Mechanics and Design

Chapter 1. Vectors and Tensors

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Risk Management

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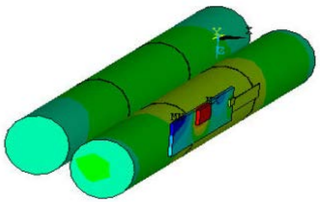

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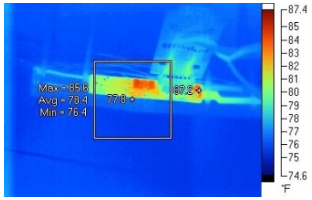

Evaluation

Engineered system Predicted measure

Vehicle Safety

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Cornering force

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Reliability

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Thermal behavior

Vectors, Vector Additions, etc.

Convention

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Base vectors

$$\mathbf{i},\quad \mathbf{j},\quad \mathbf{k} \quad \text{or} \quad \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$$

Indicial notation

$$a_i = \mathbf{a}\cdot\mathbf{e_i}$$

$$\mathbf{a} = a_i\mathbf{e_i} = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$$

Summation convention (Repeated index)

$$a_i a_i = a_1 a_1 + a_2 a_2 + a_3 a_3$$

$$a_{kk} = a_{11} + a_{22} + a_{33}$$

$$\begin{aligned} a_{ij}a_{ij} &= a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} \\ &\quad + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} \\ &\quad + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33} \end{aligned}$$

Scalar Product and Vector Product

Scalar product

The scalar product is defined as the product of the two magnitudes times the cosine of the angle between the vectors

$$\mathbf{a}\cdot\mathbf{b} = |a||b|\cos\theta \qquad (1\text{-}1)$$

From the definition of eq. (1-1), we immediately have

$$(m\mathbf{a})\cdot(n\mathbf{b}) = mn(\mathbf{a}\cdot\mathbf{b})$$

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$$

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$$

The scalar product of two different unit base vectors defined above is zero, since $\cos(90°) = 0$, that is

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$$

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

Then the scalar product becomes

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (a_x\mathbf{i}+a_y\mathbf{j}+a\mathbf{k})\cdot(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k})\\ &= a_xb_x + a_yb_y + a_zb_z\\ &= a_ib_i\end{aligned}$$

Scalar Product and Vector Product

Vector product (or cross product)

c = **a** × **b** is defined as a vector **c** perpendicular to both **a** and **b** in the sense that makes **a, b, c** a right-handed system. Magnitude of vector **c** is given by

$$|c| = |a||b|\sin(\theta)$$

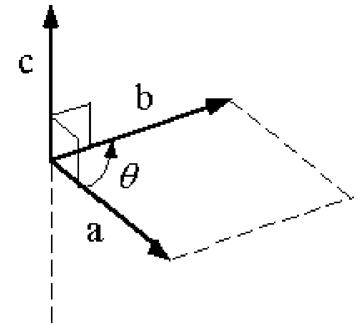


Fig. 1.1 Definition of vector product

In terms of components, vector product can be written as

$$\mathbf{a}\times\mathbf{b} = (a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k})\times(b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

The vector product is distributive as

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$$

But it is not associative as

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$$

Scalar Product and Vector Product

Scalar triple product

The scalar triple product is defined as the dot product of one of the vectors with the cross product of the other two.

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

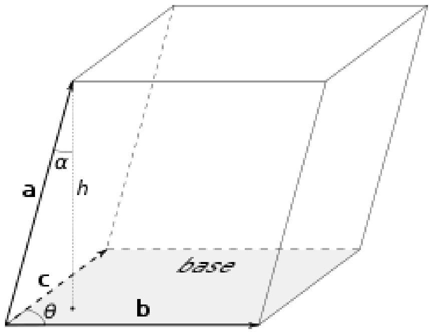


Fig. 1.2 Definition of scalar triple product

Permutation symbol ε_{mnr}

$$\varepsilon_{mnr}=\begin{cases} 0 & \text{when any two indices are equal} \\ +1 & \text{when } (m,n,r) \text{ is even permutation of } (1,2,3) \\ -1 & \text{when } (m,n,r) \text{ is odd permutation of } (1,2,3) \end{cases}$$

Where, even permutations are (1,2,3), (2,3,1), and (3,1,2)
odd permutations are (1,3,2), (2,1,3), and (3,2,1).

Using the permutation symbol, the vector product can be represented by

$$\mathbf{a}\times\mathbf{b}=\varepsilon_{pqr}a_q b_r \mathbf{i}_p$$

Scalar Product and Vector Product

Kronceker delta δ_{ij}

The Kronecker delta is a function of two variables, usually just positive integers. The function is 1 if the variables are equal, and 0 otherwise.

$$\delta_{pq} = \begin{cases} 1 \text{ if } p = q \\ 0 \text{ if } p \neq q \end{cases}$$

Some examples are given below:

$$\delta_{ii} = 3$$

$$\delta_{ij}\delta_{ij} = \delta_{ii} = 3$$

$$u_i\delta_{ij} = u_i = u_j$$

$$T_{ij}\delta_{ij} = T_{ii}$$

Rotation of Axes, etc.

Change of orthonormal basis

- Cartesian components of a vector are not changed by translation of axes.
- However, the components of a vector change when the coordinate axes rotate.

Let x_i and $\bar{x}_i$ be the two coordinate systems, as show in Fig. 1.3, which have the same origins.
Also, let the orientation of the two coordinate systems is given by the direction cosines as

	$\bar{\mathbf{i}}_1$	$\bar{\mathbf{i}}_2$	$\bar{\mathbf{i}}_3$
$\mathbf{i}_1$	α_1^1	α_2^1	α_3^1
$\mathbf{i}_2$	α_1^2	α_2^2	α_3^2
$\mathbf{i}_3$	α_1^3	α_2^3	α_3^3

a_i^j $j \longrightarrow$ original, $i \longrightarrow$ transformed

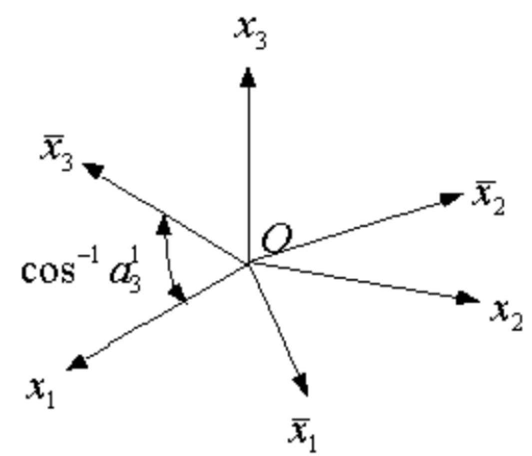


Fig. 1.3 Coordinate transformation

Rotation of Axes, etc.

2D coordinate transformation

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix} \quad \text{or} \quad \mathbf{u} = A\bar{\mathbf{u}}$$

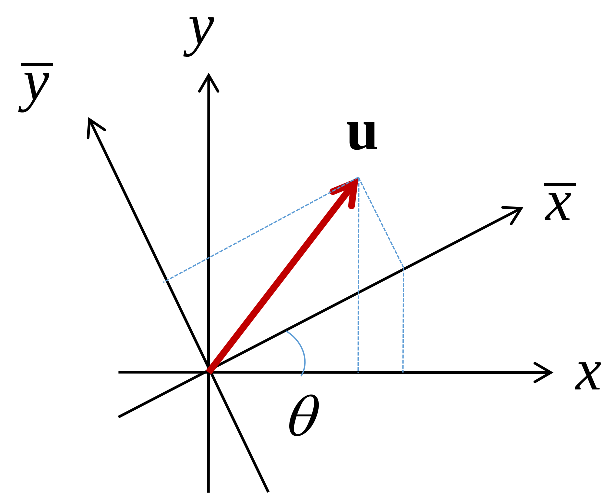


Fig. 1.4 2D coordinate transformation

Inverse of 2D coordinate transformation

$$\begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad \text{or} \quad \bar{\mathbf{u}} = A^{-1}\mathbf{u} = A^T\mathbf{u}$$

Coordinate transformation of vector

Let,

$$\mathbf{A} = \begin{bmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{bmatrix}$$

Then,

$$\mathbf{v} = \mathbf{A}\bar{\mathbf{v}} \quad \text{or} \quad \bar{\mathbf{v}} = \mathbf{A}^T\mathbf{v}$$

$$\mathbf{A} = \begin{bmatrix} \bar{\mathbf{a}}_1 & \bar{\mathbf{a}}_2 & \bar{\mathbf{a}}_3 \end{bmatrix}$$

Note that $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$ $\longrightarrow$ So **A** is “Orthogonal matrix.” (with orthonormal basis)

Rotation of Axes, etc.

Second order tensors as linear vector functions (or transformations)

The second order tensor may be expressed by tensor product or open product of two vectors.

Let, $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$

$$\begin{aligned}
\mathrm{T} = \mathbf{ab} = \mathbf{a}\otimes\mathbf{b} &= (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\otimes(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\
&= a_1b_1\ \mathbf{e}_1\otimes\mathbf{e}_1 + a_1b_2\ \mathbf{e}_1\otimes\mathbf{e}_2 + a_1b_3\ \mathbf{e}_1\otimes\mathbf{e}_3 \\
&\quad + a_2b_1\ \mathbf{e}_2\otimes\mathbf{e}_1 + a_2b_2\ \mathbf{e}_2\otimes\mathbf{e}_2 + a_2b_3\ \mathbf{e}_2\otimes\mathbf{e}_3 \\
&\quad + a_3b_1\ \mathbf{e}_3\otimes\mathbf{e}_1 + a_3b_2\ \mathbf{e}_3\otimes\mathbf{e}_2 + a_3b_3\ \mathbf{e}_3\otimes\mathbf{e}_3
\end{aligned}$$

or

$$\begin{aligned}
\mathrm{T} = {} & T_{11}\ \mathbf{e}_1\otimes\mathbf{e}_1 + T_{12}\ \mathbf{e}_1\otimes\mathbf{e}_2 + T_{13}\mathbf{e}_1\otimes\mathbf{e}_3 \\
& + T_{21}\ \mathbf{e}_2\otimes\mathbf{e}_1 + T_{22}\ \mathbf{e}_2\otimes\mathbf{e}_2 + T_{23}\ \mathbf{e}_2\otimes\mathbf{e}_3 \\
& + T_{31}\ \mathbf{e}_3\otimes\mathbf{e}_1 + T_{32}\ \mathbf{e}_3\otimes\mathbf{e}_2 + T_{33}\ \mathbf{e}_3\otimes\mathbf{e}_3
\end{aligned}$$

Here, we may consider $\mathbf{e}_i\otimes\mathbf{e}_j \neq \mathbf{e}_j\otimes\mathbf{e}_i$ as a base of the second order tensor, and T_{ij} is a component of the second order tensor T.

Rotation of Axes, etc.

It can be seen that the second order tensor map a vector to another vector, that is,

$$\begin{aligned}\mathbf{u} = T \cdot \mathbf{v} = \ &(T_{11}\ \mathbf{e}_1 \otimes \mathbf{e}_1 + T_{12}\ \mathbf{e}_1 \otimes \mathbf{e}_2 + T_{13}\mathbf{e}_1 \otimes \mathbf{e}_3 \\ &+ T_{21}\ \mathbf{e}_2 \otimes \mathbf{e}_1 + T_{22}\ \mathbf{e}_2 \otimes \mathbf{e}_2 + T_{23}\ \mathbf{e}_2 \otimes \mathbf{e}_3 \\ &+ T_{31}\ \mathbf{e}_3 \otimes \mathbf{e}_1 + T_{32}\ \mathbf{e}_3 \otimes \mathbf{e}_2 + T_{33}\ \mathbf{e}_3 \otimes \mathbf{e}_3) \cdot (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\ = \ &(T_{11}v_1 + T_{12}v_2 + T_{13}v_3)\mathbf{e}_1 + (T_{21}v_1 + T_{22}v_2 + T_{23}v_3)\mathbf{e}_2 \\ &+ (T_{31}v_1 + T_{32}v_2 + T_{33}v_3)\mathbf{e}_3 \\ = \ &T_{ij}v_j\mathbf{e}_i\end{aligned}$$

Symmetric tensor and skew-symmetric tensor

Symmetric tensor $\longrightarrow$ $T_{ij} = T_{ji}$

Skew-symmetric
or Antisymmetric tensor $\longrightarrow$ $T_{ij} = -T_{ji}$

Rotation of Axes, etc.

Rotation of axes, change of tensor components

Let $u_i = T_{ip} v_p$ and $\bar{u}_i = \bar{T}_{ip} \bar{v}_p$
where u_i and $\bar{u}_i$ are the same vector but decomposed into two different coordinate systems, x_i and $\bar{x}_i$. The same applies to v_i and $\bar{v}_i$.
Then by the transformation of vector, we get

$$\bar{u}_i = a_i^j u_j = a_i^j T_{jq} v_q$$
$$= a_i^j T_{jq} a_p^q \bar{v}_p$$

Therefore, transformation matrix can be expressed as,

$$\bar{T}_{ip} = a_i^j a_p^q T_{jq}$$

In matrix form, we have,

$$\bar{T} = A^T T A \quad \text{or} \quad T = A \bar{T} A^T$$

Rotation of Axes, etc.

Scalar product of two tensors

$$\mathrm{T} : \mathrm{U} = T_{ij} U_{ij}$$

$$\mathrm{T} \cdot\cdot \mathrm{U} = T_{ij} U_{ji}$$

If we list all the terms of tensor product, we get

$$\begin{aligned} \mathrm{T} : \mathrm{U} = T_{ij} U_{ij} \\ &= T_{11} U_{11} + T_{12} U_{12} + T_{13} U_{13} \\ &\quad + T_{21} U_{21} + T_{22} U_{22} + T_{23} U_{23} \\ &\quad + T_{31} U_{31} + T_{32} U_{32} + T_{33} U_{33} \end{aligned}$$

The product of two second-order tensors

$$\mathrm{T} \cdot \mathrm{U}$$

$$(\mathrm{T} \cdot \mathrm{U}) \cdot \mathrm{v} = \mathrm{T} \cdot (\mathrm{U} \cdot \mathrm{v})$$

If $\mathrm{P} = \mathrm{T} \cdot \mathrm{U}$, then $P_{ij} = T_{ik} U_{kj}$

Rotation of Axes, etc.

The trace

Definition) $tr(\mathrm{T}) = T_{kk}$

Note that

(1) $\mathrm{A} \cdot\cdot \mathrm{B} = tr(\mathrm{A} \cdot \mathrm{B})$

(2) $\mathrm{A} : \mathrm{B} = tr(\mathrm{A} \cdot \mathrm{B}^T) = tr(\mathrm{A}^\mathrm{T} \cdot \mathrm{B})$

(3) $tr(\mathrm{A} \cdot \mathrm{B}) = tr(\mathrm{B} \cdot \mathrm{A})$

(4) $tr(\mathrm{A} \cdot \mathrm{B} \cdot \mathrm{C}) = tr(\mathrm{B} \cdot \mathrm{C} \cdot \mathrm{A}) = tr(\mathrm{C} \cdot \mathrm{A} \cdot \mathrm{B})$ Cyclic property of trace

Proof)

(1) $\mathrm{A} \cdot\cdot \mathrm{B} = tr(\mathrm{A} \cdot \mathrm{B})$

Let, $\mathrm{P} = \mathrm{A} \cdot \mathrm{B}$, i.e., $P_{ij} = A_{ik} B_{kj}$

Then, $tr(\mathrm{P}) = P_{ii} = A_{ik} B_{ki} = \mathrm{A} \cdot\cdot \mathrm{B}$

(2) $\mathrm{A} : \mathrm{B} = tr(\mathrm{A} \cdot \mathrm{B}^T) = tr(\mathrm{A}^\mathrm{T} \cdot \mathrm{B})$

Let, $\mathrm{P} = \mathrm{A} \cdot \mathrm{B}^\mathrm{T}$, i.e., $P_{ij} = A_{ik} B_{jk}$

Then, $tr(\mathrm{A} \cdot \mathrm{B}^T) = P_{ii} = A_{ik} B_{ik} = \mathrm{A} : \mathrm{B}$

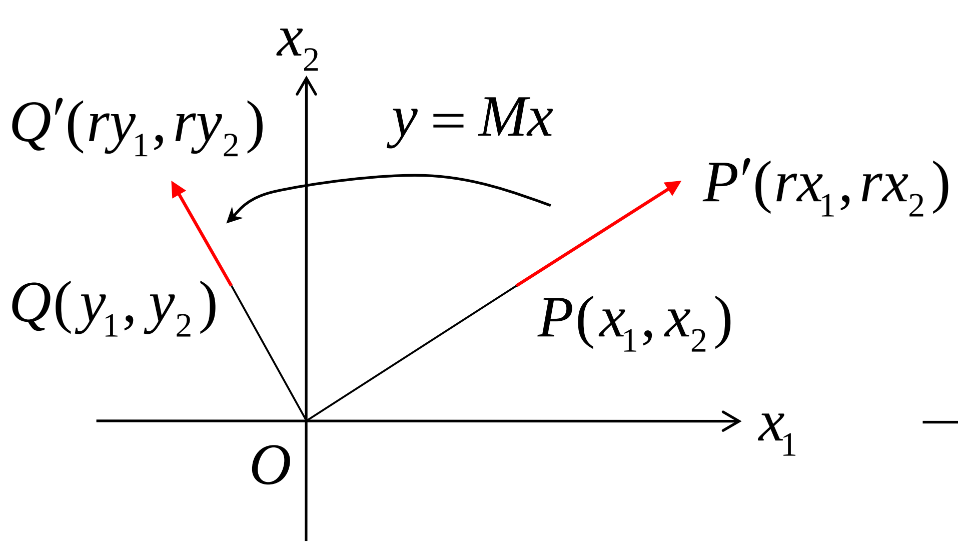
Review of Elementary Matrix Concepts

Eigenvectors and eigenvalues of a matrix

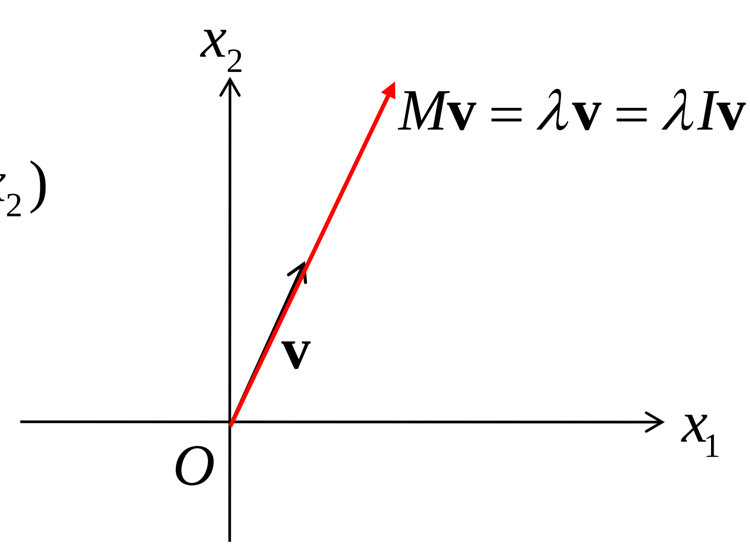
Linear transformation, $y = Mx$, associates, to each point $P(x_1, x_2, x_3)$, another point $Q(y_1, y_2, y_3)$. Also, it associates to any other point (rx_1, rx_2, rx_3) on the line OP, another point (ry_1, ry_2, ry_3) on the line OQ.
So we may consider the transformation to be a transformation of the line OP into the line OQ. (left figure)

Now any line transformed into itself is called an eigenvector of the matrix **M.**
That is,

$$\mathbf{Mx} = \lambda \mathbf{x} = \lambda \mathbf{I}\ \mathbf{x}$$



Linear Transformation



M maps $\mathbf{v}$ into the same vector $\mathbf{v}$.

Review of Elementary Matrix Concepts

A nontrivial solution will exist if and only if the determinant vanishes:

$$\left|(\mathbf{M} - \lambda \mathbf{I})\right| = 0$$

Note that 3x3 determinant is expanded, it will be a cubic polynomial equation with real coefficients. The roots of this equation are called the eigenvalues of the matrix. Upon solving the equation, some of the roots could be complex numbers.

Proof)

For some λ, if $\det(M - \lambda I) = 0$, then $\exists v \in R^n \backslash \{\mathbf{0}\}$ s.t. $(M - \lambda I)v = 0$

It is equivalent to $Mv = \lambda v$ which is definition of eigenvalue.

Therefore, λ is an eigenvalue of M if λ is root of $\det(M - \lambda I) = 0$

Review of Elementary Matrix Concepts

A real symmetric matrix has only real eigenvalues

If there were a complex root λ , then its complex conjugate $\bar{\lambda}$ is also a root. Therefore,

$$\mathbf{M}\mathbf{x} = \lambda \mathbf{x}$$

$$\bar{\mathbf{M}}\mathbf{x} = \bar{\lambda} \mathbf{x}$$

These equations can be written as

$$\bar{\mathbf{x}}^T \mathbf{M}\mathbf{x} = \lambda \bar{\mathbf{x}}^T \mathbf{x}$$

$$\mathbf{x}^T \mathbf{M}\bar{\mathbf{x}} = \bar{\lambda} \mathbf{x}^T \bar{\mathbf{x}}$$

Note that $\bar{\mathbf{x}}^T \mathbf{x} = x_k \bar{x}_k = \mathbf{x}^T \bar{\mathbf{x}}$ and since $\mathbf{M}$ is symmetric, we get

$$\begin{aligned} \bar{\mathbf{x}}^T \mathbf{M}\mathbf{x} &= M_{ij} \bar{x}_i x_j \\ &= M_{ji} \bar{x}_j x_i && \text{(by interchanging the dummy indices)} \\ &= M_{ij} x_i \bar{x}_j = \mathbf{x}^T \mathbf{M} \bar{\mathbf{x}} && \text{(by symmetry of M)} \end{aligned}$$

Review of Elementary Matrix Concepts

Subtracting them, we get

$$(\lambda - \bar{\lambda})\bar{\mathbf{x}}^T \mathbf{x} = 0$$

Since x is nontrivial, $\bar{\mathbf{x}}^T \mathbf{x} \neq 0$. Therefore, we should have

$$\lambda = \bar{\lambda}$$

So that λ must be real.

We can obtain the eigenvector associated to each eigenvalue by substituting each eigenvalue into the matrix equation.

When eigenvalues are all distinct : ??
Two of the eigenvalues are equal : ??
All of the eigenvalues are equal : ??

Reference (Example)

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[2] S. Yin and H. Luo, "A review on basic data-driven approaches for industrial process monitoring," IEEE Transactions on Industrial Electronics, vol. 61, no. 11, pp. 6418-6428, 2014.

[3] A. Muszynska, Rotordynamics, CRC Press, 2005.

THANK YOU
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