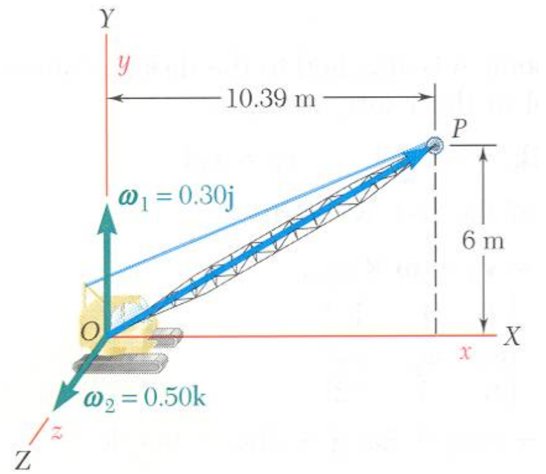
$$\vec{r} = 12(\cos 30°\vec{i} + \sin 30°\vec{j})$$
$$= 10.39\vec{i} + 6\vec{j}$$

- **Angular velocity of the boom,** $\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$

$$\vec{\alpha} = \dot{\vec{\omega}}_1 + \dot{\vec{\omega}}_2 = \dot{\vec{\omega}}_2 = \left(\dot{\vec{\omega}}_2\right)_{Oxyz} + \vec{\Omega} \times \vec{\omega}_2$$

$$= \vec{\omega}_1 \times \vec{\omega}_2$$

- **Angular acceleration of the boom,**
- **Velocity of boom tip,** $\vec{v} = \vec{\omega} \times \vec{r}$
- **Acceleration of boom tip,** $\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$



<u>MODELING and ANALYSIS</u>:

- **Angular velocity of the boom,**

$$\boxed{\vec{\omega} = (0.30\,\mathrm{rad/s})\vec{j} + (0.50\,\mathrm{rad/s})\vec{k}}$$

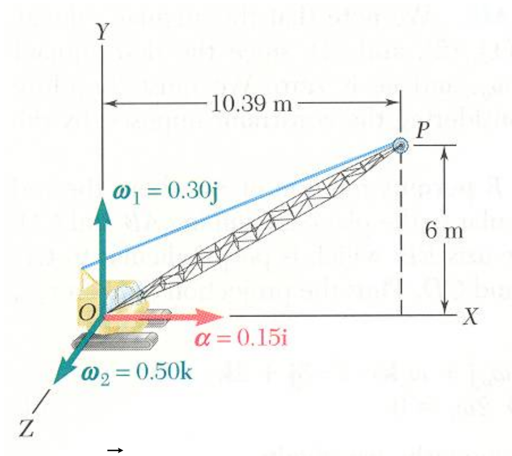
$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

- **Angular acceleration of the boom,**

$$\vec{\alpha} = \dot{\vec{\omega}}_1 + \dot{\vec{\omega}}_2 = \dot{\vec{\omega}}_2 = \left(\dot{\vec{\omega}}_2\right)_{Oxyz} + \vec{\Omega} \times \vec{\omega}_2$$

$$= \vec{\omega}_1 \times \vec{\omega}_2 = (0.30\,\mathrm{rad/s})\vec{j} \times (0.50\,\mathrm{rad/s})\vec{k}$$

$$\boxed{\vec{\alpha} = \left(0.15\,\mathrm{rad}/\mathrm{s}^2\right)\vec{i}}$$



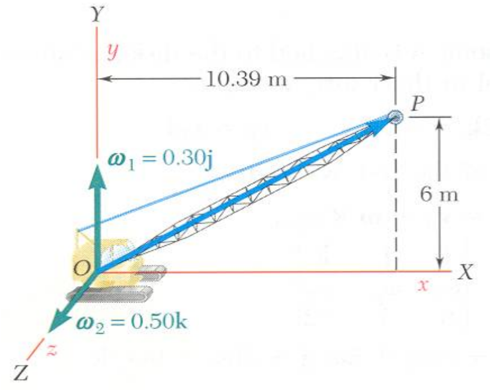
$= 0.50\vec{k}$

- **Velocity of boom tip,**

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0.3 & 0.5 \\ 10.39 & 6 & 0 \end{vmatrix}$$

$$\boxed{\vec{v} = -(3.54\,\mathrm{m/s})\vec{i} + (5.20\,\mathrm{m/s})\vec{j} - (3.12\,\mathrm{m/s})\vec{k}}$$

- **Acceleration of boom tip,**



$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

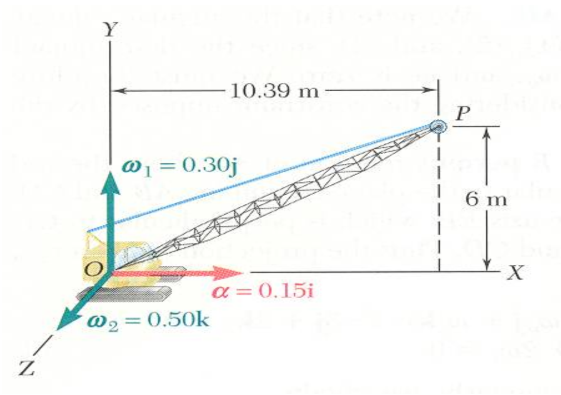
$$\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.15 & 0 & 0 \\ 10.39 & 6 & 0 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0.30 & 0.50 \\ -3 & 5.20 & -3.12 \end{vmatrix}$$

$$= 0.90\vec{k} - 0.94\vec{i} - 2.60\vec{i} - 1.50\vec{j} + 0.90\vec{k}$$

$$\boxed{\vec{a} = -\left(3.54\,\mathrm{m/s^2}\right)\vec{i} - \left(1.50\,\mathrm{m/s^2}\right)\vec{j} + \left(1.80\,\mathrm{m/s^2}\right)\vec{k}}$$

REFLECT and THINK:

- The base of the cab acts as the fixed point of the motion. Even though both components of angular velocity are constant, there is an angular acceleration due to the change in direction of the angular velocity ω_2. The angular velocity vector ω_2 changes due to the rotation of the cab, ω_1.

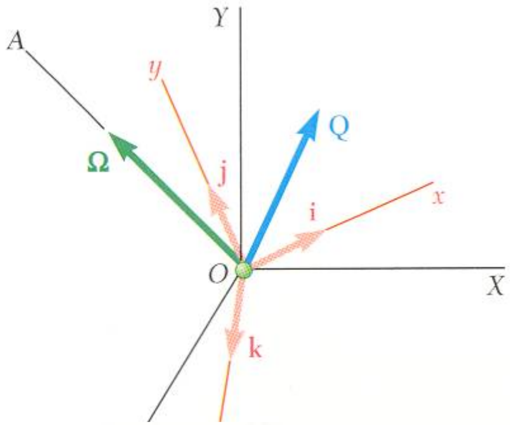


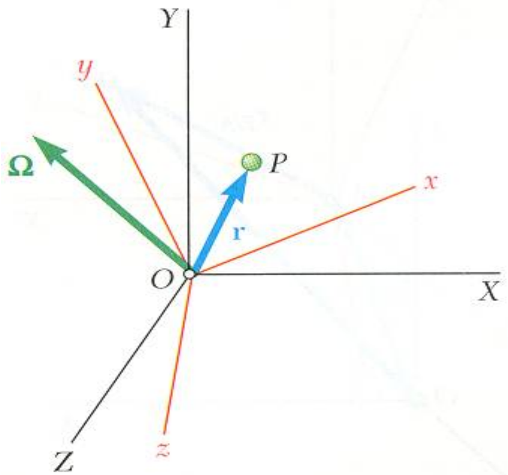
$$\vec{\omega}_1 = 0.30\vec{j} \quad \vec{\omega}_2 = 0.50\vec{k}$$

$$\vec{r} = 10.39\vec{i} + 6\vec{j}$$

15.7Motion Relative to A Moving Reference Frame

15.7A Three-Dimensional Motion Relative to a Rotating Frame





With respect to the fixed frame *OXYZ* and rotating frame *Oxyz*,

$$\left(\dot{\vec{Q}}\right)_{OXYZ} = \left(\dot{\vec{Q}}\right)_{Oxyz} + \vec{\Omega} \times \vec{Q}$$

Consider motion of particle *P* relative to a rotating frame *Oxyz* or F for short. The absolute velocity can be expressed as

$$\vec{v}_P = \vec{\Omega} \times \vec{r} + \left(\dot{\vec{r}}\right)_{Oxyz}$$

$$= \vec{v}_{P'} + \vec{v}_{P/\mathsf{F}}$$

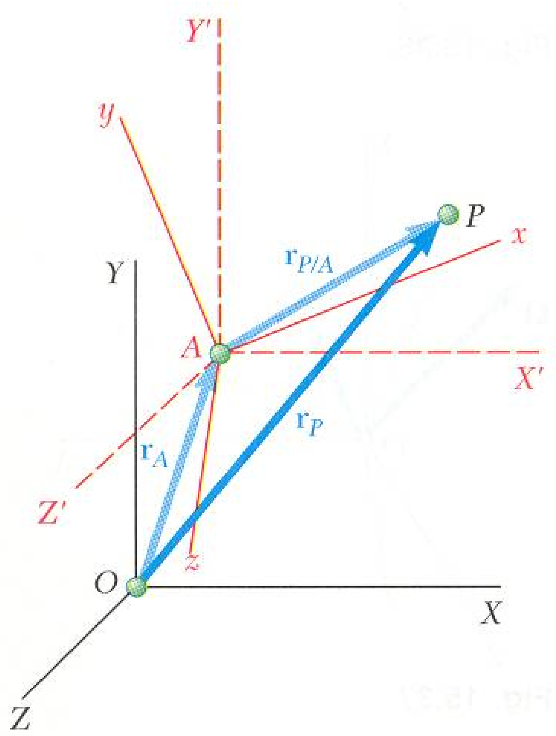
The absolute acceleration can be expressed as

$$\vec{a}_P = \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) + 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxyz} + \left(\ddot{\vec{r}}\right)_{Oxyz}$$

$$= \vec{a}_{p'} + \vec{a}_{P/\mathsf{F}} + \vec{a}_c$$

$$\vec{a}_c = 2\vec{\Omega} \times \left(\dot{\vec{r}}\right)_{Oxyz} = 2\vec{\Omega} \times \vec{v}_{P/\mathsf{F}} = \text{Coriolis acceleration}$$

15.7B Frame of Reference in General Motion



- **With respect to *OXYZ* and *AX'Y'Z'*,**

$$\vec{r}_P = \vec{r}_A + \vec{r}_{P/A}$$

$$\vec{v}_P = \vec{v}_A + \vec{v}_{P/A}$$

$$\vec{a}_P = \vec{a}_A + \vec{a}_{P/A}$$

- **The velocity and acceleration of *P* relative to *AX'Y'Z'* can be found in terms of the velocity and acceleration of *P* relative to *Axyz*.**

$$\vec{v}_P = \vec{v}_A + \vec{\Omega} \times \vec{r}_{P/A} + \left(\dot{\vec{r}}_{P/A}\right)_{Axyz}$$

$$= \vec{v}_{P'} + \vec{v}_{P/\mathrm{F}}$$

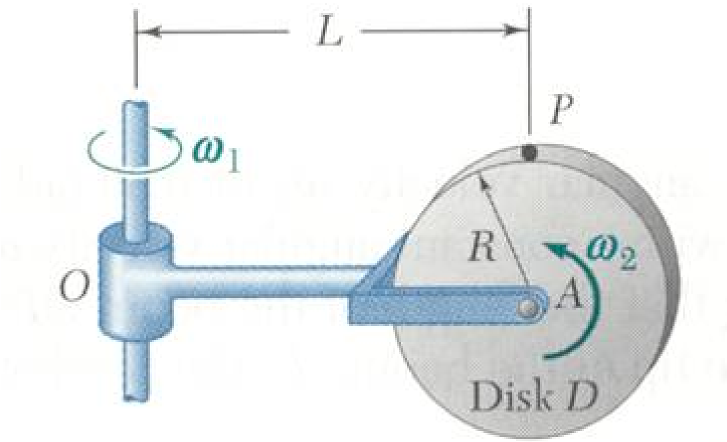
Consider:

- **fixed frame *OXYZ*,**
- **translating frame *AX'Y'Z'*, and**

– **translating and rotating frame *Axyz* or F.**

$$\vec{a}_P = \vec{a}_A + \dot{\vec{\Omega}} \times \vec{r}_{P/A} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}_{P/A}\right) + 2\vec{\Omega} \times \left(\dot{\vec{r}}_{P/A}\right)_{Axyz} + \left(\ddot{\vec{r}}_{P/A}\right)_{Axyz} = \vec{a}_{P'} + \vec{a}_{P/\mathbf{F}} + \vec{a}_c \quad (15.54)$$

Sample Problem 15.25



For the disk mounted on the arm, the indicated angular rotation rates are constant.

Determine:

- **the velocity of the point *P*,**
- **the acceleration of *P*, and**
- **angular velocity and angular acceleration of the disk.**

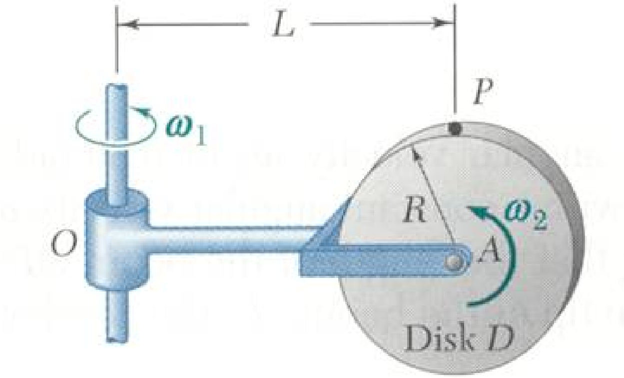
STRATEGY:

$$\vec{a}_P = \vec{a}_A + \dot{\vec{\Omega}} \times \vec{r}_{P/A} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}_{P/A}\right) + 2\vec{\Omega} \times \left(\dot{\vec{r}}_{P/A}\right)_{Axyz} + \left(\ddot{\vec{r}}_{P/A}\right)_{Axyz} = \vec{a}_{P'} + \vec{a}_{P/\mathbf{F}} + \vec{a}_c$$

- Define a fixed reference frame *OXYZ* at *O* and a moving reference frame *Axyz* or F attached to the arm at *A*.
- With *P′* of the moving reference frame coinciding with *P*, the velocity of the point *P* is found from $\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathrm{F}}$
- The acceleration of *P* is found from $\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathrm{F}} + \vec{a}_c$
- The angular velocity and angular acceleration of the disk are

$$\vec{\omega} = \vec{\Omega} + \vec{\omega}_{D/\mathrm{F}}$$

$$\vec{\alpha} = \left(\dot{\vec{\omega}}\right)_{\mathrm{F}} + \vec{\Omega} \times \vec{\omega}$$

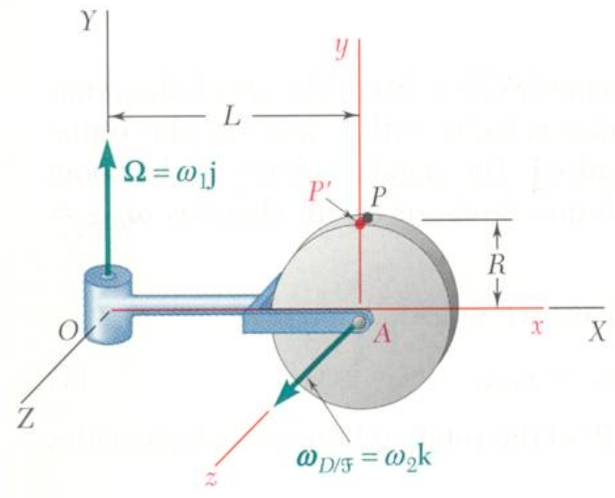


MODELING and ANALYSIS:

- Define a fixed reference frame *OXYZ* at *O* and a moving reference frame *Axyz* or F attached to the arm at *A*.

$$\vec{r} = L\vec{i} + R\vec{j} \qquad \vec{r}_{P/A} = R\vec{j}$$

$$\vec{\Omega} = \omega_1 \vec{j} \qquad \vec{\omega}_{D/\mathrm{F}} = \omega_2 \vec{k}$$



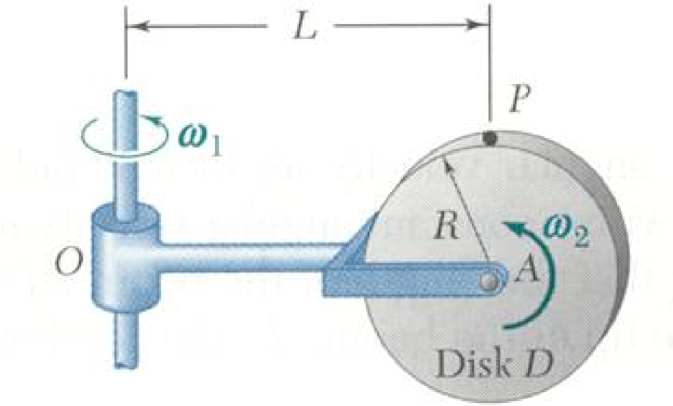
- With P' of the moving reference frame coinciding with P, the velocity of the point P is found from

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathrm{F}}$$

$$\vec{v}_{P'} = \vec{\Omega} \times \vec{r} = \omega_1 \vec{j} \times \left(L\vec{i} + R\vec{j} \right) = -\omega_1 L \vec{k}$$

$$\vec{v}_{P/\mathrm{F}} = \vec{\omega}_{D/\mathrm{F}} \times \vec{r}_{P/A} = \omega_2 \vec{k} \times R\vec{j} = -\omega_2 R \vec{i}$$

$$\boxed{\vec{v}_P = -\omega_2 R \vec{i} - \omega_1 L \vec{k}}$$



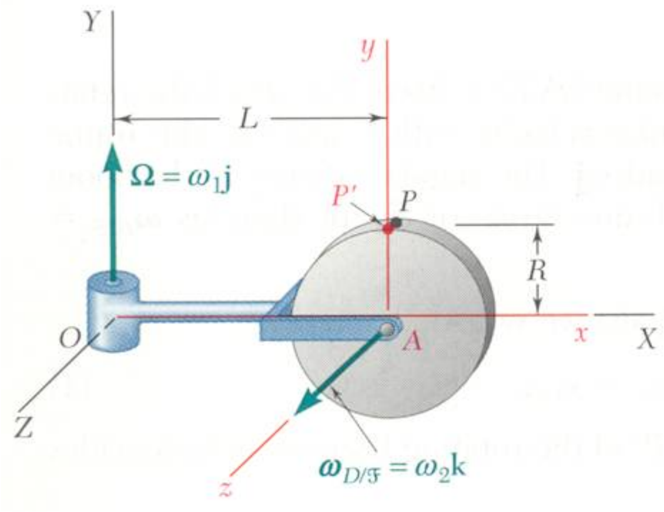
- The acceleration of P is found from

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{P/\mathrm{F}} + \vec{a}_c$$

$$\vec{a}_{P'} = \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r} \right) = \omega_1 \vec{j} \times \left(-\omega_1 L \vec{k} \right) = -\omega_1^2 L \vec{i}$$

$$\vec{a}_{P/\mathbf{F}} = \vec{\omega}_{D/\mathbf{F}} \times \left(\vec{\omega}_{D/\mathbf{F}} \times \vec{r}_{P/A} \right)$$
$$= \omega_2 \vec{k} \times \left(-\omega_2 R \vec{i} \right) = -\omega_2^2 R \vec{j}$$

$$\vec{a}_c = 2\vec{\Omega} \times \vec{v}_{P/\mathbf{F}}$$
$$= 2\omega_1 \vec{j} \times \left(-\omega_2 R \vec{i} \right) = 2\omega_1 \omega_2 R \vec{k}$$



$$\boxed{\vec{a}_P = -\omega_1^2 L\,\vec{i} - \omega_2^2 R\vec{j} + 2\omega_1\omega_2 R\vec{k}}$$

- **Angular velocity and acceleration of the disk,**

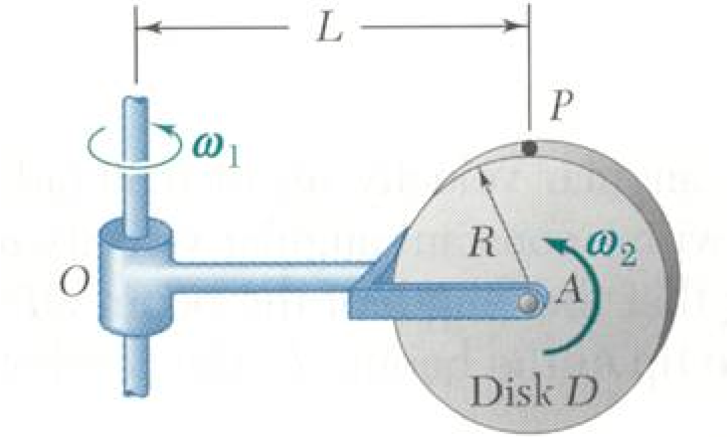
$$\vec{\omega} = \vec{\Omega} + \vec{\omega}_{D/\mathrm{F}}$$

$$\boxed{\vec{\omega} = \omega_1 \vec{j} + \omega_2 \vec{k}}$$

$$\vec{\alpha} = \left(\dot{\vec{\omega}}\right)_{\mathbf{F}} + \vec{\Omega} \times \vec{\omega}$$

$$= \omega_1 \vec{j} \times \left(\omega_1 \vec{j} + \omega_2 \vec{k}\right)$$

$$\boxed{\vec{\alpha} = \omega_1 \omega_2 \vec{i}}$$



REFLECT and THINK:

- Knowing the absolute angular velocity of the disk is equal to $\omega_1 \vec{j} + \omega_2 \vec{k}$, you could have determined the velocity of P by attaching the rotating axes to the disk and using $v_P = v_A + \Omega_D \times r_{P/A} + v_{P/A} = -\omega_1 L\vec{k} - \omega_2 R\vec{i}$,