Chapter 19

Fermi-Dirac Gases

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- Fermions

Fermi-Dirac statistics governs the behavior of indistinguishable particles of half-integer spin called fermions. Fermions obey the Pauli exclusion principle.

- Fermi-Dirac distribution

$$f_j = \frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon-\mu)/kT}+1}$$

$$\boldsymbol{f(\varepsilon) = \frac{N(\varepsilon)}{g(\varepsilon)} = \frac{1}{e^{(\varepsilon-\mu)/kT}+1}} \qquad \textbf{\textit{Fermi function}}$$

$$0 \le f(\varepsilon) \le 1$$

Chemical potential

$$if \ \ \varepsilon = \underline{\mu}, \qquad f(\varepsilon) = \frac{1}{2}$$

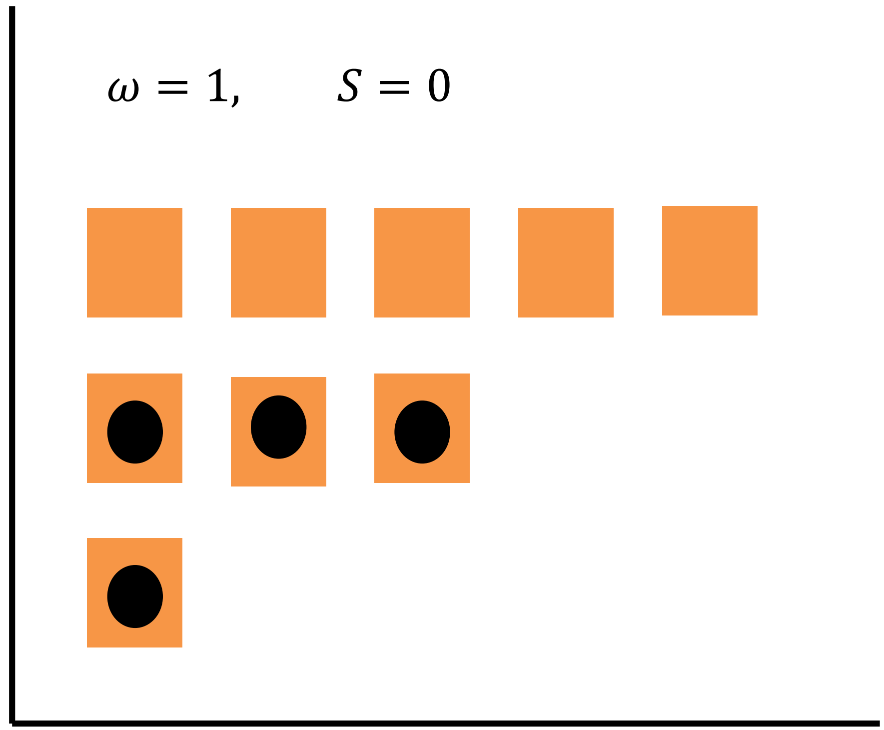
$$if \ \ T = 0, \qquad \mu(0): \ Fermi\ energy$$

occupied

$$\{\varepsilon - \mu(0)\}/kT = \begin{cases} -\infty, & \underline{f(\varepsilon) = 1} \quad if \ \ \varepsilon < \mu(0) \\ \infty, & \underline{f(\varepsilon) = 0} \quad if \ \ \varepsilon > \mu(0) \end{cases}$$

unoccupied

19.1 The Fermi Energy



At absolute zero fermions will occupy the lowest energy states available.

- For particles of spin 1/2, such as electrons, the spin factor is 2, and

$$g(\varepsilon)d\varepsilon = \underbrace{\gamma_s}_{=2}\frac{4\sqrt{2}\pi V}{h^3}m^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}d\varepsilon \qquad (from\ Eq.12.26)$$

- For conservation of particles,

$$\int_0^\infty N(\varepsilon)d\varepsilon = \int_0^\infty f(\varepsilon)g(\varepsilon)d\varepsilon = N$$

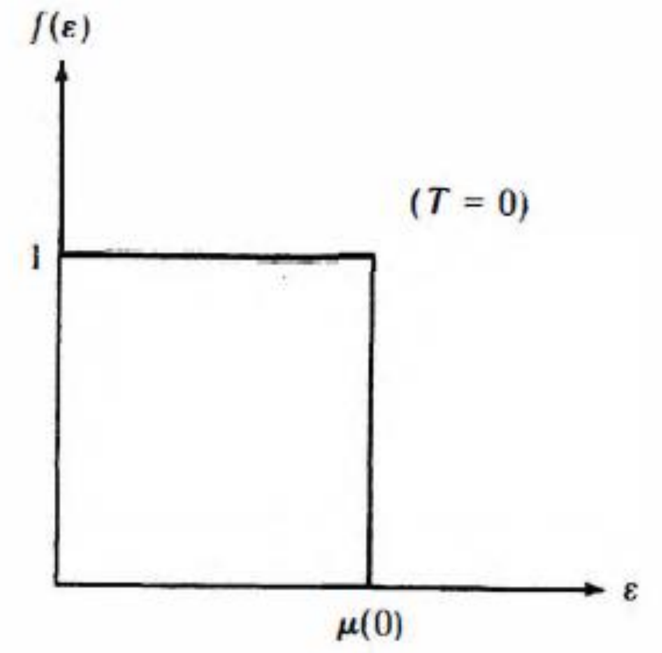


Fig. The Fermi function at T=0

- At T=0,

$$N = \int_0^{\mu(0)} g(\varepsilon) d\varepsilon$$

$$= 4\pi V \left(\frac{2m}{h^2}\right)^{\frac{2}{3}} \int_0^{\mu(0)} \varepsilon^{\frac{1}{2}} d\varepsilon$$

$$= \frac{8\pi V}{3} \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \{\mu(0)\}^{\frac{3}{2}}$$

Here, $\mu(0) = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$

- Fermi Temperature T_F such that $\mu(0) = \varepsilon_F = kT_F$

$$\boldsymbol{T_F = \frac{h^2}{2\pi mk}\left(\frac{N}{1.504V}\right)^{\frac{2}{3}}}$$

$$cf. Bose\ Temperature \quad T_B = \frac{h^2}{2\pi mk}\left(\frac{N}{2.612V}\right)^{\frac{2}{3}}$$

- The Calculation of $\mu(T)$

$$N = \int_0^\infty f(\varepsilon) g(\varepsilon) d\varepsilon = 4\pi V \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \int_0^\infty \underline{\frac{\varepsilon^{\frac{1}{2}} d\varepsilon}{e^{(\varepsilon-\mu)/kT} + 1}} \quad = \frac{2}{3}\{\mu(0)\}^{\frac{3}{2}}$$

$$At\ T = 0\,, \qquad N = 4\pi V \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \{\mu(0)\}^{\frac{3}{2}}$$

$$Let\ I = \frac{2}{3}\{\mu(0)\}^{\frac{3}{2}} = \int_0^\infty \frac{\varepsilon^{\frac{1}{2}} d\varepsilon}{e^{(\varepsilon-\mu)/kT} + 1}$$

19.2 The Calculation of $\mu(T)$

$$Then, I = \int_0^\infty \underbrace{\varepsilon^{\frac{1}{2}}}_{\frac{dF}{d\varepsilon}} \underbrace{\frac{1}{e^{(\varepsilon-\mu)/kT}+1}}_{f(\varepsilon)} d\varepsilon$$

$$= \underbrace{\left[\underbrace{\frac{2}{3}\varepsilon^{\frac{3}{2}}}_{F(\varepsilon)} \underbrace{\frac{1}{e^{(\varepsilon-\mu)/kT}+1}}_{f(\varepsilon)}\right]_0^\infty}_{=0} - \int_0^\infty \underbrace{\frac{2}{3}\varepsilon^{\frac{3}{2}}}_{F(\varepsilon)} \frac{-\frac{1}{kT}e^{(\varepsilon-\mu)/kT}}{[e^{(\varepsilon-\mu)/kT}+1]^2} d\varepsilon$$

$$F(\varepsilon) \approx F(\mu) + F'(\mu)(\varepsilon-\mu) + \frac{1}{2!}F''(\mu)(\varepsilon-\mu)^2 + \cdots$$

$$\approx \frac{2}{3}\mu^{\frac{3}{2}} + \mu^{\frac{1}{2}}(\varepsilon-\mu) + \frac{1}{4}\mu^{-\frac{1}{2}}(\varepsilon-\mu)^2 + \cdots$$

$$\therefore F(\varepsilon) \approx \frac{2}{3}\mu^{\frac{3}{2}} + \mu^{\frac{1}{2}}(\varepsilon-\mu) + \frac{1}{4}\mu^{-\frac{1}{2}}(\varepsilon-\mu)^2$$

- Set $y = (\varepsilon - \mu)/kT$

$$y = (\varepsilon - \mu)/kT$$

$$dy = \frac{d\varepsilon}{kT}$$

$$I = \int_{-\frac{\mu}{kT}}^{\infty} \left\{ \frac{2}{3}\mu^{\frac{3}{2}} + \mu^{\frac{1}{2}}(kT)y + \frac{(kT)^2}{4\mu^{\frac{1}{2}}} y^2 \right\} \frac{e^y}{(e^y+1)^2} dy$$

$\approx -\infty$

$$= \frac{2}{3}\mu^{\frac{3}{2}} + 0 + \frac{\pi^2}{12}\frac{(kT)^2}{\mu^{\frac{1}{2}}}$$

$$= \frac{2}{3}\{\mu(0)\}^{\frac{3}{2}}$$

$$\frac{2}{3}\mu^{\frac{3}{2}}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]=\frac{2}{3}\{\mu(0)\}^{\frac{3}{2}}$$

$$\frac{2}{3}t^3+\frac{\pi}{12}\frac{(kT)^2}{t}=\frac{2}{3}\{\mu(0)\}^{\frac{3}{2}}$$

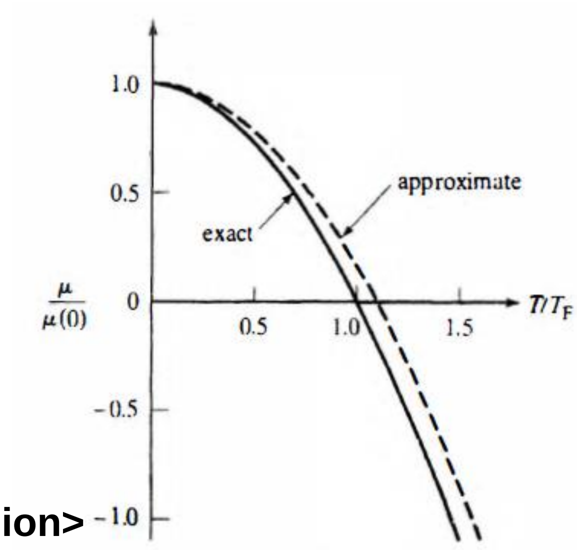
$$\frac{2}{3}t^4-\frac{2}{3}\{\mu(0)\}^{\frac{3}{2}}t+\frac{\pi(kT)^2}{12}=0$$

$$t^3+\frac{\pi}{8}\frac{(kT)^2}{t}=\{\mu(0)\}^{\frac{3}{2}}$$

$$\mu=\mu(0)\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]^{-\frac{2}{3}}$$

$$\cong\mu(0)\left[1-\frac{\pi^2}{12}\underbrace{\left(\frac{kT}{\mu}\right)^2}_{=\left(\frac{T}{T_F}\right)^2}\right]$$

$$\mu=\mu(0)\left[1-\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]\qquad T\ll T_F$$



<Fermion>

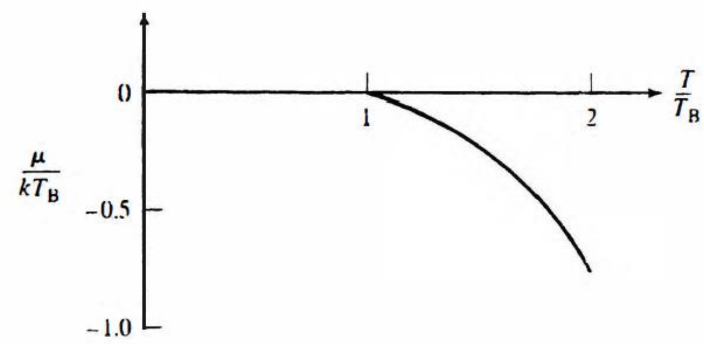


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Fig. Exact and approximate calculations of μ / $\mu(0)$ versus T/T_F.

- When $T < T_F, \;\; \mu > 0$
- When $T > T_F, \;\; \mu < 0$

More and more of the fermions are in the excited states and the mean occupancy of the ground state falls below $1/2$. In this region,

$$\boldsymbol{f(0)} = \frac{\mathbf{1}}{\boldsymbol{e^{-\mu/kT}} + \mathbf{1}} < \frac{\mathbf{1}}{\mathbf{2}}$$

$$\frac{\boldsymbol{\mu}}{\boldsymbol{kT}} < 0$$

- When $T < T_F, \ \mu > 0$
- When $T > T_F, \ \mu < 0$

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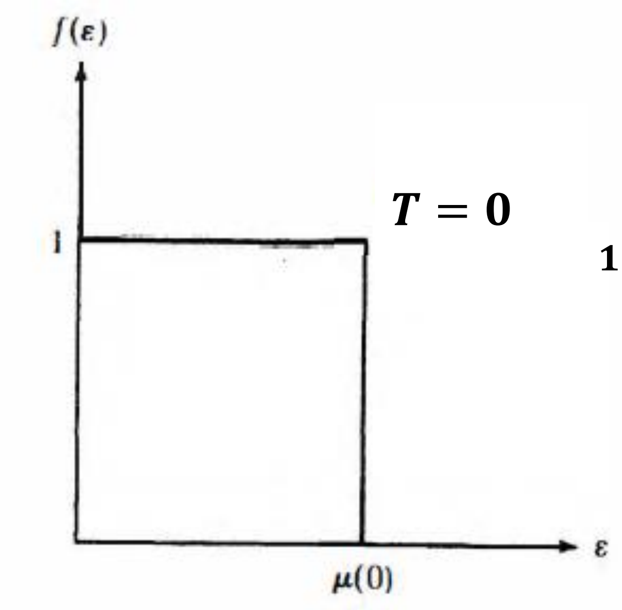
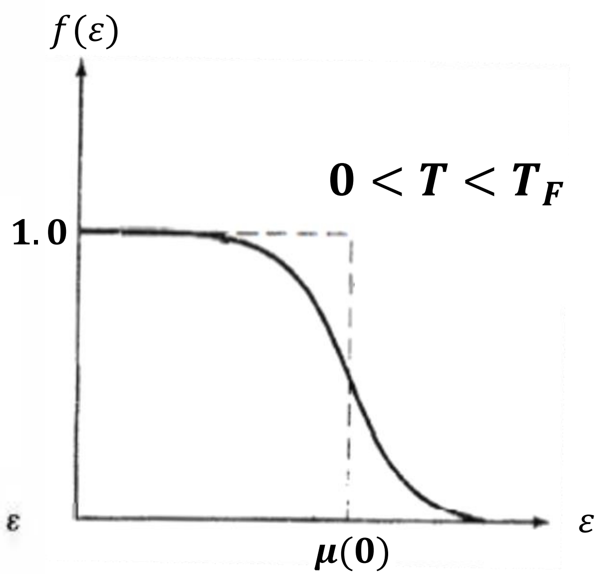
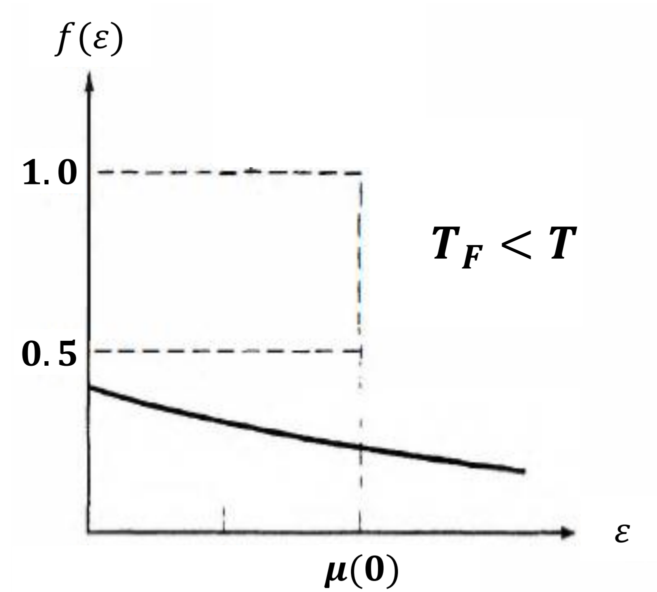

$f(\varepsilon)$
1
$T = 0$
$\mu(0)$
ε


$f(\varepsilon)$
1.0
$0 < T < T_F$
$\mu(0)$
ε


$f(\varepsilon)$
1.0
$T_F < T$
0.5
$\mu(0)$
ε

19.4 Properties of a Fermion Gas

$$U = \int_0^\infty \varepsilon N(\varepsilon) d\varepsilon$$

$$= \int_0^\infty \varepsilon f(\varepsilon) g(\varepsilon) d\varepsilon$$

$$= 4\pi V \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \int_0^\infty \frac{\varepsilon^{\frac{3}{2}} d\varepsilon}{e^{(\varepsilon-\mu)/kT} + 1}$$

$$\approx \frac{3}{5} N \underbrace{\varepsilon_F}_{= \mu(0)} \left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 - \frac{\pi^4}{16}\left(\frac{T}{T_F}\right)^4 + \cdots\right] \qquad , T_F = \frac{\varepsilon_F}{k}$$

$$C_e = \frac{dU}{dT}$$