Ch. 2

Introduction to Mechanics of Deformable Bodies

2.1 Analysis of deformable bodies (Principles of the mechanics)

▶ Steps for the principles of mechanics for deformable body

i) Study of forces and equilibrium requirements

ii) Study of deformation and conditions of geometric fit

iii) Application of force-deformation relations

▶ Example 2.2 Suppose that a man steps up on the middle of the plank and begins to walk slowly toward one end. We should like to know how far he can walk before one end of the plank touches the ground; that is, estimate the distance b in Fig. 2.2b, when the right end E of the plank is just in contact with the ground (with two similar springs of spring constant k).

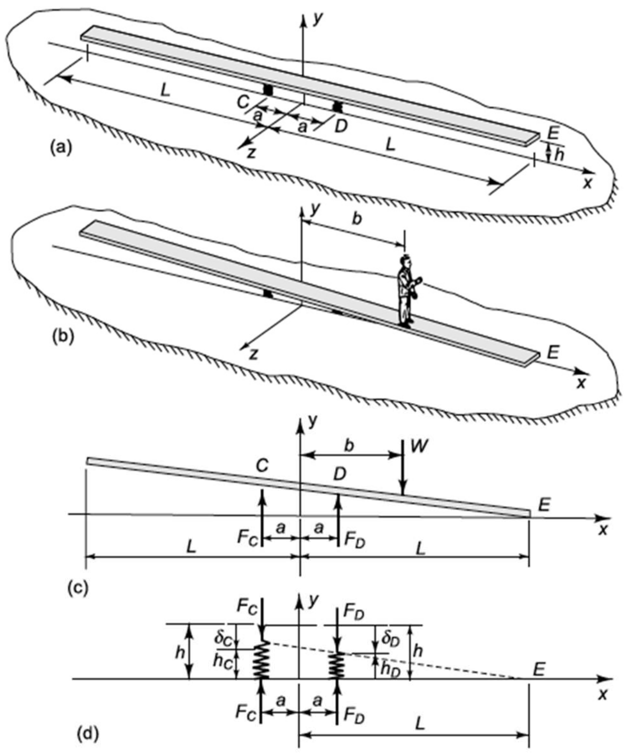


Fig. 2.2 Example 2.2

▷ Assumption

i) The wood plank is rigid body

ii) Neglect the plank's own weight

▷ Equilibrium

$\sum F_y = F_C + F_D - W = 0$ (a)

$\sum M_C = 2aF_D - (a+b)W = 0$ (b)

▷ Geometry

Since $(L+a) : h_C = (L-a) : h_D$

$\therefore \quad \dfrac{h_C}{h_D} = \dfrac{L+a}{L-a}$ (c)

Now, $\delta_C = h - h_C$ & $\delta_D = h - h_D$ (d)

▷ Relations;

$F_C = k\delta_C$ & $F_D = k\delta_D$ (e)

→ Five unknowns (F_C, F_D, δ_C, δ_D b) with five equations (a), (b), (c), (d), and (e).

From eqs. (a)~(e)

→ $\mathrm{b} = \frac{\mathrm{a}^2}{L}(\frac{2kh}{W} - 1)$ (f)

From eqs. (a),(b),(e), eliminate $F_C,\ F_D$

→ $\delta_\mathrm{C} = \frac{W}{2k}(1 - \frac{b}{a})$ & $\delta_\mathrm{D} = \frac{W}{2k}(1 + \frac{b}{a})$ (g)

∴ In case $\mathrm{b} > \mathrm{a}$, C spring is under the tension

▶ <u>Example 2.3</u> Determine the deflections in the three springs as functions of the load position parameter λ

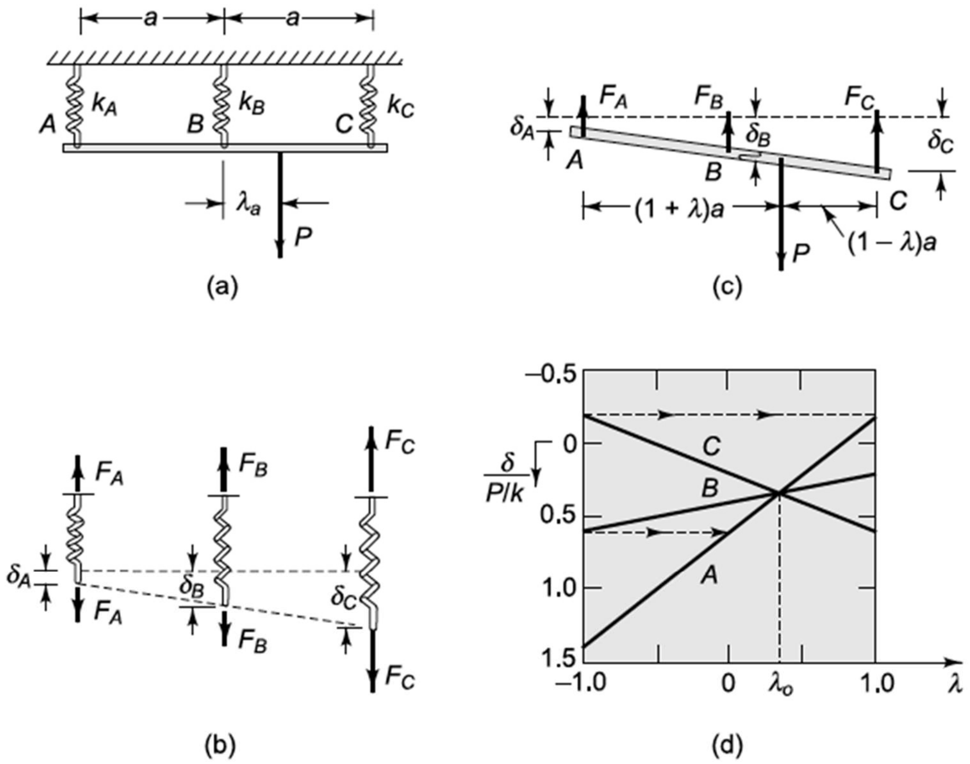


Fig. 2.3 *Example 2.3(a)*

▷ Assumptions

i) Before the load P is applied, the bar is horizontal

ii) The system is modeled by a rigid weightless bar and three linear elastic springs

▷ Equilibrium

→ We note that there are three unknown parallel forces acting on the bar in Fig. 2.3 (b) and only two independent equilibrium requirements ($\therefore$ Statically indeterminate)

$$\begin{aligned} \sum M_C = 0 \ ; \quad & 2aF_A = (1-\lambda)aP - aF_B \\ \sum M_A = 0 \ ; \quad & 2aF_C = (1+\lambda)aP - aF_B \end{aligned} \tag{a}$$

▷ Geometry

$$\delta_B = \frac{\delta_A + \delta_C}{2} \tag{b}$$

▷ F-δ Relations

$$\delta_A = \frac{F_A}{k_A}, \ \delta_B = \frac{F_B}{k_B}, \ \delta_C = \frac{F_C}{k_C} \tag{c}$$

i) Equations (a), (b), and (c) are six independent relations among the six unknowns the three forces and the three deflections.

ii) By substituting (a) into (c), obtain all the deflections in terms of F_B

iii) Inserting these deflections into (b) to obtain a single equation for F_B.

iv) Once F_B is known, F_A and F_C are given by (a)

$$\begin{aligned} &\rightarrow \ \delta_A = P\frac{2k_C - \lambda(k_B + 2k_C)}{k_A k_B + 4k_A k_C + k_B k_C} \\ &\rightarrow \ \delta_B = P\frac{k_A + k_C + \lambda(k_A - k_C)}{k_A k_B + 4k_A k_C + k_B k_C} \\ &\rightarrow \ \delta_C = P\frac{2k_A + \lambda(k_B + 2k_A)}{k_A k_B + 4k_A k_C + k_B k_C} \end{aligned} \tag{d}$$

cf. It is interesting to observe that when the load is at the position indicated by λ_0 in Fig. 2.3 (d), all three spring deflections are equal.

This means that the bar deflects without tipping when the load is applied at this position.

2.2 Uniaxial loading & deformation

▶ Uniaxial loading

→ The deformation of three rods of identical material, but having different lengths and cross-sectional areas as Fig. 2.5 (a)

→ Assume that for each bar the load is gradually increased from zero, and at several values of the load a measurement is made of the elongation δ.

→ Assume that the maximum elongation is a tiny fraction of the bar length. The results of the three tests will be represented by a plot like Fig. 2.5(b) or like Fig. 2.5(c).

→ Plotting load over area (stress) as ordinate and elongation over original length (strain) as abscissa, the test results for the three bars can be represented by a single curve, as shown in Fig. 2.6 (a) or (b).

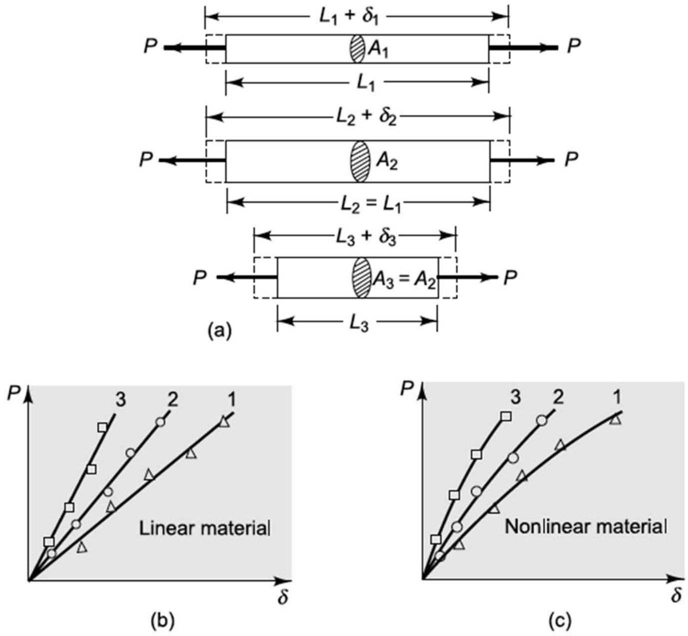


Fig. 2.5 *Uniaxial loading*

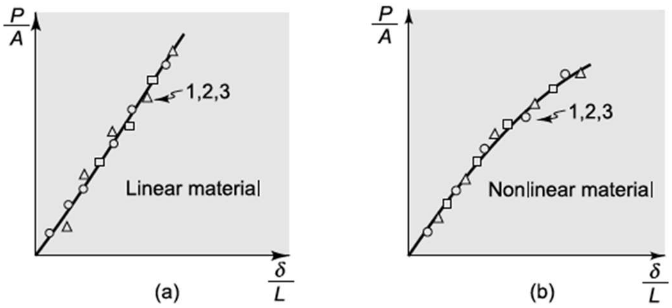


Fig. 2.6 *Uniaxial-loading data of Fig. 2.5(b) and c plotted as P/A versus δ/L*

▶ Hooke's law

▷ If the uniaxial load-elongation relation of the material is linear

→ The slope in Fig. 2.6 (a) is called the *modulus of elasticity* and is usually denoted by the symbol E.

$$E = \frac{P/\ A}{\delta/\ L} = \frac{\sigma}{\epsilon}$$

$$\therefore\ \delta = \frac{PL}{AE} \tag{2.2}$$

→ Unit is $[\mathrm{N/m^2}]$, $[\mathrm{lb/in^2}]$, [psi]

→ Unit is $[\mathrm{N/m^2}]$, $[\mathrm{lb/in^2}]$, [psi]

→ $P = \dfrac{AE}{L}\delta = k\delta$

cf. Typical values of E for a few materials are given in Table 2.1

Table 2.1

Material	E, psi	E, kN/m²
Tungsten carbide	$60–100 \times 10^6$	$410–690 \times 10^6$
Tungsten	58×10^6	400×10^6
Molybdenum	40×10^6	275×10^6
Aluminum oxide	47×10^6	325×10^6
Steel and iron	$28–30 \times 10^6$	$194–205 \times 10^6$
Brass	15×10^6	103×10^6
Aluminum	10×10^6	69×10^6
Glass	10×10^6	69×10^6
Cast iron	$10–20 \times 10^6$	$69–138 \times 10^6$
Wood	$1–2 \times 10^6$	$6.9–13.8 \times 10^6$
Nylon, epoxy, etc.	$4–8 \times 10^4$	$27.5–55 \times 10^4$
Collagen	$2–15 \times 10^3$	$13.8–103 \times 10^3$
Soft rubber	$2–8 \times 10^2$	$13.8–55 \times 10^2$
Smooth muscle	2–150	13.8–1034
Elastin	50–100	345–690

1 N/m² = pascal (Pa)

2.3 Statically determinate situation

▶ <u>Example 2.4</u> Estimate the displacement at the point D due to the 20 kN load carried by the chain hoist.

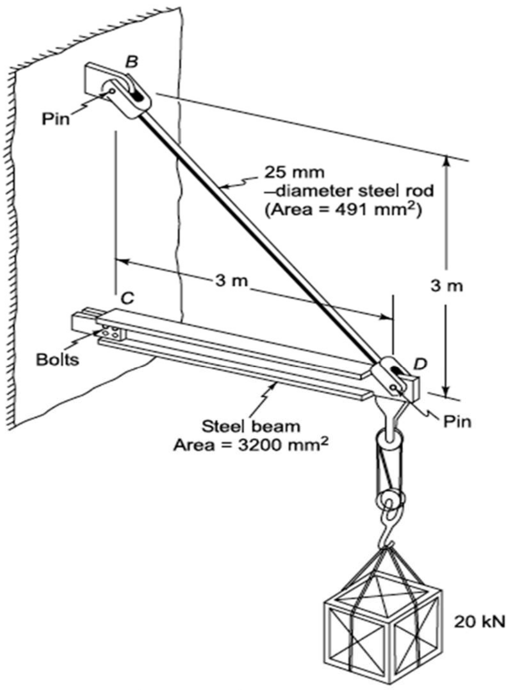


Fig. 2.7 *Example 2.4*

▷ Assumption

i) The bolted connection in C is treated as a frictionless pinned joint

<u>The equilibrium requirements of the first step should be satisfied in the deformed equilibrium configuration</u>

▷ F.B.D. (in Figs. 1.24 and 2.8)

▷ Force-deformation relation

$$\delta_{BD} = \left(\frac{FL}{AE}\right)_{BD}$$

$$= \frac{28.3(4.242\times10^{3})}{0.491\times10^{-3}(205\times10^{6})}$$

$$= 1.19 \ \text{mm} \qquad \text{(a)}$$

$$\delta_{CD} = \left(\frac{FL}{AE}\right)_{CD}$$

$$= \frac{20.0(3.000\times10^{3})}{3.200\times10^{-3}(205\times10^{6})}$$

$$= 0.0915 \ \text{mm}$$

▷ Geometry

$$\delta_H = \delta_{CD} = 0.0915 \ mm$$

$$\begin{aligned} \delta_V &= D_2F + FD_4 \\ &= DG + FG \end{aligned} \qquad \text{(b)}$$

$$= \sqrt{2}\delta_{BD} + \delta_{CD} = \mathbf{1.77} \ \boldsymbol{mm}$$

cf. If the equilibrium requirements are applied to the deformed shape of Fig. 2.8 (d), F_{BD} is decreased by 12 N and F_{CD} is decreased by 0.6 N.

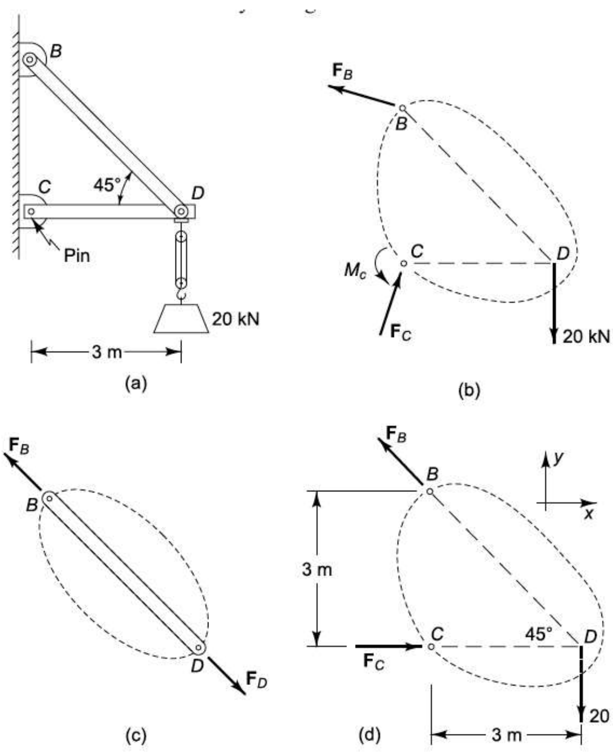


Fig. 1.24 *Idealized model of system of Fig. 1.23*

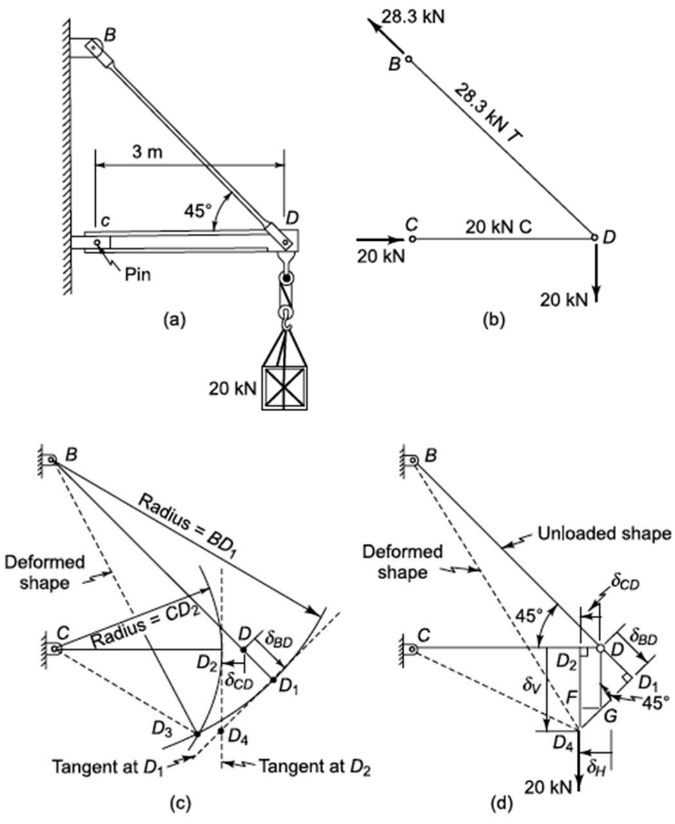


Fig. 2.8 *Example 2.4*

▶ Example 2.6 (Stiff horizontal beam)

Find out where to locate the roller so that the beam will still be horizontal in the deflected position. Also, we should like to know if the location would be the same if the load is increased from 150 kN to 300 kN. (Fundamentally the same as that treated in Example 2.2)

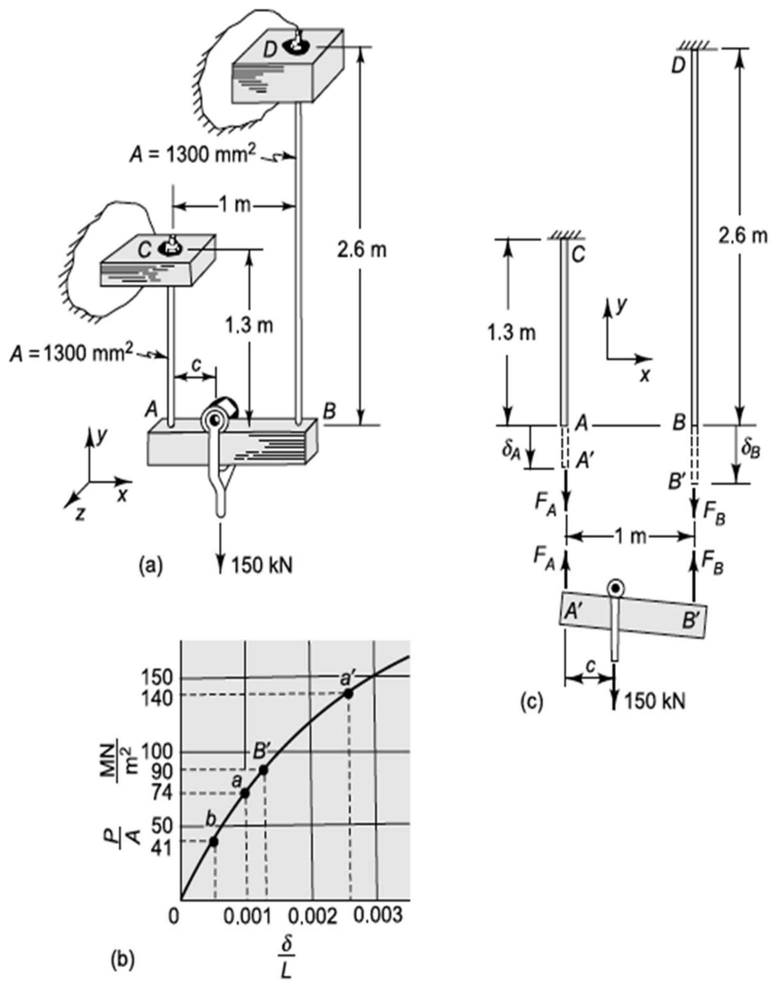


Fig. 2.10 Example 2.6(a)

▷ Assumption

i) The points A and B deflect vertically to A' and B'.

ii) The beam is considered rigid

iii) There are no horizontal forces or couples acting between the beam and the bars

▷ Equilibrium

$$\begin{aligned} \sum F_y &= F_A + F_B - 150 = 0 \\ \sum M_{A'} &= F_B - c(150) = 0 \end{aligned} \quad \text{(a)}$$

▷ Geometry

$$\delta_A = \delta_B \qquad \text{(b)}$$

$$\frac{\delta_A}{L_A} = 2\frac{\delta_B}{L_B} \qquad \text{(c)}$$

▷ Force-deformation relation

Dividing the first of Eq. (a) by A_A

$$\frac{F_A}{A_A} + \frac{F_B}{A_B} = \frac{150}{A_A} = 115 \ MN/m^2 \qquad (\because A_A = A_B) \qquad \text{(e)}$$

▷ Trial & error calculation from Fig. 2-9 (c)

i) Select an arbitrary value of δ_B/L_B.

ii) Using Eq. (c), obtain δ_A/L_A.

iii)Enter the diagram in Fig. 2.10 (b) and obtain F_A/A_A.

iv) Check to see if these values satisfy Eq. (e).

v) If (e) is not satisfied, we make a new guess for δ_A/L_A and obtain new values for F_A/A_A. That is, retrial step i), ii), and iii) until step iv) is valid.

In here, we get

$$F_A/A_A = 74 \ MN/\ m^2, \quad F_A = 96.2 \ kN$$
$$F_B/A_B = 41 \ MN/\ m^2, \quad F_B = 53.3 \ kN \tag{f}$$
$$\delta_A/A_A = 0.001 \ , \quad \delta_A = \delta_B = 1.3 \ mm$$

∴ From the second of Eq. (a), we obtain the required location of the roller

$$c = 0.355 \ \text{m} \tag{g}$$

cf. If we repeated the analysis for a load of 300 kN,

$$c = 0.393 \ \text{m} \tag{h}$$

▶ Example 2.7 Determine the forces in the ring and the deformation of the ring due to the internal pressure

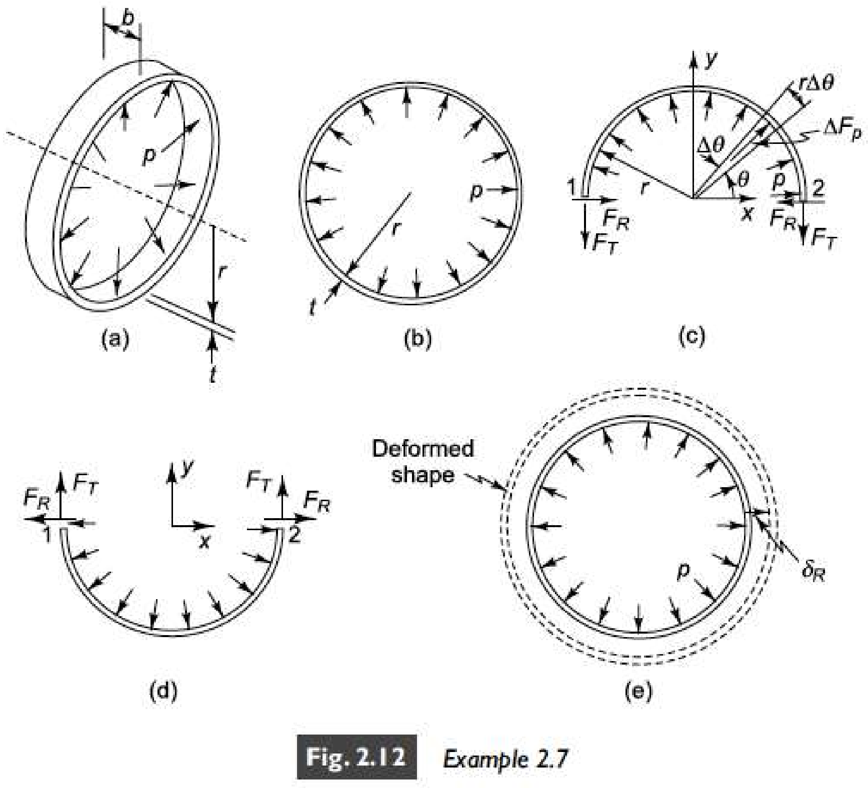


Fig. 2.12 Example 2.7

▷ F.B.D (In Fig. 2.12 (c) and (d)).

We observe that the forces F_T act in similar manner on the two halves of the hoop, but the forces F_R act inward on the upper half and outward on the lower half. This action of the forces F_R violates the symmetry which we expect to find in the two halves of the hoop.

∴ The radial forces F_R are zero, and that on any radial cut made across the hoop there is acting only a tangential force F_T.

▷ Equilibrium

Considering an arc length $r\Delta\theta$

$$\Delta F_p = p[b(r\Delta\theta)] \tag{a}$$

$$\Delta \mathrm{F_y} = \Delta F_p \sin\theta = p[b(r\Delta\theta)]\sin\theta \tag{b}$$

In the limit as $\Delta\theta \to 0$

$$\sum F_y = \int_{\theta=0}^{\theta=\pi} pbr\,sin\,\theta\,d\theta - 2F_T = 0 \tag{c}$$

Integrating (c) we find

$$F_T = prb \tag{d}$$

cf. $[(r\Delta\theta)\,sin\,\theta]$ in (b) is the projection on the x axis of the arc length $r\Delta\theta$

$$\sum F_y = p\ \ (2rb) - 2F_T = 0 \tag{e}$$

▷ Force-deformation relation

Since $\delta = \mathrm{FL/AE}$,

$$\delta_\mathrm{T} = \frac{\mathrm{F_T}[2\pi(r+t/2)]}{(bt)E} = \frac{2\pi pr^2}{tE}\left(1+\frac{t}{2r}\right) \tag{f}$$

▷ Geometry

$$\delta_R = \frac{\delta_T}{2\pi} = \frac{pr^2}{tE}\left(1+\frac{t}{2r}\right) \qquad \text{(g)(h)}$$

cf. $\delta_\mathrm{D} = \delta_T/\pi$

If $t/2r \ll 1,\ \delta_R = \frac{pr^2}{tE}$

→ This approximate solutions are good when $t/r < 0.1$

▶ <u>Example 2.8</u> Predict how much elongation there will be in the section AB of the brake band when the braking force is such that there is a tension of 40 kN in the section BC of the band

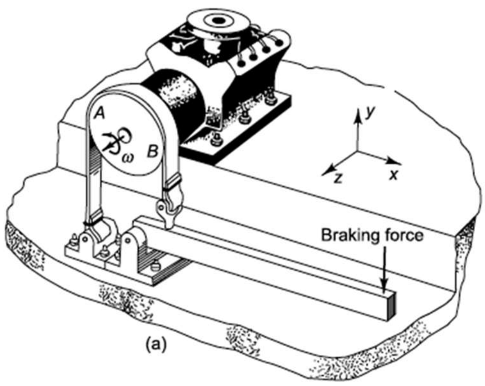


Fig. 2.13 *Example 2.8*

▷ Data

i) The brake band is 1.6 mm thick and 50 mm wide

ii) A kinetic coefficient $f = 0.4$

▷ Schematic

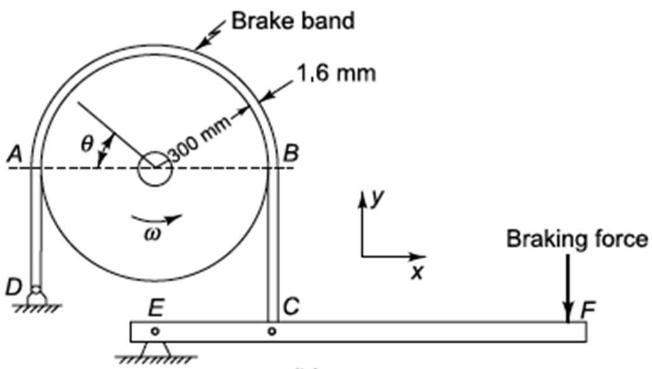


Fig. 2.13 *Example 2.8*

▷ F.B.D.

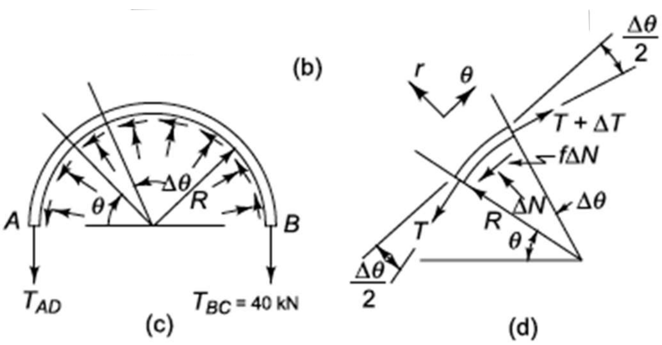


Fig. 2.13 *Example 2.8*

▷ Equilibrium

$$\sum F_r = \Delta N - T\,sin\frac{\Delta\theta}{2} - (T+\Delta T)\,sin\frac{\Delta\theta}{2} = 0$$
$$\sum F_\theta = (T+\Delta T)\,cos\frac{\Delta\theta}{2} - T\,cos\frac{\Delta\theta}{2} - f\Delta N = 0 \qquad \text{(a)}$$

The angle $\Delta\theta$ is small (in the limit), and for small angles it is frequently convenient to make the following approximations.

$$\begin{cases}\sin\theta \approx \theta \\ \cos\theta \approx 1 \\ \tan\theta \approx \theta\end{cases}$$

Table 2.2

θ		sin θ	tan θ	cos θ
Degrees	*Radians*			
0	0	0	0	1
5	0.0873	0.0872	0.0875	0.9962
10	0.1745	0.1736	0.1763	0.9848
15	0.2618	0.2588	0.2679	0.9659

∴ Eq. (a) is

$$\Delta N - T\frac{\Delta\theta}{2} - (T+\Delta T)\frac{\Delta\theta}{2} = 0 \;; \quad \therefore \Delta N - T\Delta\theta = 0$$
$$(T+\Delta T) - T - f\Delta N = 0 \;; \quad \therefore \frac{\Delta T}{f} - \Delta N = 0 \qquad \text{(b)}$$

$$\therefore \frac{\Delta T}{f} - T\Delta\theta = 0 \rightarrow \frac{\Delta T}{\Delta\theta} = fT \qquad \text{(c)}$$

For $\Delta\theta \to 0$,

$$\frac{dT}{d\theta} = fT \qquad \text{(d)}$$

Integrating (d),

$$\mathrm{dT/T = f\ d\theta}$$

$$\int_{T_0}^{T} dT/T = \int_0^{\theta} f\ d\theta \quad \rightarrow \quad ln(T/T_0) = f\theta + C$$

$$\therefore\ T = T_0 e^{f\theta}$$

Applying the boundary condition $T = T_{AD}$ at $\theta = 0$

$$T = T_{AD}e^{f\theta} \qquad \text{(e)}$$

Applying the boundary condition $T = T_{BC} = 40\ kN$ at $\theta = \pi$

$$T = 11.38\ e^{0.4\theta}\ kN \qquad \text{(f)}$$

▷ Force-deformation relation

$$\Delta\delta = \frac{T(R\Delta\theta)}{AE} \tag{g}$$

∴ See that the elongation varies with position along the band.

▷ Geometry

In the limit as $\Delta\theta \rightarrow 0$, this sum becomes the following integral:

$$\delta_{AB} = \int_{\theta=0}^{\theta=\pi} d\delta = \int_0^{\pi} \frac{TRd\theta}{AE} \tag{h}$$

$$= \frac{T_{AD}R}{AE}\int_0^{\pi} e^{f\theta}\, d\theta = \frac{T_{AD}R}{AEf}(e^{f\pi}-1) = \frac{11.38\times300\times(e^{0.4\pi}-1)}{1.6\ (50)(10^{-6})\ (205\times10^{6})} = 1.31\ \ mm$$

2.4 Statically indeterminate situation

→ We must examine the deformation of the system in order to determine the manner in which the forces are distributed within the system.

▶ Example 2.9 Figure 2.14 (a) shows the pendulum of a clock which has a 12-N weight suspended by three rods of 760 mm length. Two of the rods are made of brass and the third of steel. We wish to know how much of the 12-N suspended weight is carried by each rod. Our model of the system is shown in Fig 2.14 (b).

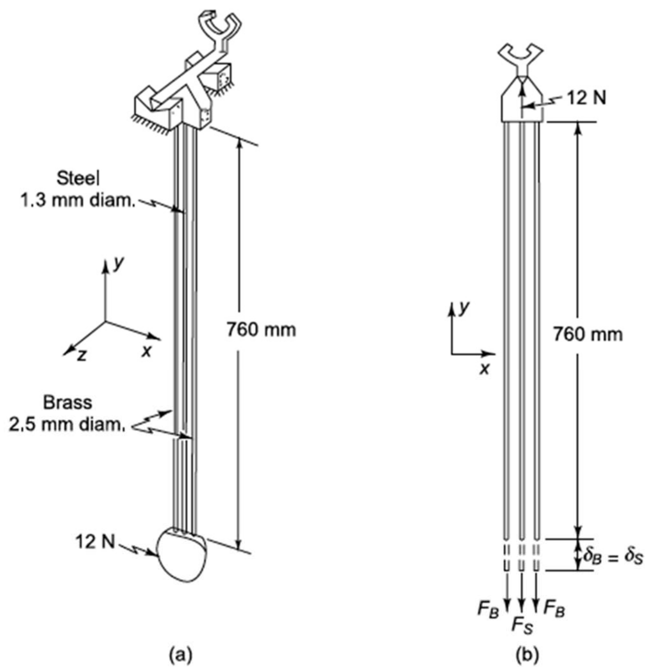


Fig. 2.14 Example 2.9

▷ Equilibrium

$$\sum F_y = 12 - F_S - 2F_B = 0 \tag{a}$$

▷ Geometry

$$\delta_S = \delta_B \tag{b}$$

▷ Force-deformation relation

$$\delta_S = \left(\frac{FL}{AE}\right)_S$$
$$\delta_B = \left(\frac{FL}{AE}\right)_B \qquad \text{(c)}$$

From Eq. (b) & (c)

$$F_S = \frac{A_S E_S L_B}{A_B E_B L_S} F_B = \frac{(1.3)^2(200)760}{(2.5)^2(100)760} F_B = 0.541 \ \ F_B \qquad \text{(d)}$$

Combining (a) and (d), we find

$$F_S = 2.55N \ \ \& \ \ F_B = 4.72N$$

▶ <u>Example 2.10</u> Figure 2.15 (a) shows an instrument suspension consisting of two aluminum bars and one steel rod mounted in a stiff frame, together with a spring EA which is inclined at 45° to BA. In assembly the nut on the steel rod at D is tightened so there is no slack in the line BAD, and then the spring EA is installed with sufficient extension to produce a force of 50 N. We wish to find the deflection of the joint A (relative to the frame) caused by the spring loading.

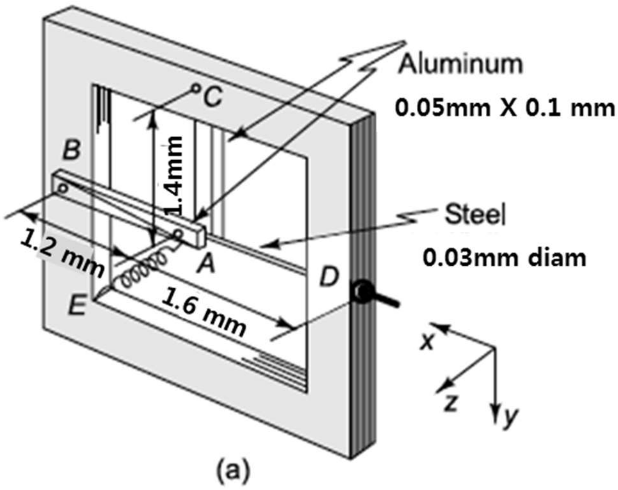


Fig. 2.15 Example 2.10(a)

▷ Assumption

i) The frame is essentially rigid compared to the aluminum bars and the steel rod

ii) Consider the steel rod to be pinned at point D

▷ F.B.D

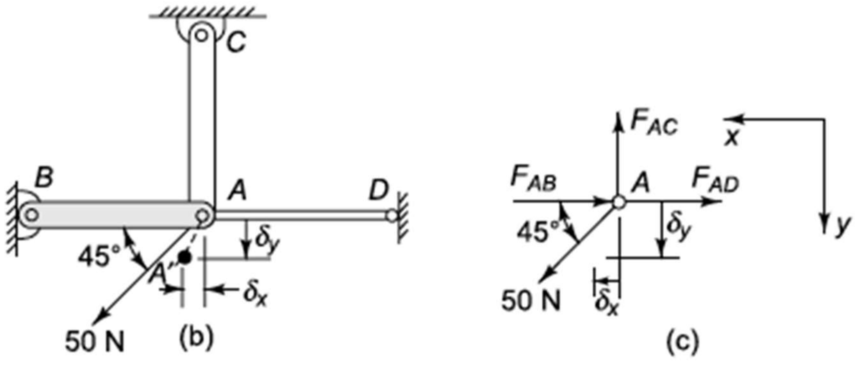


Fig. 2.15 *Example 2.10(a)*

▷ Equilibrium

$$\sum F_x = \frac{50}{\sqrt{2}} - F_{AD} - F_{AB} = 0$$
$$\sum F_y = \frac{50}{\sqrt{2}} - F_{AC} = 0 \tag{a}$$

▷ Geometry

$$\delta_{AC} = \delta_y \ (+)$$
$$\delta_{AD} = \delta_x \ (+) \tag{b}$$
$$\delta_{AB} = \delta_x \ (-)$$

▷ Force-deformation relation

$$\delta_{AC} = \left(\frac{FL}{AE}\right)_{AC} = \frac{35.3553 \ (1.4)}{0.005 \ (69\times10^3)} = 0.1435 \ mm$$

$$\delta_{AD} = \left(\frac{FL}{AE}\right)_{AD} = \frac{F_{AD} \ (1.6)}{0.00071 \ (205\times10^3)}$$

$$\delta_{AB} = \delta_{AD} = \left(\frac{FL}{AE}\right)_{AB} = \frac{F_{AB} \ (1.2)}{0.005 \ (69\times10^3)}$$

From Eqs. (a)~(b)

$$F_{AD} = 8.498 \ N \ (+) \quad \& \quad F_{AB} = 26.8573 \ N \ (-)$$

$$\delta_y = 0.1435 \ mm \quad \& \quad \delta_x = 0.0934 \ mm$$

cf. Skip the chapter 2.5 (computer analysis) → M2794.001000

2.6 Elastic energy; Castigliano's theorem

<<Potential Energy>>

$dW = \mathbf{F} \cdot \mathrm{d}\mathbf{s} = F \cos\theta \, ds$

→ Total work done by $\mathbf{F}$ is; $W = \int \mathbf{F} \cdot \mathrm{d}\mathbf{s}$

1) When work is done by an external force on certain systems, their internal geometric states are altered in such a way that they have the potential to give back equal amounts of work whenever they are returned to their original configurations.

   cf. Such systems are called conservative, and the work done on them is said to be stored in the form of potential energy.

   cf. The system should be elastic, but not necessarily linear.

2) That is, Total work $W = Potential\ \ Energy$ → $Conservative$

   $W = \int \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_0^\delta F \ \ d\delta = U$ (2.3)

   (where δ is elongation)

   cf. This relationship appears in Fig. 2.19 (b)

3) $U = f \ \ (\delta)$

   From Fig. 2.19 (b)

   If this spring should happen to be part of a larger elastic system, it will always contribute the energy (2.3) to the total stored energy of the system whenever its individual elongation is δ.

4) Total work done by all the external loads = Total potential energy U stored by all the internal elastic members

   $U(\mathbf{s}_i) = \sum_i \int_0^{\mathbf{s}_i} \mathbf{P}_i \cdot d\mathbf{s}_i = U$ (2.4)

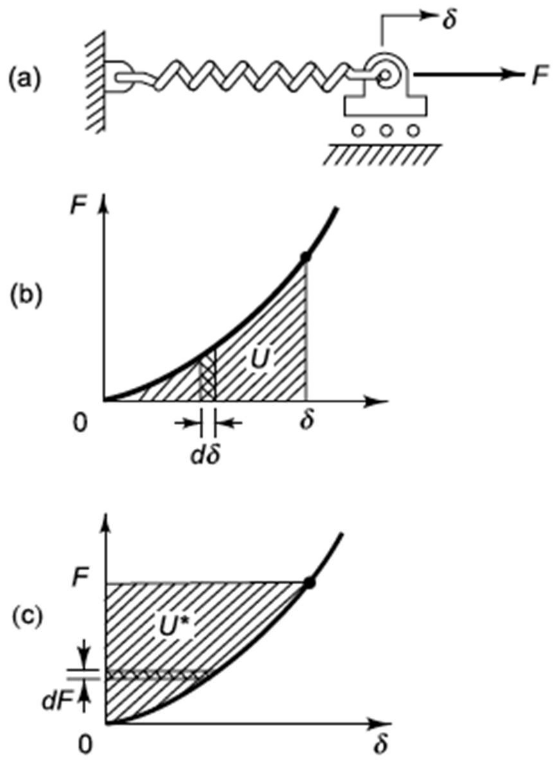


Fig. 2.19

<<Complementary Energy>>

$dW^* = \mathbf{s} \cdot d\mathbf{F} = s \cos\theta \, dF$

→ $W^* = \int \mathbf{s} \cdot d\mathbf{F}$

1) When complementary work is done on certain systems, their internal force states are altered in such a way that they are capable of giving up equal amounts of complementary work when they are returned to their original force states.

cf. The class of systems which store complementary energy include all elastic system for which the equilibrium requirements can be applied in the un-deformed configuration.

2) $W^* = \int \mathbf{s} \cdot d\mathbf{F} = \int_0^F \delta \ dF = U^*$ (2.5)

3) $U^* = f(F)$

From Fig. 2.19 (c), if this spring should happen to be part of a larger elastic system, it will always contribute the complementary energy (2.5) to the total system complementary energy whenever the force in it has the value F.

4) Total complementary work done by all the external loads = Total complementary energy U^* stored by all the internal elastic members

$$U^*(\mathbf{P}_i) = \sum_i \int_0^{\mathbf{P}_i} \mathbf{s} \cdot d\mathbf{P}_i = \sum_i \int_0^{\mathbf{P}_i} \delta_i \cdot dP_i \tag{2.6}$$

<<Castigliano's Theorem>>

▶ 1^{st} Theorem

Force increment (ΔP_i^*) → Internal force change → Increment of complementary work → Increment of complementary energy

→ From $\delta_i \Delta P_i^* = \Delta U^*$

$$\Delta U^* / \ \Delta P_i = \delta_i \qquad \therefore \ \partial U^* / \partial P_i = \delta_i \tag{2.7}$$

→ If the total complementary energy U* of a loaded elastic system is expressed in terms of the loads, the in-line deflection at any particular loading point is obtained by differentiating U* with respect to the load at that point.

cf. The theorem can be extended to include moment loads

$$\therefore \partial U^* / \partial M_i = \phi_i \tag{2.8}$$

▷ In linear system, the force-deformation relation is linear in Fig. 2.19; that is, $U^* = U$

i) $\frac{1}{2} k\delta^2 (= U) = \frac{1}{2} F\delta = \frac{F^2}{2k} (= U^*)$ (2.10)

ii) $\frac{EA}{2L} (= U) = \frac{1}{2} P\delta = \frac{P^2 L}{2EA} (= U^*)$ (2.11)

→ To apply Castigliano's theorem to a linear-elastic system it is necessary to express the total elastic energy of the system in terms of the loads.

▷ In Fig. 2.20, if we denote the $\mathbf{s}_i$-direction component of $\mathbf{P}_i$ as f_i,

$$f_i = \frac{\partial U}{\partial s_i}$$

((proof)

$$\mathbf{P}_i \cdot \Delta\mathbf{s}_i = \Delta U$$

$$\mathbf{P}_i \cdot \Delta\mathbf{s}_i = P_i \Delta s_i \cos\theta = f_i \Delta s_i$$

$$\therefore f_i \Delta s_i = \Delta U$$

→ $f_i = \partial U / \partial s_i$ (1^{st} theorem)

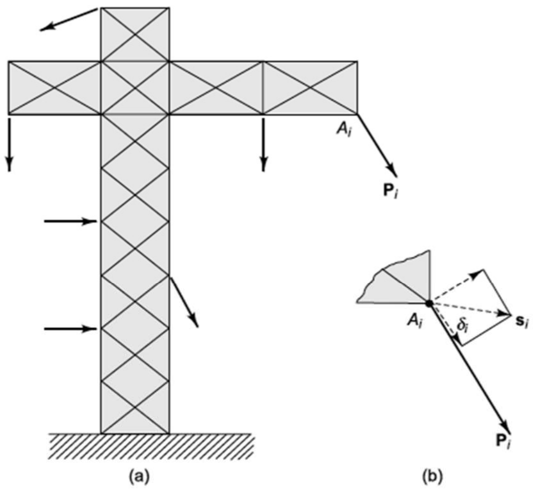


Fig. 2.20 General elastic structure (a) subjected to loads $\mathbf{P}_i$ applied at points A_i; (b) enlarged view showing displacements $\mathbf{s}_i$ at points A_i

▶ 2^{nd} Theorem

For linear elastic system,

$$\delta_i = \partial U / \partial P_i \tag{2.12}$$

▶ Example 2.11 Consider the system of two springs shown in Fig. 2.21 We shall use Castigliano's theorem to obtain the deflections δ_1 and δ_2 which are due to the external loads P_1 and P_2.

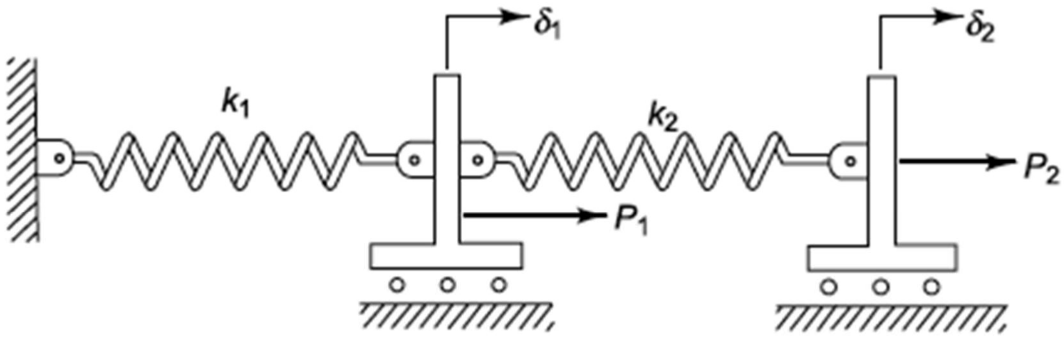


Fig. 2.21 Example 2.11

To satisfy the equilibrium requirements,

$$F_1 = P_1 + P_2 \tag{a}$$

$$F_2 = P_2$$

From Eq. (2.10),

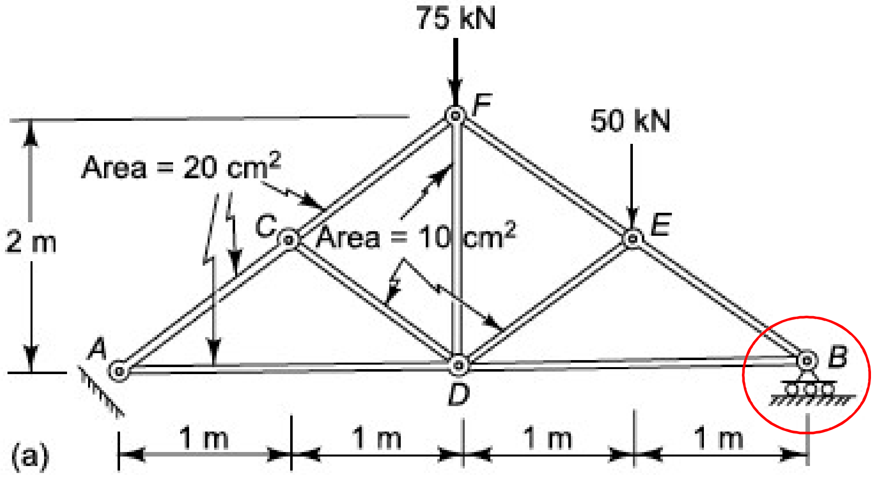
$$\mathrm{U} = \mathrm{U}_1 + U_2 = (P_1 + P_2)^2/(2k_1) + P_2^2/(2k_2)$$

$\therefore \delta_1 = \partial U/\partial P_1 = (P_1 + P_2)/k_1$

$\delta_2 = \partial U/\partial P_2 = (P_1 + P_2)/k_1 + P_2/k_2$

▶ Example 2.13 Determine deflections in the direction of P and reaction force Q

1) Statically determinate situation (B: roller, which makes reaction force Q be zero)



Recall $\therefore\ \delta = \dfrac{PL}{AE}$ (2.2)

The energy stored in the i^{th} member is

$U_i = F_i^2 L_i/(2A_i E_i)$ (for linear system) (a)

In this case,

$U = \sum_{i=1}^{9} U_i$ (b)

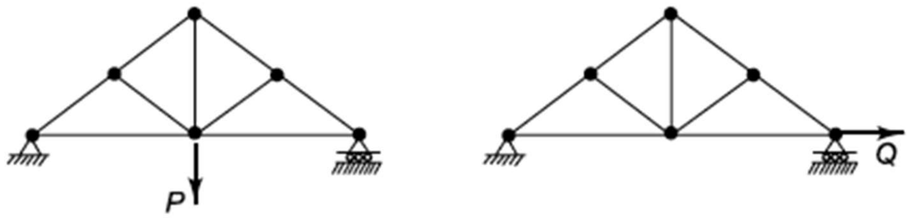


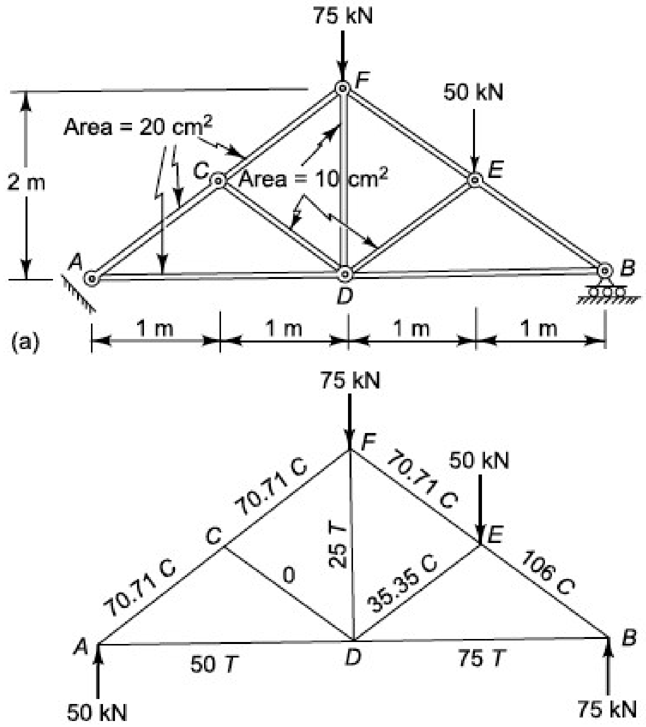
Fig. 2.25 *Unit loads on truss of Example 2.13*

$$\delta_P = \partial U/\partial P = \frac{\partial}{\partial P}\sum_{i=1}^{n}\frac{F_i^2 L_i}{2A_i E_i} = \sum_{i=1}^{n} F_i \frac{L_i}{A_i E_i}\frac{\partial F_i}{\partial P} \quad \text{(c)}$$

$$= \frac{\partial U}{\partial F}\frac{\partial F}{\partial P} = \frac{FL}{EI}\frac{\partial F}{\partial P}$$

$Where\ \ U = F^2/(2k) = F^2L/(2EI)$ (d)

The quantity $\partial \mathrm{F_i}/\partial \mathrm{P}$ which represents the rate of change of the force in the i^{th} member with load P, can be thought of as the load in the i^{th} member due to a unit load P.



In Table 2.4 we have tabulated the individual quantities in (c) as well as their products.

$$\delta_P = \sum_{i=1}^{n} F_i \frac{L_i}{A_i E_i} \frac{\partial F_i}{\partial P} = F \frac{L}{EI} \frac{\partial F}{\partial P}$$

Cf. At the case of statically determinate situation, substitute Q values of Table 2.4 to zero.

Table 2.4 *Truss solution by energy methods*

i	F_i kN	(L/AE) m/kN	$\frac{\partial F_i}{\partial P}$	$\frac{\partial F_i}{\partial Q}$	$\left(\frac{FL}{AE}\frac{\partial F}{\partial P}\right)_i$	$\left(\frac{FL}{AE}\frac{\partial F}{\partial Q}\right)_i$
1	$+50+Q$	1.142×10^{-5}	1/2	+1	2.855×10^{-4}	5.71×10^{-4}
2	$+75+Q$	1.142×10^{-5}	1/2	+1	4.282×10^{-4}	8.565×10^{-4}
3	-106	8.08×10^{-6}	$-1/\sqrt{2}$	0	6.056×10^{-4}	
4	-70.71	8.08×10^{-6}	$-1/\sqrt{2}$	0	4.04×10^{-4}	
5	-70.71	8.08×10^{-6}	$-1/\sqrt{2}$	0	4.04×10^{-4}	
6	-70.71	8.08×10^{-6}	$-1/\sqrt{2}$	0	4.04×10^{-4}	
7	0	2.02×10^{-5}	0	0	0	
8	-35.35	2.02×10^{-5}	0	0	0	
9	$+25$	2.85×10^{-5}	1	0	7.125×10^{-4}	
					$\Sigma=32.438\times10^{-4}$ $=\delta_y$	$\Sigma=14.275\times10^{-4}$ $=\delta_x$

2) Statically indeterminate situation (B: not a roller, which makes reaction force Q be non-zero)

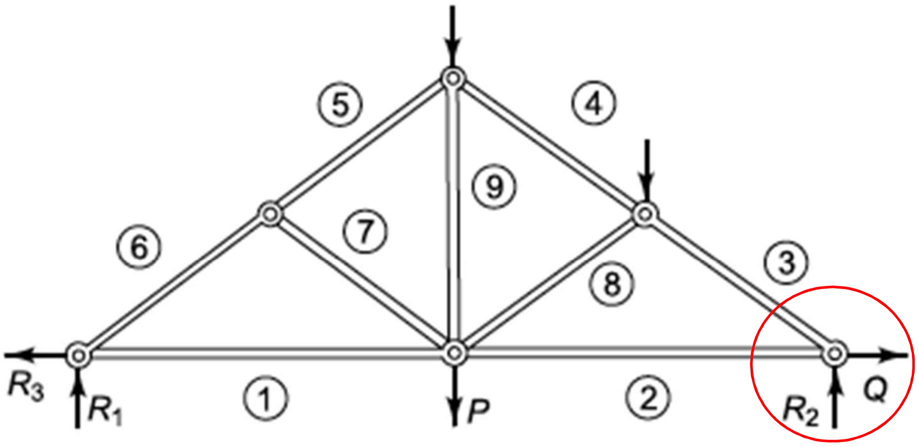


Fig. 2.24 *Example 2.13*

We simply require that $\partial U/\partial Q = 0$ as there is no horizontal motion at the point at which Q acts.

As there is no horizontal motion at the point at which Q acts, $\partial U/\partial Q = 0$

Thus, from Eq. (c) and Table 2.4,

$$\delta_B = \partial U/\partial Q = \sum F_i \frac{L_i}{A_i E_i}\frac{\partial F_i}{\partial Q} = 0$$

$$= \sum F_i \frac{L_i}{A_i E_i}\frac{\partial F_i}{\partial Q} = [50 + Q + 75 + Q]\ [1.142 \times 10^{-5}] = 0$$

$\rightarrow Q = -62.5\ kN$

If now we wish to solve for the deflection at P, we must reevaluate the products in rows 1 and 2 of Table 2.4 with Q at its actual value as determined above.

$\therefore \delta_P = \partial U/\partial P = 32.438 \times 10^{-4} - 1.142Q \times 10^{-5} = 32.438 \times 10^{-4} - 7.1375 \times 10^{-4}$
$\therefore \delta_P = 2.53 \times 10^{-3}\ m$

i	$\left(\frac{FL}{AE}\frac{\partial F}{\partial P}\right)_{i,Q\neq 0}$
1	$2.855 \times 10^{-4} + 0.571\ Q \times 10^{-5}$
2	$4.282 \times 10^{-4} + 0.571\ Q \times 10^{-5}$